Dual-Quaternion-Based Spacecraft Autonomous Rendezvous and Docking Under Six-Degree-of-Freedom Motion Constraints

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This paper addresses the integrated attitude and position control problem for the final phase proximity operations of spacecraft autonomous rendezvous and docking, in which important motion constraints of the chaser spacecraft are considered. On the one hand, to ensure reliable real-time measurements of the relative attitude and position information between two spacecraft, the relevant sensor system of the chaser spacecraft is required to continuously point toward the target; on the other hand, for the proximity safety concerns, the chaser also needs to follow a specified approach path constraint. A special dual-quaternion-based artificial potential function is presented to encode information regarding these motion constraints. Using this potential function, a novel six-degree-of-freedom control method is proposed to ensure the arrival of the chaser at the docking port of the target with a desired relative attitude, while strictly complying with all constraints. The closed-loop stability is demonstrated by a Lyapunov-based method in conjunction with the special properties of the artificial potential function. The local minimum problem associated with the artificial potential function can be addressed by selection of control parameters that satisfies a mild condition.

Simulation results of prototypical spacecraft rendezvous and docking missions are provided to illustrate the effectiveness of the proposed method.

I. Introduction

AUTONOMOUS spacecraft rendezvous and docking (RVD) operations have drawn significant attention during the last several decades, with wide engineering applications, such as on-orbit refueling [1] and assembly [2]. As an interesting case study, experiment results of the RVD mission of Engineering Test Satellite VII were provided in [3]. A safe trajectory design method was introduced in [4]. Singla et al. [5] developed an adaptive output feedback control law for spacecraft RVD under measurement uncertainties. An optimal trajectory planning scheme for proximity operations was presented in [6].

Typically, spacecraft RVD operations usually have several main phases [7], including homing phase [8], closing phase [9], final phase, and proximity operations. Among these, the last two phases are crucial from the point of view of mission complexity and safety requirement. Philip and Ananthasayanam [7] introduced relative estimation and control schemes for the final phase of an RVD mission. A model predictive control (MPC) method for proximity operations for low-thrust spacecraft was presented in [10]. Unlike the traditional and fully decoupled attitude or position control problems [11,12] (three-degree-of-freedom control problems) of spacecraft, in RVD missions, especially in final phase and proximity operations, we are usually presented with six-degree-of-freedom (6-DOF) control problems, in which a chaser spacecraft should synchronously track both the time-varying relative positions and the desired attitude trajectories accurately with respect to a target spacecraft. Sun and Huo [13] presented a 6-DOF feedback controller for spacecraft proximity operations with parametric uncertainties. Lv et al. [14] further addressed the saturation problem for a 6-DOF synchronized control problem of spacecraft. A control scheme for spacecraft docking using only the intersatellite electromagnetic force was introduced in [15]. In recent years, the dual-quaternion formalism has aroused extensive interest in the literature regarding 6-DOF spacecraft dynamics modeling and controller design. Dual quaternions are derived from the traditional Euler quaternions and inherit most of their elegant mathematical properties, but still different in the sense that dual quaternions can be used to describe, not only the rotational motion, but also the translational motion of rigid bodies. Compared with other 6-DOF description and modeling methods of spacecraft motion, the dual-quaternion formalism has clear physical meaning and compact mathematical description, and the dual-quaternion-based dynamics can automatically take into account the motion couplings between the rotational and translational motion. Wang et al. [16] applied the dual-quaternion-based model to spacecraft dynamics and presented several finite time control laws based upon the sliding mode method. By employing dual quaternions, Filipe and Tsiotras [17] introduced a robust control method with additional mass and inertia identification mechanisms, and a velocity-free control law for spacecraft pose tracking problems was proposed in [18]. In [19,20], fault-tolerant issues under similar mathematical background are further addressed.

When viewed from the practical standpoint of final phase and proximity operations in an RVD mission, the chaser spacecraft needs to follow some important motion constraints. On the one hand, per safety requirements, to avoid collisions during proximity operations, the translational guidance is required to guarantee the chaser can approach the docking port of the target from a safe docking direction (which is usually referred to as the approach path constraint in literature). Zhang et al. [21] proposed a fuzzy-logic control method for a spacecraft docking task, in which the approach path constraint was taken into account. Liu and Lu [22] solved the fuel-optimal problem for spacecraft RVD missions subject to an approach cone restriction and actuator constraints. On the other hand, spacecraft
II. Mathematical Preliminaries

In this section, necessary mathematical foundations of quaternions and dual quaternions are introduced, and then a dual-quaternion-based relative motion model of spacecraft is presented.

A. Quaternions

The Euler quaternion is a widely used method for rotation description of rigid bodies. The definition of a unit quaternion is

\[ q = [q, \xi] \in \mathbb{H} \]

(throughout the paper, \( \mathbb{H} \) is used to denote the set of unit quaternions); \( q \in \mathbb{R} \) and \( \xi = [\xi_1, \xi_2, \xi_3] \in \mathbb{R}^3 \) are, respectively, called the scalar part and the vector part of \( q \), with \( q^2 + \xi^2 = 1 \). Some important operations of unit quaternions are given as follows:

\[
q_1 \otimes q_2 = \left[ q_1, \xi_1 \right] \left[ q_2, \xi_2 \right] = \left[ q_1 q_2 - \xi_1^T \xi_2, (q_1 \xi_2 + q_2^T \xi_1) \right] \triangleq [q_{12}]_{0}\xi_2
\]

where

\[
[q]_{0} = \left[ \eta, -\xi^T \right] \quad S(\xi) = \begin{bmatrix}
0 & -\xi_3 & \xi_2 \\
\xi_3 & 0 & -\xi_1 \\
-\xi_2 & \xi_1 & 0
\end{bmatrix}
\]

\[
q_1^* = [q_1, -\xi_1^T] \quad q_1^* \otimes q_1 \otimes q_2 = q_2
\]

for all \( q_1, q_2 \in \mathbb{H} \). Also, for any \( v_1, v_2 \in \mathbb{R}^3 \) and \( q \in \mathbb{H} \), it is guaranteed that [26]

\[
v_1^T v_2 = (v_1 \otimes q)^T (v_2 \otimes q) = (q \otimes v_1)^T (q \otimes v_2)
\]

It should be mentioned that, throughout the paper, when any three-dimensional vector is operated with a quaternion, the vector should be regarded as an equivalent embedded quaternion with a vanishing zero scalar part, to guarantee the dimensions are matched, for example, \( v_1 \otimes [0, v_1]^T \).

Consider an inertial frame \( \mathcal{N} \) and also the body-fixed frames \( B \) and \( T \) of the chaser and target spacecraft, respectively. Using \( q_t \) to denote the unit quaternion of frame \( B \) with respect to frame \( \mathcal{N} \), and then the coordinate transformation of any vector \( a \in \mathbb{R}^3 \) from frame \( \mathcal{N} \) to frame \( B \), can be represented by the following equation:

\[
a^B = q_t^* \otimes a \otimes q_t
\]

Throughout the paper, superscripts (such as \( b, n, t \)) are employed to denote variables that are expressed in the corresponding frames (\( B, N, T \)).

Similarly, using \( q_t \) to denote the unit quaternion of frame \( T \) with respect to frame \( N \), we can further have

\[
a^B = q_{ht}^* \otimes a^t \otimes q_{ht}
\]

Note that \( q_{ht} = q_t^* \otimes a^t \otimes q_t \) is defined as the relative quaternion of frame \( B \) with respect to frame \( T \).

B. Dual Numbers and Dual Vectors

The concept of dual quaternions was introduced by Clifford [27] and subsequently perfected by Study [28]. Before giving the definition of dual quaternions, dual numbers should be introduced first. The definition of dual numbers is

\[
\hat{a} = a_0 + ea_d
\]

wherein \( a_0, a_d \in \mathbb{R} \) are called the real part and the dual part of \( \hat{a} \), respectively. Throughout the paper, \( \mathbb{R} \) is employed to denote the set of real numbers, and the superscript implies the corresponding quantity belongs to \( \mathbb{R}^4 \) (or \( \mathbb{R}^{4n} \) in vector cases). Furthermore, \( e \) is called the dual unit, satisfying

\[
e \neq 0 \quad \text{but} \quad e^2 = 0
\]

Next, the definition of dual vectors is

\[
\hat{a} = a_e + ea_d
\]

in which \( a_e, a_d \in \mathbb{R}^m \).

require tight relative attitude and position knowledge to support relative 6-DOF motion control, and the primary means to enforce these specifications is through the adoption of vision sensors [23] and similar photoelectric sensor systems. To permit reliable real-time measurements, the target must always stay within the field of view of the vision sensor system of the chaser. However, due to structural constraints, vision sensors usually only have limited fields of view (i.e., limited line-of-sight angles), and therefore, the attitude motion of the chaser also needs to be properly controlled to satisfy this requirement (referred to as the field-of-view constraint). Garcia and How [24] considered the field-of-view constraint and proposed a trajectory optimization scheme for the reconfiguration of spacecraft formations. Combining both the approach path constraint and the field-of-view constraint with the original RVD mission will lead to a challenging 6-DOF constrained control problem, which is typically difficult to be solved by many existing conventional control methods, and most of the existing literature can only solve a part of this complex problem (see, for example, [21,22,24]). It is important to mention that Lee and Mesbahi [25,26] addressed a somewhat similar complex problem (see, for example, [21,22,24]). It is important to mention that, throughout the paper, when any three-dimensional vector is operated with a quaternion, the vector should be regarded as an equivalent embedded quaternion with a vanishing zero scalar part, to guarantee the dimensions are matched, for example, \( v_1 \otimes [0, v_1]^T \).

Consider an inertial frame \( \mathcal{N} \) and also the body-fixed frames \( B \) and \( T \) of the chaser and target spacecraft, respectively. Using \( q_t \) to denote the unit quaternion of frame \( B \) with respect to frame \( \mathcal{N} \), and then the coordinate transformation of any vector \( a \in \mathbb{R}^3 \) from frame \( \mathcal{N} \) to frame \( B \), can be represented by the following equation:

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\]

Throughout the paper, superscripts (such as \( b, n, t \)) are employed to denote variables that are expressed in the corresponding frames (\( B, N, T \)).

Similarly, using \( q_t \) to denote the unit quaternion of frame \( T \) with respect to frame \( N \), we can further have

\[
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\[
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\]

Next, the definition of dual vectors is

\[
\hat{a} = a_e + ea_d
\]

in which \( a_e, a_d \in \mathbb{R}^m \).
Some basic operations of dual numbers and dual vectors used in this paper are introduced in Appendix A, following the results from [16,17].

C. Dual Quaternions

Dual quaternions can be regarded as the combination of dual numbers and traditional Euler quaternions. The dual quaternion of frame $B$ with respect to frame $N$ is defined as

$$\hat{q}_b = q_b + \varepsilon \frac{1}{2} q_b \otimes r^b_t = q_b + \varepsilon \frac{1}{2} r^b_t \otimes q_b$$

(10)

wherein $r_b$ is the relative position vector of the two frames. Similarly, the relative dual quaternion of frame $B$ with respect to frame $T$ is given as

$$\hat{q}_{bT} = q_{bT} + \varepsilon \frac{1}{2} q_{bT} \otimes r^b_t = q_{bT} + \varepsilon \frac{1}{2} r^b_t \otimes q_{bT}$$

(11)

where $r^b_{bt}$ and $r^b_{bT}$ are the relative position vector of $B$ with respect to $T$, and expressed in frames $B$ and $T$, respectively.

Throughout the paper, we use $\hat{\mathbb{H}}$ to denote the set of dual quaternions. A dual quaternion $\hat{q} \in \hat{\mathbb{H}}$ can also be written as $\hat{q} = [\tilde{q}, \xi]^T$, where $\tilde{q} \in \mathbb{R}$ and $\xi = [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3]^T \in \mathbb{R}^3$ are the scalar part and the vector part of $\hat{q}$, respectively.

Dual quaternions follow the operations of dual vectors (refer to Appendix A), and some other commonly used operations and properties of dual quaternions are introduced as follows:

$$\text{vec}(\hat{q}) = \tilde{\xi}$$

(12)

$$\hat{q}^* \equiv \begin{bmatrix} \tilde{\eta} & -\tilde{\xi}^T \end{bmatrix}^T$$

(13)

$$\hat{q}_1 \otimes \hat{q}_2 = \begin{bmatrix} \hat{q}_1 \tilde{q}_2 - \tilde{\xi}_1 \hat{q}_2 - \tilde{\xi}_2 \hat{q}_1 + \tilde{\xi}_3 \hat{\xi}_2 - \tilde{\xi}_2 \hat{\xi}_3 \end{bmatrix} \triangleq [\hat{q}_1]_{a} \hat{q}_2$$

(14)

where $\hat{q}_1 = [\tilde{q}_1, \tilde{\xi}_1]^T \in \hat{\mathbb{H}}$ and $\hat{q}_2 = [\tilde{q}_2, \tilde{\xi}_2]^T \in \hat{\mathbb{H}}$, and

$$[\hat{q}]_{a} = \begin{bmatrix} \tilde{\eta} & -\tilde{\xi}^T \end{bmatrix} \in \mathbb{H}$$

$$S(\xi) = \begin{bmatrix} 0 & -\tilde{\xi}_3 & \tilde{\xi}_2 \\ \tilde{\xi}_3 & 0 & -\tilde{\xi}_1 \\ -\tilde{\xi}_2 & \tilde{\xi}_1 & 0 \end{bmatrix}$$

(15)

D. Relative Motion Kinematics and Dynamics

Based on the dual-quaternion formulation, the relative motion kinematics and dynamics of frame $B$ with respect to frame $T$ are given as follows [16,17]:

$$\ddot{\hat{q}}_{bT} = \frac{1}{2} \dot{\hat{q}}_{bT} \otimes \ddot{\hat{q}}^b_{bT}$$

(16)

$$\dot{J}_b \ddot{\hat{q}}_{bT} = -\omega^b_{bT} \times (\dot{J}_b \omega^b_{bT}) + \hat{J}_b (\omega^b_{bT} \times \omega^b_{bT} - \dot{q}_{bT} \otimes \dot{\hat{q}}^b_{bT} \otimes \ddot{q}_{bT}) + \ddot{u}$$

(17)

wherein $\omega^b_{bT} = \omega^b_{bT} + \varepsilon (\dot{r}^b_t, \omega^b_{bT} \times \dot{r}^b_t)$ is called the relative dual angular velocity between the two frames, with $\omega^b_{bT}$ and $\dot{r}^b_t$ as the relative angular velocity and the relative linear velocity of $B$ with respect to $T$, respectively. Similarly, $\dot{\omega}^T = \dot{\omega}^T + \varepsilon (\dot{r}^T, \omega^T \times \dot{r}^T)$ is the dual angular velocity of $T$ with respect to $N$, $\dot{\omega}^T = \dot{\omega}^T \otimes \dot{\omega}^T \otimes \dot{q}_{bT}$, and the variable $\dot{\omega}^T$ is the time derivative of $\omega^T$. A reasonable assumption is that both $\dot{\omega}^T$ and $\dot{\omega}^T$ are bounded. Furthermore, $\ddot{u} = f^T + \varepsilon \tau^T$ is called the total dual control input applied to the chaser, and here $f^T \in \mathbb{R}^3$ and $\tau^T \in \mathbb{R}^3$ are the total force and the total torque applied to the chaser spacecraft, respectively. $\hat{J}_b$ is the dual inertia of the chaser, with the definition

$$\dot{J}_b = m_b I_b \frac{d}{de} + \varepsilon J_b$$

(18)

in which $m_b \in \mathbb{R}$ is the mass of the chaser, and $J_b \in \mathbb{R}^{3x3}$ is the mass moment of inertia of the chaser in frame $B$. By this definition, for any dual vector $\tilde{a} = a_r + \varepsilon a_d \in \mathbb{R}^3$, we have

$$\dot{\hat{J}}_b \ddot{a} = m_b a_d + \varepsilon J_b a_r$$

(19)

An important feature of dual inertias is that they do not, in general, satisfy the law of association, formalized as

$$\tilde{a}^T (\hat{J}_b \tilde{b}) \neq (\tilde{a}^T \hat{J}_b) \tilde{b}$$

(20)

for $\tilde{a}, \tilde{b} \in \mathbb{R}^3$. However, one has that

$$\hat{J}_b (\tilde{J}_b^{-1} \tilde{a}) = \hat{J}_b (\tilde{J}_b \tilde{a}) = \tilde{a}$$

(21)

and here

$$\hat{J}_b^{-1} = J_b^{-1} \frac{d}{de} + \frac{1}{m_b} I_b \varepsilon \tau$$

defined as the inverse of $\hat{J}_b$.

III. Motion Constraint Description

As mentioned in Sec. I, during the proximity process, the chaser should follow both the field-of-view constraint and the approach path restriction. In this section, these constraints are further discussed and formulated in terms of dual quaternions.

A. Field-of-View Constraint

Reliable measurements of relative motion states are crucial in the final phase and proximity operations of spacecraft RVD, and the vision-based sensor system is one of the primary means to achieve this goal. For example, docking experiments were conducted with the Tiangong-2 target spacecraft of China in 2016, in which charge-coupled device optical imaging sensors were employed for final proximity operations. However, due to structural restrictions (e.g., the view limitation of the lens), the vision sensor system usually just has a limited field of view (FOV). Although certain special optical devices may have relatively larger fields of view (still limited), noise levels typically increase significantly when measurements fall near the edge of the fields of view [23]. It is therefore important to ensure that the vision sensor system of the chaser continuously points to the target and keeps the target within a specified FOV. This is the so-called field-of-view constraint, as illustrated in Fig. 1.

In Fig. 1, $\mathbf{e}_b$ is the unit central line-of-sight vector of the vision sensor system, and $\mathbf{a}_{max}$ is the maximum allowable (half) line-of-sight angle. As discussed, to satisfy the constraint, $-r^b_{bT}$ should always stay within $\mathbf{a}_{max}$ about $\mathbf{e}_b^b$, which can be formulated as

$$\hat{J}_b \ddot{\hat{q}}_{bT} = m_b I_b \frac{d}{de} + \varepsilon J_b$$

(18)
\[ -r^b_{ht} \cdot c^b > \| r^b_{ht} \| \cos \alpha_m \]  

(22)

Notice throughout the paper that the notation \( \| \cdot \| \) indicates the Euclidean norm. By the property of unit quaternions given in Eq. (4), one has

\[ r^b_{ht} \cdot c^b = (q_{bt} \otimes r^b_{ht} \otimes \hat{c}^b) \]

(23)

and Eq. (23) can be further rewritten to a dual-quaternion form

\[ (q_{bt} \otimes r^b_{ht} \otimes \hat{c}^b)^T (q_{bt} \otimes c^b) = \hat{q}_{bt} \star (\hat{\Xi} \hat{q}_{ht}) \]

(24)

in which

\[ \Xi = \begin{bmatrix} 0 & e^b \\ e^b & -S(c^b) \end{bmatrix}, \quad \hat{\Xi} = \Xi \frac{d}{dt} \frac{d}{dt} + \epsilon \Xi \]

Furthermore, by the definition of dual quaternions, one can readily show that

\[ \| r^b_{ht} \| \cos \alpha_m = 2(\epsilon_I 4_{2x4}) \star \hat{q}_{bt} \| \cos \alpha_m \]

(25)

Building upon the foregoing analysis, define the following scalar function

\[ f_1(\hat{q}_{bt}) = \frac{\hat{q}_{bt} \star (\hat{\Xi} \hat{q}_{ht})}{\| (2\epsilon_I 4_{2x4}) \star \hat{q}_{bt} \| \cos \alpha_m} + \cos \alpha_m \]

(26)

Thus, by Eq. (22), the field-of-view constraint can be guaranteed if \( \forall \ t \geq 0, f_1(t) < 0 \).

B. Approach Path Constraint

In spacecraft RVD missions, the target (like a space station) may have a large scale with many mission-critical components. As part of the final proximity operations, whenever the relative distances between the two spacecraft are close, in the absence of proper control, the chaser may collide with some components of the target while approaching the docking port. To solve this problem, the chaser should get close to the docking port from a direction ensuring safe docking, which is conventionally mentioned as the final approach corridor in literature.

In this paper, a cone-like final approach corridor is considered, as shown in Fig. 2, in which \( p^b \) is the unit direction vector of the docking port, and \( \beta_m \) is the maximum allowable (half) cone angle of the corridor. To avoid collision, the chaser must approach the docking port within this cone-like corridor. To satisfy this constraint, we should have

\[ r^b_{ht} \cdot p^b > \| r^b_{ht} \| \cos \beta_m \]

(27)

Similar to the condition in Eq. (26), the constraint given in Eq. (27) can be guaranteed by ensuring \( f_2(\hat{q}_{bt}) < 0 \), and therefore,

\[ f_2(\hat{q}_{bt}) = -\frac{\hat{q}_{bt} \star (\hat{\Omega} \hat{q}_{bt})}{\| (2\epsilon_I 4_{2x4}) \star \hat{q}_{bt} \| \cos \beta_m} + \cos \beta_m \]

(28)

wherein

\[ \hat{\Omega} = \Omega^T \frac{d}{dt} + \epsilon \Omega \]

and

\[ \Omega = \begin{bmatrix} 0 & -p^b^T \cdot S(p^b) \end{bmatrix} \]

IV. Potential Function Design and Control Law Development

A. Potential Function Design

To encode constraint information into the controller design, the APF method is employed. The APF method is derived from the concept of potential energies in physics and has been widely applied to develop guidance and control schemes for nonlinear systems, like robots [29] and spacecraft [12,30]. In this subsection, a special dual-quaternion-based APF is designed.

The main control objective of this paper is that the chaser can finally arrive at the docking port of the target with a desired relative attitude. To describe this goal, consider the following constant dual quaternion, which represents the desired final relative pose between the chaser and the target:

\[ \hat{q}_d = q_d + \epsilon \frac{1}{2} r_d \otimes q_d \]

(29)

Here, \( r_d \) is the position vector of the docking port of the target (represented in frame \( T \)) and \( q_d \) is the desired final relative attitude. So, the control objective is to guarantee \( \lim_{t \to \infty} \hat{q}_{bt} = q_d \), while strictly complying with the specified 6-DOF motion constraints. Moreover, \( q_d \) should be properly designed, such that, when \( q_{bt} = q_d \), all constraints must be complied.

Then, further define the following error dual quaternion:

\[ \hat{e} = \hat{q}_{bt} \otimes \hat{q}_{bt} - q_e \]

(30)

wherein \( q_e = q_d \otimes q_d \) and \( r_e = r_{bt} - q_d^b \otimes q_d^b \otimes q_{bt} = r_{bt} - r_d \) are called the error quaternion and the error relative position, respectively. Thus, if \( e = \hat{e} \), one has \( \hat{q}_{bt} = \hat{q}_d \otimes \hat{q}_d = q_d \), and the relative dual quaternion \( \hat{q}_{bt} \) gets the desired value.

Notice that, since \( \hat{q}_d \) is a constant dual quaternion, by Eq. (11), we have

\[ \dot{\hat{e}} = \hat{q}_d \otimes \hat{q}_d = \frac{1}{2} \hat{q}_d \otimes \hat{q}_d \otimes \hat{q}_d^b = \frac{1}{2} \hat{q}_e \otimes \hat{q}_e^b \]

(31)

Under the foregoing definition for \( \hat{e} \), since \( \hat{q}_{bt} = \hat{q}_d \otimes q_e \), \( f_1 \) and \( f_2 \) can be regarded as the functions of \( \hat{q}_e \), given as follows:

\[ f_1(\hat{q}_e) = \frac{(\hat{q}_e \otimes \hat{q}_e) \star (\hat{\Xi} (\hat{q}_d \otimes \hat{q}_d))}{\| (2\epsilon_I 4_{2x4}) \star (\hat{q}_d \otimes \hat{q}_d) \| \cos \alpha_m} + \cos \alpha_m \]

(32)

\[ f_2(\hat{q}_e) = -\frac{\hat{q}_e \star (\hat{\Omega} (\hat{q}_d \otimes \hat{q}_d))}{\| (2\epsilon_I 4_{2x4}) \star (\hat{q}_d \otimes \hat{q}_d) \| \cos \beta_m} + \cos \beta_m \]

(33)

Building upon the foregoing constructions, a scalar artificial potential function is designed as
\[ V_p(\dot{q}_i) = k_p (\dot{q}_i - \dot{q}^*_i) \ast (\dot{q}_i - \dot{q}^*_i) + k_f (\dot{q}_i - \dot{q}^*_i) \ast (\dot{q}_i - \dot{q}^*_i) \left( \frac{1}{f_1(\dot{q}_i)} + \frac{1}{f_2(\dot{q}_i)} \right) \]  

(34)

where \( k_p \) and \( k_f \) are user-defined positive constants. Notice that, if \( f_1 = 0 \) or \( f_2 = 0 \), \( V_p \to \infty \). Therefore, when \( f_1(\dot{q}_i(0)) < 0 \) and \( f_2(\dot{q}_i(0)) < 0 \), if we can ensure \( V_p \in L_\infty \) by employing a proper closed-loop control law, then it can be guaranteed that all constraints are satisfied.

Some important facts of \( V_p \) are given in the following proposition.

**Proposition 1**: When \( \alpha_m, \beta_m \in (-\pi/2, 0) \cup (0, \pi/2) \), for all \( \tilde{q}_i \in \mathbb{D} \), \( \mathbb{D} \) is an empty set, which is an unrealistic situation. Second, from a practical point of view, very few, if any, on-orbit vision-based sensors have a half line-of-sight angle bigger than \( \pi/2 \). Likewise, a cone-like approach corridor with a half-cone angle larger than \( \pi/2 \) usually makes very little practical sense.

### B. Control Law Development and Stability Analysis

Based on the APF designed in the previous subsection, an integrated 6-DOF control method is summarized in the following theorem.

**Theorem 1**: Given the dual-quaternion-based 6-DOF relative motion model in Eqs. (16) and (17), and the potential function \( V_p(\dot{q}_i) \) in Eq. (34), consider the following feedback controller:

\[ \dot{\omega}_b = \frac{1}{2} \text{vec}(\dot{\omega}_b \times (\nabla V_p)) - k_d \dot{\omega}_b - k_b (\dot{q}_b \times (\dot{\omega}_b \times \dot{q}_b)) + \omega_b \times (J_b \dot{\omega}_b) \]  

(36)

in which \( k_d > 0 \) is an user-specified constant. If \( \alpha_m, \beta_m \in (-\pi/2, 0) \cup (0, \pi/2) \), and \( k_p, k_f \) are characterized to satisfy the condition in Eq. (35), then it can be guaranteed that \( \lim_{t \to \infty} \dot{q}_i(t) = \dot{q}^*_i \) and \( \lim_{t \to \infty} \omega_b(t) = 0 \) for all \( \dot{q}_i(0) \in \mathbb{D} \), and furthermore, \( \forall t \geq 0, f_1(\dot{q}_i(t)) < 0 \) and \( f_2(\dot{q}_i(t)) < 0 \).

**Proof**: Consider the following Lyapunov-like candidate function:

\[ V = V_p + \frac{1}{2} (\dot{\omega}_b) \ast (J_b \dot{\omega}_b) \]  

(37)

By the definition of the “\( \ast \)” product, it can be readily demonstrated that \( V \geq 0 \), and \( V = 0 \) only when \( \dot{q}_i = \dot{q}^*_i \) and \( \dot{\omega}_b = 0 \), and so \( V \) is a valid Lyapunov-like candidate function.

The time derivative of \( V \) is

\[ \dot{V} = \nabla V_p \ast \dot{q}_i + \dot{\omega}_b \ast (J_b \dot{\omega}_b) \]  

(38)

Notice that, for any \( \dot{q}_1, \dot{q}_2, \dot{q}_3 \in \mathbb{H} \), it is proved that [18]

\[ \dot{q}_1 \ast (\dot{q}_2 \times \dot{q}_3) = \dot{q}_1 \ast (\dot{q}_2 \times \dot{q}_3) \]  

(39)

By this property, and further consider Eq. (17), Eq. (38) can be rewritten as

\[ \dot{V} = \frac{1}{2} (\dot{\omega}_b) \ast \dot{q}_i \ast (\nabla V_p) + (\dot{\omega}_b) \ast (J_b \dot{\omega}_b) \]  

(40)

Notice that, since \( \dot{\omega}_b = \dot{q}_i \times \dot{q}^*_i \), we have

\[ \dot{\omega}_b \times (J_b \dot{\omega}_b) = \dot{q}_i \times (J_b \dot{\omega}_b) + \dot{q}_i \times (J_b \dot{\omega}_b) + \dot{q}_i \times (J_b \dot{\omega}_b) \]  

(42)

Substituting Eq. (41) into Eq. (40) yields

\[ \dot{V} = -k_d \dot{\omega}_b - \dot{\omega}_b \ast (\dot{\omega}_b) - \dot{\omega}_b \ast (J_b \dot{\omega}_b) \]  

(43)

where the constant \( \dot{\omega}_b = \alpha + \beta \), and then by the definition of \( \dot{\omega}_b \), we have

\[ (\dot{\omega}_b) \ast (\dot{\omega}_b) = (\dot{\omega}_b) \ast (\dot{\omega}_b) \]  

in which the proper \( a \cdot (b \times c) = b \cdot (c \times a) \), for any \( a, b, c \in \mathbb{R}^3 \) is employed. Furthermore, using \( \dot{\omega}_b \) and \( \dot{\omega}_b \) to denote the real part and dual part of \( \dot{\omega}_b \), respectively, while using \( \dot{\omega}_b \) and \( \dot{\omega}_b \) to denote the real part and dual part of \( \dot{\omega}_b \), respectively, we have

\[ \dot{\omega}_b \ast \dot{\omega}_b (J_b \dot{\omega}_b) + \dot{\omega}_b \ast (J_b \dot{\omega}_b) \]  

(44)

Notice \( S(\dot{\omega}_b J_b + J_b S(\dot{\omega}_b)) \) is a skew-symmetric matrix, so that

\[ (\dot{\omega}_b) \ast (\dot{\omega}_b J_b + J_b S(\dot{\omega}_b)) = 0 \]  

(45)

Accordingly, Eq. (42) simplifies to

\[ \dot{V} = \dot{\omega}_b \ast \dot{\omega}_b \]  

(46)

Since \( V \geq 0, \dot{V} \leq 0 \), we have \( V \in L_\infty \), which guarantees \( V_p, \dot{q}_i \), and \( \dot{\omega}_b \in L_\infty \). Equation (46) also shows

\[ \int_0^\infty V(t) \, dt \]  

exists and is finite, and so \( \dot{\omega}_b \in L_2 \). Recalling that both \( \dot{\omega}_b \) and \( \dot{\omega}_b \) are bounded, by Eq. (17), we have \( \dot{\omega}_b \in L_\infty \). Through using Barbalat’s lemma, we can guarantee \( \lim_{t \to \infty} \omega_b(t) = 0 \).

Next, taking the derivative for both sides of Eq. (17), we can further obtain \( \dot{\omega}_b \in L_\infty \), by Barbalat’s Lemma, this result shows that
the local minimum problem indeed occurs as outlined in the [disregarding Eq. (35)]; then, if the system states converge to the local minimum issue when the condition on the control parameters are satisfied such as, for example, the adaptive controllers given in [17–31].

Remark 3: Theorem 1 shows that, when the gains \( k_p \) and \( k_d \) satisfy the condition given in Eq. (35), the local minimum problem (i.e., the chaser converges to a state that is different from the final desired state) can be successfully avoided. Some additional discussions regarding the local minimum issue when the condition on the control parameters in Eq. (35) does not hold are now in order. As shown in Proposition 1, under the “vector” presentation of dual quaternions, \( \mathbf{V}_V = 0 \) equivalent to

\[
\left[ (2 k_p + 2 k_f V_f) I_{0x8} + k_f \mathbf{V}_f (\hat{q}_e - \hat{q}_i)^T \right] (\hat{q}_e - \hat{q}_i) = 0
\]

Denote \( (2 k_p + 2 k_f V_f) I_{0x8} + k_f \mathbf{V}_f (\hat{q}_e - \hat{q}_i)^T \equiv \mathbf{Q} \), and notice that \( \text{rank}(\mathbf{Q}) = 7 \). Accordingly, unless \( \hat{q}_e = \hat{q}_i \), there is at most one undesired potential local minimum, and \( \hat{q}_i \) could converge to this local minimum only when all the following conditions happen: 1) \( \text{rank}(\mathbf{Q}) = 7 \); 2) \( \hat{q}_e - \hat{q}_i \) belongs to the null space of \( \mathbf{Q} \); and 3) the solution of \( \hat{q}_i \) satisfying \( \mathbf{Q} (\hat{q}_e - \hat{q}_i) = 0 \) is a dual quaternion that also belongs to \( \mathfrak{H} \). These conditions suggest that the local minimum problem will usually be unlikely to occur from a practical standpoint. As an alternative to constraining the control gains from Eq. (35), an effective and easily implementable application of the proposed method could be as follows. First, choose \( k_p, k_f, k_d \) by only considering the closed-loop performance of the proposed method [disregarding Eq. (35)]; then, if the system states converge to the desired values, control objectives are achieved. On the other hand, if the local minimum problem indeed occurs as outlined in the foregoing discussion, one can instantaneously modify \( k_p \) and \( k_d \) to satisfy Eq. (35) in such a way to release the system states from the undesirable local minimum.

V. Numerical Simulations

In this section, numerical simulation examples are presented to illustrate the effectiveness of the proposed method. Considering a target–chaser spacecraft formation, the inertial frame \( \mathcal{N} \) is the Earth-centered inertial frame, and the target spacecraft is running on a Molniya orbit, with the orbital elements given in Table 1. At the beginning, the body-fixed frame of the target is fully coincident with its local-vertical/local-horizontal (LV/LH) frame. The information related to the motion constraints of the chaser spacecraft are presented in Table 2.

The mass and inertia of the chaser are [32]

\[
m = 15 \text{ kg}, \quad J = \begin{bmatrix} 3.0514 & 0 & 0 \\ 0 & 2.6628 & 0 \\ 0 & 0 & 2.1879 \end{bmatrix} \text{ kg} \cdot \text{m}^2
\]

Table 1 Orbital elements of the target spacecraft

<table>
<thead>
<tr>
<th>Orbital elements</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semimajor axis, km</td>
<td>26553.937</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>0.729767</td>
</tr>
<tr>
<td>Inclination, deg</td>
<td>63.4</td>
</tr>
<tr>
<td>Argument of perigee, deg</td>
<td>−90</td>
</tr>
<tr>
<td>True anomaly, deg</td>
<td>0</td>
</tr>
</tbody>
</table>

For all the simulations, control inputs are required to be bounded by \(|f|^2 \leq 30 \text{ N} \cdot \text{m} \) and \(|r_j|^2 \leq 1 \text{ N} \cdot \text{m} \cdot \text{s} \), \( j = 1, 2, 3 \), where \( f^j \) and \( r_j^j \) are the entries of \( f^j \) and \( r_j^j \), respectively.

A. Comparison Results with the PD-Like Controller

As shown in Sec. IV, the proposed method can be somewhat regarded as the combination of a PD-like part and a special APF-based part. To show the new features of the proposed method with respect to the pure PD-like controller, comparison simulation results are presented in this subsection. In this study, the initial relative quaternion and relative position between two spacecraft are \( q_{0j}(0) = [1, 0, 0, 0]^T \) and \( r_{0j} = [0, −250, 0]^T \) m, and therefore, \( q^0_{0j} = [0, 0, 0, 0]^T \) m, which renders \( q_j = [1, 0, 0, 0]^T \) and \( r_j = [0, −10, 0]^T \) m, respectively. The desired final relative attitude and position are chosen as \( q_{fj} = [1, 0, 0, 0]^T \) and \( r_{fj} = [0, −5, 0]^T \) m, which renders \( q_j = [1, 0, 0, 0]^T \) and \( r_j = [0, −10, 0]^T \) m, respectively. The figures demonstrate that, under the PD-like controller, both the field-of-view constraint and the approach path constraint are violated, whereas by employing the proposed method, the constraints are strictly satisfied during the entire proximity process, which confirms the effectiveness of the APF designed in this paper. In Fig. 5, the figures show the time histories of the control input and the control errors, which have a significant advantage over the PD-like controller. In Fig. 6, the radii of circular coordinates are the amplitudes of \( q(t) \) or \( r(t) \), and the broken circles denote the maximum allowable bounds of the corresponding angles. It can be seen that, under the proposed controller, the trajectories of \( q(t) \) and \( r(t) \) are both within the broken circles, which means no constraint violation happens, whereas the PD-like controller cannot guarantee these requirements.

In addition, three-dimensional (3-D) illustrations are given in Figs. 6–8. In Figs. 6 and 7, to show the relative 6-DOF motion between the chaser and the target, simulation results are shown on the
LVLH frame of the target, and then the translational motion trajectories of the chaser of the first 30 s in simulations are illustrated. (The results under the PD-like method and the APF-based method are given in Figs. 6 and 7, respectively.) In these two figures, the target and the chaser are denoted by small spheres, the instantaneous positions of the chaser at different simulation times are given, and mutually perpendicular lines pointing from the chaser are employed to denote the instantaneous axis directions of frame $B$. Furthermore, the field of view of the chaser is presented by cones whose half-cone angles are equal to $\alpha_{\text{max}}$. Because the target is tumbling, the chaser needs to approach the target through “roundabout” paths. It is shown that, under the PD-like controller, the chaser loses the position of the target after 20.5 s, whereas the proposed method can guarantee that the target is always in the field of view of the chaser. These figures can also explain that the slight fluctuations seen in Fig. 3b are because, under the proposed method, the chaser always tries to keep the target
in the specified field of view of itself, otherwise it will lose the target (like in the PD-like case). A more comprehensive 3-D illustration is given in Fig. 8. To show the approach path constraint in a “static” way, motion trajectories of the chaser are transformed into frame $T$ (so that the approach restriction zone is fixed in Fig. 8, but notice that, actually, the target is tumbling with respect to $N$), and the fields of view of the chaser at different times are also provided by cones. One can see that, at the beginning, the two methods have similar performance, but when the chaser has the trend to cross the bound of the restriction zones, the proposed method can “pull” the chaser back, whereas the PD-like method would violate the constraints. Finally, the control inputs of the proposed method are reported in Fig. 9.

B. Comparison Results with the MPC Method

As mentioned in the Introduction, Lee and Mesbahi [26] considered a somewhat similar problem as this paper (albeit a special case) and the MPC technique was used. In this subsection, simulation results are demonstrated to compare and contrast the APF-based feedback method presented in this paper and the suboptimal results given in [26]. In this regard, it is important to recognize that, unlike our results, the methods proposed in [26] cannot deal with tracking problems. Therefore, to proceed with the comparison, in this simulation case, we set $\omega^r_t = \dot{\omega}^r_t = 0$ (so that the tracking problem is transferred to an “almost” stabilization problem). The initial relative conditions are reset to be $q^r_t(0) = [0.8, 0, -0.6, 0]^T$, $r^r_t(0) = [0, -100, 45]^T$ m,
Simulation results of $\mathbf{q}$ and $\mathbf{r}$ under the APF-based method and the MPC method are illustrated in Figs. 10a and 10b, respectively. The time responses of the line-of-sight angle $\alpha(t)$ and the approaching cone angle $\beta(t)$ are provided in Fig. 11. The energy consumptions related to the attitude control and the position control (calculated by $J_f = \int (f^2(t)) dt$ and $J_t = \int (r^2(t)) dt$, respectively) are illustrated in Fig. 12.

From Figs. 10–12, in this simulation case, both the MPC method and the APF-based controller can successfully achieve control objectives without constraint violations. Qualitatively, the APF-based method delivers quicker convergence and smoother trajectories than the MPC method. On the other hand, it also needs higher energy, especially the attitude control part. These results show the different features of the two control methods. The APF-based controller proposed in this paper fully concentrates on eliminating tracking errors and complying with motion constraints, and so it has better transient responses but potentially requires more energy, whereas the MPC method tries to make a balance between tracking objectives and the energy optimization, so that it provides a suboptimal performance regarding both respects. Another important fact is that, as mentioned in the Introduction, the APF-based feedback controller is computationally much cheaper when compared with the MPC formulation. For this particular simulation case, for every $\Delta t$,

\[ J_N = \sum_{k=1}^{N-1} \left( 10||\mathbf{q}(k) - \mathbf{q}_d||^2 + 0.25||\mathbf{r}(k) - \mathbf{r}_d||^2 + 10||\mathbf{\omega}(k)||^2 + 0.5||\mathbf{p}(k)||^2 + 5||\mathbf{f}(k)||^2 + 50||\mathbf{r}(k)||^2 \right) \]  

(50)
the MPC needs to solve an optimization problem for a 6 × 6 matrix [the dimension of control input ×(tᵢ / Δt)], subject to a total of more than 50 vector/scalar constraints (these constraints come from kinematics, dynamics, input bounds, motion constraints, and so on). Notice that, even though the discretized prediction model presented in [26] has a linear form, because the matrices A, (transition matrix) and B, (input matrix) of the model ([26] Eq. (77)) are still state related, the model still has nonlinear properties. This fact, along with the nonlinear constraints (f₁, f₂ ≤ 0), leads to additional computational complexities for most of the conventional programming solvers. On the other hand, the most expensive computation for the APF-based feedback controller is simply calculating the partial differential of a scalar (the potential function) with respect to an 8 × 1 variable (the error dual quaternion), which can be easily obtained, and the other calculations are all basic scalar variable algebraic operations (additions and multiplications). Furthermore, recall that the proposed method can be applied to not only stabilization problems but also tracking problems, along with rigorous assurances of the closed-loop stability. From all these points of view, the proposed method brings many advantages compared with the MPC formulation of [26], albeit at the cost of additional energy usage.

C. Monte Carlo Simulations

Finally, to provide comprehensive insight into the performance of the proposed controller, Monte Carlo style simulations are conducted, in which initial conditions and control parameters are randomized. Table 3 shows the detailed information about the randomized variables and their ranges.

In the simulations, the gravity gradient torque τ₆₆ and the J₂ perturbation force f₆₆ (due to the nonspherical nature of Earth) are employed as external disturbances, given by [17]

\[
τ_{66} = 3μ J_2 \frac{r^2 × (J_2 r^1)}{\|r^2\|^3}
\]

Table 3  Randomized parameters and initial values

<table>
<thead>
<tr>
<th>Variables</th>
<th>Ranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>k₂</td>
<td>[0.05, 2]</td>
</tr>
<tr>
<td>k₃</td>
<td>[0.3, 8]</td>
</tr>
<tr>
<td>k₅</td>
<td>[0.002, 0.03]</td>
</tr>
<tr>
<td>qₑ(0)</td>
<td>[−0.05, 0.05]</td>
</tr>
<tr>
<td></td>
<td>[−0.05, 0.05] × [−0.05, 0.05] × [−0.05, 0.05]</td>
</tr>
<tr>
<td>vₑ(0)</td>
<td>[−5.5] × [−5.5] × [−5.5]</td>
</tr>
</tbody>
</table>

where J₃ = 6378.137 km is the mean equatorial radius of the Earth; and xₖ, yₖ, and zₖ are the corresponding components of rₑ.

To proceed with Monte Carlo simulations, first, a set of randomized parameters and initial conditions is generated, which satisfies the ranges stated in Table 3. Second, combine these new generated parameters and initial conditions with other simulation conditions, as given in Sec. V.A, to conduct a single simulation and record the related data. Then, repeat the entire process. Based on this guideline, 1500 Monte Carlo runs are conducted. To summarize the results, Fig. 13 illustrates the steady tracking error distribution of rₑ and ξₑ (every point in the figure represents a single simulation). The distributions of the maximum line-of-sight angle α(t) and approaching cone angle β(t) (shown on the corresponding x−z planes) of every single simulation are given in Fig. 14. These figures indicate that the overwhelming majority of simulation cases deliver acceptable performance and can render the relative tracking errors to a small region around the origin (on average, the steady errors of qₑ,
and $r_4^2$ are around $4 \times 10^{-5}$ and $2 \times 10^{-3}$ m, respectively) without constraint violations ($\max\{\alpha(t)\}, \max\{\beta(t)\} < 0.5$ rad).

Moreover, these figures also show that there are about 35 simulation cases (out of the total 1500 test cases) that fail to achieve the control objectives. By carefully analyzing these failed cases, we find that they are essentially caused by extreme initial conditions that are compounded by the boundedness property of control inputs. For example, for the highest point depicted in Fig. 13, the initial position corresponding to this case is $r_{4i}(0) = [201.7699, -430.6187, 117.2563]^T$ m (which has been transformed to frame $T$), with a velocity $\dot{r}_{4i}(0) = [4.3541, -0.2857, 1.0829]^T$ m/s. For this case, this indicates that the initial position of the chaser very nearly violates the approach path constraint. In this situation, when additionally subject to the initial velocity along with the requirement $|\dot{r}_{4j}| < 30$ N, it is practically impossible to stop the chaser from entering the forbidden zone [$|\beta(t)| > 0.5$ rad is seen to happen within 1 s]. These failed simulations also show a limitation of the proposed method. To be specific, the controller presented in this paper cannot adequately deal with the saturation problem of actuators when the initial conditions are near the edge of the constrained zones.

VI. Conclusions
A six-degree-of-freedom control method for final proximity operations of a spacecraft rendezvous and docking mission is proposed in this paper. By employing the dual-quaternion-based description, the field-of-view constraint and the approach path constraint are described in a unified fashion, and then a special artificial potential function is presented to encode this constraint information into controller design. Subsequently, a feedback control law is proposed to ensure the arrival of the chaser at the docking port of the target without any constraint violation. Asymptotic stability of the closed-loop system is guaranteed through a Lyapunov-based method while exploiting the special properties of the underlying artificial potential function. The effectiveness of the proposed method is illustrated by extensive numerical simulations.

Appendix A: Basic Operations of Dual Numbers and Dual Vectors
For any $\lambda \in \mathbb{R}$, $\hat{a}_1 = a_1 + \varepsilon a_1d \in \mathbb{R}^n$, and $\hat{a}_2 = a_2 + \varepsilon a_2d \in \mathbb{R}^m$, one has

\[
\hat{\lambda} = \lambda \hat{a}_1 + \varepsilon \lambda a_1d \quad \text{and} \quad \hat{a}_1^T = a_1^T + \varepsilon a_1^Td \quad \text{(A1) and (A2)}
\]

\[
\hat{a}_1 = a_1 + \varepsilon a_1d, \quad (\hat{a}_1^T)^T = \hat{a} \quad \text{(A3)}
\]
\[
\hat{a}_1 \pm \hat{a}_2 = a_1 \pm a_2 + \varepsilon (a_1d \pm a_2d) \quad \text{(A4)}
\]
\[
\hat{a}_1 \cdot \hat{a}_2 = a_1 \cdot a_2 + \varepsilon (a_1d \cdot a_2d + a_1 \cdot a_2d) \quad \text{(A5)}
\]
\[
\hat{a}_1 \circ \hat{a}_2 = a_1^T \cdot a_2d + a_1^T \cdot a_2 \quad \text{(A6)}
\]
\[
\|\hat{a}\| = \|a_1\| + \varepsilon \|a_2\| \quad \text{(A7)}
\]

Fig. 14 Distributions of maximum $\alpha(t)$ and $\beta(t)\)
with
\[
\Xi = \begin{bmatrix} 0 & \Xi^T \\ \Xi & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & \Omega^T \\ \Omega & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 \\ 0 & I_{4\times4} \end{bmatrix}
\]

(B4)

Moreover, the potential function also has a corresponding equivalent expression:
\[
\begin{align*}
V_p(\dot{q}_e) &= k_p(\dot{q}_e - \ddot{q})^T(\dot{q}_e - \ddot{q}) \\
&\quad + k_f(\dot{q}_e - \ddot{q})^T(\dot{q}_e - \ddot{q})(1/f_1(\dot{q}_e) + 1/f_2^2(\dot{q}_e))
\end{align*}
\]

(B5)

It should be emphasized that, although \( \dot{q}_e \) and \( \ddot{q} \) are represented in different forms, they are essentially equivalent representations of dual quaternions and follow the same algebraic structure. Both of them are commonly used in literature involved in the dual-quaternion formulation, and the representation \( \dot{q}_e \) is generally seen to be more popular due to its clearer physical meaning, which allows for more concise algebraic operations. The reason why we temporarily employ \( \ddot{q} \) here is that, by using this vector representation, we can analyze the differential of \( V_p \) completely in the real-number domain, which lends itself for algebraic convenience for our subsequent analysis. We emphasize that, for the controller design, we still employ the traditional form of dual quaternions as given in Sec. III.

By the definition of \( V_p, \) \( \nabla V_p(\dot{q}_e) = 0 \) indicates
\[
2k_p(\dot{q}_e - \ddot{q}) + 2k_f(\dot{q}_e - \ddot{q})V_f + k_f(\dot{q}_e - \ddot{q})V_f(1/f_1(\dot{q}_e) + 1/f_2^2(\dot{q}_e)) = 0
\]

(B6)

where \( V_f = (1/f_1 + 1/f_2^2) \) is defined for ease of notation, and \( \nabla V_f \) denotes the gradient of \( V_f \) with respect to \( \dot{q}_e \), and so
\[
\nabla V_f = -\frac{2}{f_1} f_1 - \frac{2}{f_2} f_2
\]

and here \( \nabla f_1 \) and \( \nabla f_2 \) are, respectively, the gradient of \( f_1 \) and \( f_2 \) with respect to \( \dot{q}_e \), satisfying
\[
\begin{align*}
\nabla f_1 &= \frac{2q_1(\dot{q}_e \otimes \ddot{q}_e) - 4(\dot{q}_e \otimes \ddot{q}_e) \dot{q}_e}{\|2\ddot{q}_e \otimes \ddot{q}_e\|^2} \\
\nabla f_2 &= -\frac{2q_1(\dot{q}_e \otimes \ddot{q}_e) - 4(\dot{q}_e \otimes \dddot{q}_e) \dot{q}_e}{\|2\ddot{q}_e \otimes \ddot{q}_e\|^2}
\end{align*}
\]

(B7)

(B8)

It is obvious that \( \dot{q}_e = \dddot{q}_e \) is a solution for Eq. (B6). Next, we will show that it is the unique solution when parameters \( k_p \) and \( k_f \) satisfy the condition given in Eq. (35). Premultiplying both sides of Eq. (B6) with \( \dot{q}_e - \dddot{q}_e \), and considering Eqs. (B3), (B7), and (B8), one has
\[
\begin{align*}
2k_p(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q}) + 2k_f(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q})V_f
&= k_f(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q})\left(2f_1(f_1 - \cos \alpha_m) + 2f_2(f_2 - \cos \beta_m)\right)
\end{align*}
\]

(B9)

Notice that \( (\dot{q}_e - \dddot{q}_e)^T(\ddot{q}_e - \dddot{q}_e) - (\dot{q}_e - \ddot{q})^T(\ddot{q}_e - \dddot{q}) = 1 - \eta \geq 0 \), where \( \eta \) is the scalar part of the quaternion \( \dot{q}_e \), and so then we have
\[
\begin{align*}
2k_p(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q}) &= 2k_f(1 - \eta)V_f \\
&\quad + k_f(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q})\left(-\frac{1}{f_1} \cos \alpha_m - \frac{1}{f_2} \cos \beta_m\right)
\end{align*}
\]

(B10)

Recall that, when \( \dot{q}_e \in \mathbb{D} \) and \( \alpha_m, \beta_m \in (-\pi/2, 0) \cup (0, \pi/2) \), we have \( f_1, f_2 < 0 \) and \( \cos \alpha_m, \cos \beta_m > 0 \), and so the right-hand side of Eq. (B10) is always positive, and it has a minimum value given by
\[
2k_f(q_e - \ddot{q})^T(\dot{q}_e - \dddot{q})\left(\frac{\cos \alpha_m}{(1 - \cos \alpha_m)^2} + \frac{\cos \beta_m}{(1 - \cos \beta_m)^2}\right)
\]

(B11)

When \( q_e \neq q_1 \), one can readily guarantee that
\[
2\dot{q}_e^T(q_e - \dddot{q}) > (q_e - \ddot{q})^T(q_e - \dddot{q}),
\]

and so the right-hand side of Eq. (B11) is always negative.

References
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