Abstract—This paper presents a practical discrete-time fractional order terminal sliding mode (DFOTSM) control strategy for high-precision tracking tasks based on a linear motor. In particular, the practical parametric uncertainties involving sliding friction, uncertain payload, and disturbance in tracking tasks are considered in this paper. Combining Grünwald–Letnikov fractional order definition and terminal sliding mode technique, the proposed method synthesizes a novel DFOTSM control law to drive the sliding mode dynamics into the stable region in finite steps theoretically, even though the system is suffering from uncertainties and disturbances, and the motion on the surface can guarantee higher tracking precision than the conventional discrete-time terminal sliding surface by selecting suitable controller parameters. The theoretical analyses give out the guideline of parameter selection, and experiments are carried out on the linear-motor-based test platform to demonstrate that the proposed controller is easily implemented and can achieve high-precision tracking, fast response, and considerable robustness to uncertainties.

Index Terms—Discrete-time fractional order terminal sliding mode (DFOTSM), linear motor (LM), motion control, robustness.

I. INTRODUCTION

PERSISTENT magnet linear motors (LMs) have been playing a critical role in industrial production and daily life due to high-precision, high-speed, and low-noise performance of linear movement and have successfully replaced the traditional permanent magnet rotary machines in urban rail transit [1], machining operations [2], and positioning platform [3].

As the requirement of LM-based apparatuses arises, many researchers commit to work on enhancing the performance of motion control system. Some investigations on the innovative designs of LM have analyzed the advantages in some specific purposes and provide the users with specific solutions to different payloads and air gaps. These schemes focus on different payload applications, and the electrical steel prototypes could maintain a satisfactory performance with different payloads and air gaps. These schemes focus on different purposes and provide the users with specific solutions to naturally improve performance.

In practical engineering, besides enhancing the ability of LM according to specific application, advanced control strategies can also enhance the performance of the motion control system directly, i.e., Hu et al. in [7] utilized the learning adaptive robust control to overcome the strongly coupled dynamics with parametric uncertainties and disturbances in LM-based multiaxes motion coordination and realized the excellent contouring accuracy in high-speed and large-curvature contouring tasks. Wang et al. [8] developed the robust feedback control with parametric adaption and radial basis function neural network to compensate the random uncertainties in LM-based precision tracking motion. Dong et al. [9] employed the extended Kalman filter to estimate the position of moving stage and achieved a sensorless approach to the LM-based motion control without position sensor. An adaptive controller was designed to deal with the variation in real disturbance, and the feed-forward part of the controller was based on a lumped disturbance model consisting of force ripple and sliding friction [10]. Besides, many other different control strategies for seeking high-precision and fast-response tracking performance are also reported recently, i.e., time optimal controller [11], adaptive robust controller [12], iterative-learning controller [13], and sliding mode control (SMC) [14], [15].

In all of the aforementioned methods, SMC strategies are attractive and suitable to motion control systems due to fast response, insensitiveness to parametric uncertainties, and easy implement [16], [17]. In particular, SMC can theoretically determine the final tracking precision according to the construct of reaching law and sliding surface even if controlled systems are suffering from uncertainties and disturbances, i.e., Zheng et al. applied a continuous fast nonsingular terminal sliding mode (FNTSM) control on the LM-based motion, which is subjected to parametric uncertainties and disturbances, and analyzed the theoretical tracking precision by utilizing the stability condition of the terminal sliding surface technique [16]. In order to gain high precision, fractional order terminal sliding mode (FOTSM) controllers were investigated for robotic manipulators subjected to uncertainties [17], [18]. The fractional order sliding mode control owns some distinctive features such as global memory,
evidently physical significance, and slight chattering [19], [20] and, hence, has been applied widely to the industrial applications, such as lighting control [21] and speed tuning of motor [20], [22], [23]. Typically, these fractional order SMC strategies are designed based on continuous time for controlled objects, and are directly tested by using digital computer system, which leads to precision loss due to ignoring the effect of sampling interval [24]. Moreover, there exists the nonphysical initial condition in experiments if the fractional order operator is defined by Riemann–Liouville definition [19]. Therefore, Caputo’s fractional calculus is presented to solve this issue mathematically, but in engineering applications Caputo’s definition can only be implemented in the approximation methods based on Laplace transform, which will introduce the extra approximating error into control system [25]. All these flaws will exacerbate the uncertainties existing in the LM-based motion control systems.

Motivated by the previous issues, in this paper, we constructed a novel discrete-time fractional order terminal sliding mode (DFOTSM) controller for high-precision LM-based motion tracking tasks based on digital computer system. To reduce the disparities between theoretical design and practice application by digital computer system, the proposed controller is designed based on the discrete-time plant model, which is derived by Euler’s discretization. Accordingly, to retain the distinctive features of fractional order operators, i.e., global memory, Grünwald–Letnikov fractional order difference is utilized to construct the discrete-time sliding surface. The bounded parametric uncertainties and disturbances are also considered in this paper, and a new terminal-sliding-mode-type reaching law is presented to drive the sliding mode dynamics into a region within finite steps, which owns fast convergence speed if system suffers from a large uncertainty. The theoretical analyses give the final tracking precision, and the experiments are carried out on a real LM-based test platform. To the best of our knowledge, the DFOTSM controller is presented and applied on LM-based motion for the first time.

The paper is organized as follows. Section II describes the discrete-time plant mode with parametric uncertainties in the form of Euler’s difference. The DFOTSM controller design is shown in Section III, where the final tracking precision is analyzed in details. At last, the experimental results are presented in Section IV to verify the effectiveness of the DFOTSM controller. Section V summarizes this paper.

II. Dynamic Model of Linear Motor

The experimental setup used in this paper is the LM-based motion control system (by Akribis System Company) shown in Fig. 1. The moving stage consists of the iron-core translator and the inertia load, and travels along the linear guidance with the travel range of 700 mm. The continuous-time dynamic equations are given by [26]

\[
\begin{align*}
\dot{p} &= v \\
m\dot{v} &= u - f - d \\
f &= k_c \text{sgn}(\dot{p}) + k_v \dot{v}
\end{align*}
\]  

(1)

where \( p, v, \) and \( \dot{v} \) represent the absolute position, velocity, and acceleration of the moving stage, respectively. The system input \( u \) is the force generated by the LM. The mass of the iron-core translator and inertia load is denoted as \( m \), and the lumped uncertainty \( d \) contains bounded unknown disturbance and measurement noise. The nonlinear friction \( f \) is related to the velocity of moving stage, and the viscous friction coefficient and Coulomb friction coefficient are denoted by \( k_v \) and \( k_c \), respectively. To design the digital motion control system, it is essential to discretize the plant model to yield a nominal discrete-time plant model [27]

\[
\begin{align*}
\Delta p(k) &= v(k) \\
m\Delta v(k) &= u(k) - (f(k) + \delta f(k)) - d(k) - \delta d(k) \\
f(k) &= k_c \text{sgn}(\dot{v}(k)) + k_v v(k)
\end{align*}
\]  

(2)–(4)

where \( \delta f(k) \) means the friction uncertainty, and \( \delta d(k) \) describes the possible parameter variation, discretization errors and measurement errors.

Assumption 1: [27] The uncertain terms \( d(k), \delta f(k), \) and \( \delta d(k) \) are assumed to be bounded and own the following boundary:

\[
|d(k)| \leq d_m, \quad |\delta f(k)| \leq \theta_f, \quad |\delta d(k)| \leq \theta_d.
\]  

(5)

Using Euler’s discretization in [28], the forward difference of position signal at sampling point \( k \) denoted by \( \Delta p(k) \) can be expressed by

\[
\Delta p(k) = \frac{p(k+1) - p(k)}{h}
\]  

(6)

where \( h \) represents the sampling interval and \( k \in \mathbb{N} \).

III. DFOTSM Controller Design

A. Preliminaries

To design the DFOTSM controller for LM-based motion, some basic notions and definitions for fractional order operators are recalled, and some essential lemmas are also given.
**Definition 1:** The Grünwald–Letnikov fractional order difference with arbitrary order \( \alpha \in \mathbb{R} \) is described by

\[
\Delta^\alpha y(k) = \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} y(k-j)
\]  

(7)

where \( h \) represents the sampling interval and the number of samples is denoted as \( k \). \( \binom{\alpha}{j} \) is the notation of the binomial coefficients, which can be calculated according to

\[
\binom{\alpha}{j} = \frac{1}{\alpha(\alpha-1)\cdots(\alpha-j+1)} j = 0, 1, 2, 3, \ldots
\]  

(8)

Note that it is inefficient for the real-time motion controller to record and calculate all the sampling data to obtain Grünwald–Letnikov fractional order difference in engineering applications. To conserve computing resources, the finite recording number \( L \) is introduced into the definition of fractional order difference given by

\[
\Delta^\alpha y(k) = \frac{1}{h^\alpha} \sum_{j=0}^{L} (-1)^j \binom{\alpha}{j} y(k-j).
\]  

(9)

**Lemma 1:** The relation between \( \binom{\alpha}{j} \) and \( \Gamma(x) \) can be established by

\[
\sum_{j=0}^{L} (-1)^j \binom{\alpha}{j} = \frac{\Gamma(L+1-\alpha)}{\Gamma(1-\alpha)\Gamma(L+1)}
\]  

(10)

where \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \).

Note: Considering the bounded sequence \( y(k), k = 0, 1, \ldots \) with \( \max\{|y(k-1)|\} \leq \rho_y \), the corresponding fractional order difference is bounded by

\[
\frac{1}{h^{\alpha-1}} \sum_{j=1}^{L} (-1)^j \binom{\alpha-1}{j} y(k-j) \leq \frac{(K-1)\rho_y}{h^{\alpha-1}}
\]  

(11)

where \( K = \frac{\Gamma(L+2-\alpha)}{\Gamma(2-\alpha)\Gamma(L+1)} \geq 1 \).

**B. Control System Design**

To realize digital engineering application, the DFOTSM control is constructed in this section. Different from the continuous-time controller, some properties of the DFOTSM controller cater to digital systems, and will be analyzed in this section.

The continuous-time FOTSM scheme for tracking control of robotic manipulator have been investigated in [18], and the discrete-time FOTSM surface can be given by

\[
z(k) = \Delta e(k) + \xi_1 \Delta^{\alpha-1} \left[ e(k) \right]^{\lambda_1}
\]  

(12)

where \( e(k) = p(k) - \rho_k(k) \) is the error between the position signal \( p(k) \) and the reference signal \( \rho_k(k) \). The notation \( \left[ e(k) \right]^{\lambda_1} \triangleq \left| e(k) \right|^{\lambda_1} \text{sgn}(e(k)) \) with \( 0 < \lambda_1 < 1 \) denoting a ratio of odd integers, and \( \xi_1 \in \mathbb{R}^+ \). Taking the difference of \( z(k) \) yields

\[
z(k+1) = z(k) + h\Delta \left[ e(k) - \Delta p_k(k) + \xi_1 \Delta^{\alpha-1} \left[ e(k) \right]^{\lambda_1} \right]
\]

\[
= z(k) + h\Delta v(k) - h\Delta \left[ \Delta p_k(k) - \xi_1 \Delta^{\alpha-1} \left[ e(k) \right]^{\lambda_1} \right]
\]

\[
= z(k) + \frac{h}{m} \left[ u(k) - k_e \text{sgn}(v(k)) - k_v v(k) - d(k) \right] - \delta f(k) - \delta d(k) - h \left[ \Delta^2 p_k(k) - \xi_1 \Delta \left( \Delta^{\alpha-1} \left[ e(k) \right]^{\lambda_1} \right) \right]
\]  

(13)

and based on the nominal model, the equivalent control input \( u_{eq} \) can be given by

\[
u_{eq}(k) = m \left[ \Delta^2 p_k(k) - \xi_1 \Delta \left( \Delta^{\alpha-1} \left[ e(k) \right]^{\lambda_1} \right) \right] + k_e \text{sgn}(v(k)) + k_v v(k).
\]  

(14)

Furthermore, we introduce a novel terminal-sliding-mode-type reaching law given by

\[
u_{sw}(k) = -m \left[ \xi_2 z^3(k) + \xi_3 |z(k)|^{\lambda_2} \right]
\]  

(15)

where \( 0 < \lambda_2 < 1 \), \( \xi_2 \in \mathbb{R}^+ \), and \( \xi_3 \in \mathbb{R}^+ \). Combining the equivalent control and the reaching law produces the actual control law

\[
u(k) = u_{eq}(k) + u_{sw}(k)
\]  

(16)

and substituting it into (13) gives rise to

\[
z(k+1) = z(k) + h\rho_0(k) - h\xi_2 z^3(k) - h\xi_3 |e(k)|^{\lambda_2}
\]  

(17)

with

\[
\rho_0(k) = \frac{1}{m} \left[ \delta d(k) + \delta f(k) + d(k) \right].
\]

To complete the stability analyses of our discrete-time control strategy, the following assumption about the sliding surface are required.

**Assumption 2:** For a practice discrete-time control system, the sampling interval and sliding surface should satisfy

\[
z^2(k) < \frac{1 - h\xi_3}{h\xi_2}
\]  

(18)

with \( 1 - h\xi_3 > 0 \).

At this point, the DFOTSM controller can be constructed by the following theorem.

**Theorem 1:** Considering the nominal discrete-time plant model subjected to uncertainties shown in (3) and sliding surface in (12) with Assumption 2, we can employ the novel control law in (16) to guarantee the following sliding mode dynamics.

1) The discrete-time sliding mode dynamics can be driven into a region \( \Omega \) within \( \eta \) steps, and the region \( \Omega \) can be described by

\[
\Omega = \{z(k)||z(k)|<\gamma\}
\]  

(19)

with

\[
\gamma = \hat{\rho} \lambda(\lambda_2), \eta = \frac{z^2(0) - \gamma^2}{\mu^2} + 1
\]  

(20)
where

\[ \hat{\rho} = \max \left\{ \frac{\tilde{\rho}_0}{\xi_3}, \left( \frac{h\xi_3}{1-h\xi_2} \right)^{1/2} \right\} \quad (21) \]

\[ \Lambda(y) = 1 + y^{\frac{1}{\alpha}} - y \quad (22) \]

\[ \mu = h\xi_2 \gamma^3 + \left[ \Lambda^2(\lambda) - 1 \right] |h\hat{\rho}_0| \quad (23) \]

and \( \hat{\rho}_0 = d_m + \theta_f + \theta_d \).

2) Once the sliding mode dynamics enter the region \( \Omega \), it will stay there and not escape in the subsequent time, which means \( |z(k+1)| \leq \gamma \) if \( |z(k)| \leq \gamma \).

**Proof:** Before giving out the proof, we have to review a critical property of \( \Lambda'(y) \), that is, for any \( 0 \leq x \leq 1 \) and \( 0 < y < 1 \), we have the following relations \([24]\):

\[ 1 < \Lambda(y) < 2, x\Lambda'(y) - x^y \Lambda''(y) + \Lambda'(y) \geq 1. \quad (24) \]

The following properties will also be frequently used in this proof

\[ |h\hat{\rho}_0| \leq h\xi_3 \hat{\rho}^2 \leq (1 - h\xi_2)\hat{\rho} \quad (25) \]

the proof of which can be seen in the Appendix.

1) Select the discrete-time Lyapunov function \( V(k) = \psi^2(k) \), and we have

\[ V(k+1) - V(k) = - |z(k) - z(k+1)| [z(k) + z(k+1)] \quad (26) \]

Then, we make classified discussions to prove that the sliding mode dynamics can reach the region \( \Omega \) with the condition mentioned in Assumption 2 if \( z(k) \notin \Omega \).

**Case 1:** Consider \( z(k) > \gamma > 0 \), and it is clear that

\[ z(k) - z(k+1) = h\xi_2 \psi^2(k) + h\xi_3 |z(k)|^{2\omega} - h\hat{\rho}_0(k) \geq h\xi_2 \gamma^3 + h\xi_2 \hat{\rho}^2 \Lambda^2(\lambda) - |h\hat{\rho}_0|. \quad (27) \]

Using (25), we can obtain

\[ z(k) - z(k+1) \geq h\xi_2 \gamma^3 + [\Lambda^2(\lambda) - 1] |h\hat{\rho}_0| \geq \mu. \quad (28) \]

Next, we proceed to examine the property of \( z(k) + z(k+1) \)

\[ z(k) + z(k+1) = h\hat{\rho}_0(k) + 2z(k) - h\xi_2 \psi^2(k) - h\xi_3 |z(k)|^{2\omega} \geq 2z(k) - h\xi_2 \psi^2(k) - h\xi_3 |z(k)|^{2\omega} - |h\hat{\rho}_0|. \quad (29) \]

First, consider the situation that \( z(k) \geq 1 \), hence we have \( z(k) \geq |z(k)|^{2\omega} \), and based on Assumption 2, we have

\[ z(k) \geq h\xi_2 \psi^2(k) + h\xi_3 |z(k)|^{2\omega}. \quad (30) \]

and along the line from (27) and (28), (29) becomes

\[ z(k) + z(k+1) \geq h\xi_2 \psi^2(k) + h\xi_3 |z(k)|^{2\omega} - |h\hat{\rho}_0| \geq \mu. \]

Then, we consider the situation that \( 0 < z(k) < 1 \), which implies \( \psi^2(k) < z(k) \), and based on (25) and Remark 3, (29)

becomes

\[ z(k) + z(k+1) \geq 2 \left[ h\xi_3 |z(k)|^{2\omega} + h\xi_2 z(k) \right] - h\xi_3 \psi^2(k) - h\xi_2 |z(k)|^{2\omega} - |h\hat{\rho}_0| \geq h\xi_3 |z(k)|^{2\omega} + h\xi_2 z(k) - |h\hat{\rho}_0| \geq \mu. \quad (31) \]

Hence, we have \( V(k+1) - V(k) \leq -\mu^2 \) when \( z(k) > \gamma \).

**Case 2:** Considering the situation that \( z(k) < -\gamma < 0 \) with Assumption 2, along the same line from (26) to (31), we can also prove that \( V(k+1) - V(k) \leq -\mu^2 \).

According to the above analyses, the discrete-time sliding mode dynamics can converge into the region \( \Omega \), and it is reasonable to assume that the sliding mode dynamics reach the region at the \( \eta \) step, which implies \( \psi^2(\eta) \leq \psi^2(0) - \eta \mu^2 \leq \gamma^2 \).

Moreover \( \eta \in \mathbb{N}^+ \) results in \( \eta = \frac{\psi^2(0) - \gamma^2}{\mu^2} + 1 \).

At this point, we have proved that the discrete-time sliding mode dynamics can converge into the region \( \Omega \) within \( \eta \) steps.

2) Once the sliding mode dynamics enter the region \( \Omega \), we have \( |z(k)| < \gamma \) and define \( 0 \leq \beta \leq 1 \) to obtain \( |z(k)| = \gamma \beta \).

To prove \( |z(k+1)| \leq \gamma \), it is essential to divide the analyses due to the location of \( z(k) \).

**Case 1:** Proof of \( -\gamma \leq z(k+1) \leq \gamma \) with \( z(k) \geq 0 \).

First, the situation \( 0 \leq z(k) \leq \gamma \) is considered and it implies that \( \beta \Lambda(\lambda) > 0. \)

\[ z(k+1) = z(k) - h\xi_2 \psi^2(k) - h\xi_3 |z(k)|^{2\omega} + h\hat{\rho}_0(k) \]

\[ \leq z(k) - h\xi_2 \psi^2(k) - h\xi_3 (\beta \hat{\rho} \Lambda(\lambda))^{2\omega} + |h\hat{\rho}_0|. \quad (32) \]

By \( h\xi_2 \psi^2(k) > 0 \) and (25), the above expression becomes

\[ z(k+1) \leq z(k) - h\xi_3 \beta^{2\omega} \Lambda^2(\lambda) - |h\hat{\rho}_0| \]

\[ \leq z(k) - |h\hat{\rho}_0| |\beta^{2\omega} \Lambda^2(\lambda)| + |h\hat{\rho}_0| \]

\[ \leq z(k) + [1 - |\beta^{2\omega} \Lambda^2(\lambda)|] |h\hat{\rho}_0|. \quad (33) \]

If \( \beta \Lambda(\lambda) \geq 1 \), it is obvious that \( z(k+1) \leq z(k) \leq \gamma \), and \( \beta \Lambda(\lambda) < 1 \) leads to \( \beta^{2\omega} \Lambda^2(\lambda) < 1 \). Utilizing (25) and \( z(k) = \beta \hat{\rho} \Lambda(\lambda) \), (33) becomes

\[ z(k+1) \leq \beta \hat{\rho} \Lambda(\lambda) + [1 - |\beta^{2\omega} \Lambda^2(\lambda)|] |(1 - h\xi_2)\hat{\rho} \]

\[ \leq \beta \hat{\rho} [\Lambda(\lambda) - |\beta^{2\omega} \Lambda^2(\lambda)| + h\xi_2 (\beta^{2\omega} \Lambda^2(\lambda) - 1) + 1] \]

\[ \leq \beta \hat{\rho} \Lambda(\lambda) \leq \gamma. \]

Now, we have proved that \( z(k+1) \leq \gamma \) with \( z(k) \geq 0 \), and we proceed to analyze \( z(k+1) \geq -\gamma \) with \( z(k) \geq 0 \).
Considering $z(k) \geq 1$, we have $[z(k)]^{\bar{k}_2} \leq z(k) \leq z^2(k)$ and utilize Assumption 2 and (25) to get
\[
z(k+1) \\
\geq z(k) - h_\xi z(k) - h_\xi z^2(k) - |h_0| \\
\geq (1 - h_\xi z(k) - h_\xi z^2(k) - |h_0| \\
\geq -|h_0| \geq -(1 - h_\xi z) \rho > -\Lambda(\bar{\lambda}_2) \rho \geq -\gamma. \quad (34)
\]

Contrarily, considering $0 < z(k) < 1$, we have $z^3(k) \leq [z(k)]^{\bar{k}_2}$ and utilize Assumption 2 and (25) to get
\[
z(k+1) \\
\geq (1 - h_\xi z) \beta \Lambda(\bar{\lambda}_2) \rho - h_\xi z (\beta \Lambda(\bar{\lambda}_2) \rho)^{\bar{k}_2} - |h_0| \\
\geq \rho(1 - h_\xi z) (\beta \Lambda(\bar{\lambda}_2) - \beta^{\bar{k}_2} \lambda_2 (\bar{\lambda}_2) - 1) \\
\geq -\rho(1 - h_\xi z) \Lambda(\bar{\lambda}_2) > -\rho \Lambda(\bar{\lambda}_2) \geq -\gamma. \quad (35)
\]

Case 2: Here we will give out the proof about $-\gamma \leq z(k+1) \leq \gamma$ with $z(k)<0$.

First, we will prove $z(k+1) \leq \gamma$ with $z(k)<0$.

When $-1 < z(k) < 0$, utilizing $|z(k)|^{\bar{k}_2} > |z(k)| > |z(k)|^3$ and (25), (27) becomes
\[
z(k+1) \\
= z(k) + h_\xi z^2(k) + h_\xi z [z(k)]^{\bar{k}_2} + |h_0|(k) \\
\leq -|\beta \Lambda(\bar{\lambda}_2) \rho| + h_\xi [\beta \Lambda(\bar{\lambda}_2) \rho]^{\bar{k}_2} + |h_0| \\
\leq -\rho [1 - h_\xi z] [1 - |\beta \Lambda(\bar{\lambda}_2)|]^{\bar{k}_2} \\
\leq -\rho (1 - h_\xi z) [1 - |\beta \Lambda(\bar{\lambda}_2)|] + |\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2}. \\
\]

By (24) and Assumption 2
\[
z(k+1) \leq (1 - h_\xi z) \rho_0 \Lambda(\bar{\lambda}_2) \leq \gamma. \quad (36)
\]

When $z(k) \leq -1$, the relation $|z(k)|^{\bar{k}_2} \leq |z(k)| \leq |z(k)|^3$ can be satisfied, and (27) becomes
\[
z(k+1) \\
\leq -|\beta \Lambda(\bar{\lambda}_2) \rho| + h_\xi [\beta \Lambda(\bar{\lambda}_2) \rho]^{\bar{k}_2} + |h_0| \\
\leq -(1 - h_\xi z) [\beta \Lambda(\bar{\lambda}_2) \rho] \\
+ h_\xi [\beta \Lambda(\bar{\lambda}_2) \rho]^{\bar{k}_2} + |h_0|. \\
\]

By Assumption 2 and (25), we can obtain
\[
z(k+1) \leq |h_0| \leq (1 - h_\xi z) \rho < \Lambda(\bar{\lambda}_2) \rho \leq \gamma. \quad (37)
\]

After $z(k+1) \leq \gamma$ with $z(k) < 0$ proved, we will discuss $z(k+1) > -\gamma$ with $z(k) < 0$. Considering $z(k) < 0$, (27) can be rewritten as
\[
z(k+1) \\
\geq -|z(k)| + h_\xi z^2(k) + h_\xi z [z(k)]^{\bar{k}_2} - |h_0| \\
\geq -|\beta \rho \Lambda(\bar{\lambda}_2) + (|\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2} - 1) |h_0| \\
+ h_\xi [\beta \Lambda(\bar{\lambda}_2) \rho]^{\bar{k}_2} \\
\geq -|\beta \rho \Lambda(\bar{\lambda}_2) - (|\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2} - 1) |h_0|. \\
\]

If $|\beta \Lambda(\bar{\lambda}_2)| \geq 1$, we can directly obtain
\[
z(k+1) \geq -|\beta \rho \Lambda(\bar{\lambda}_2)| \geq -\gamma.
\]

If $|\beta \Lambda(\bar{\lambda}_2)| < 1$, using (25), we have
\[
z(k+1) \\
\geq -[(|\beta \rho \Lambda(\bar{\lambda}_2)| - (|\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2} - 1) (1 - h_\xi z) \rho] \\
\geq -\rho [1 - h_\xi z] (1 - |\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2}) + (|\beta \Lambda(\bar{\lambda}_2)|^{\bar{k}_2}) \\
\geq -\rho > -\beta \rho \Lambda(\bar{\lambda}_2) \geq -\gamma.
\]

To this end, we have finished the analyses about $|z(k+1)| \leq \gamma$, that is, the discrete-time sliding mode dynamics will enter the region $\Omega$ and not escape in the subsequent time by using the proposed control law.

Remark 1: This control law can accelerate the convergence to $\Omega$ when the system is with a large uncertainty leading to $\gamma > 1$.

Remark 2: In practice, Assumption 2 is easy to satisfy. According to the requirement of control performance, a sampling interval $h$ is set first, i.e., for this paper $h = 1$ ms, and we proceed to select suitable $\xi_2$ and $\xi_3$ to make $z(0)$ initially satisfy Assumption 1. The principle of selection of $\xi_2$ and $\xi_3$ is to make $\gamma^2 < \frac{1}{h_\xi z_2}$, because this is based on the assumption is proposed to guarantee reaching condition to hold. If $\gamma^2 < \frac{z^2(0)}{h_\xi z_2}$, we can activate the reaching phase, which leads to a decreasing $z^2(k)$, and the subsequent $z(k)$ will satisfy the assumption directly. If $z^2(0) < \gamma^2$, the sliding phase works, and $|z^2(k)| < \gamma^2 < \frac{1}{h_\xi z_2}$ can also be guaranteed.

Next, we will discuss the tracking precision when the sliding mode dynamics stay in the region $\Omega$. Before moving on, a necessary lemma about discrete-time terminal sliding surface is exhibited.

Lemma 2 [24]: Considering the dynamics on the discrete-time terminal sliding surface $\Delta e(k) + \xi_1 e(k)^{\bar{k}_1} = g(k)$ with $|g(k)| \leq \gamma$, there is a finite step to guarantee
\[
|e(k)| \leq \Lambda(\bar{\lambda}_1) \cdot \max \left\{ \left( \frac{\gamma}{\xi_1} \right)^{1/\bar{k}_1}, \left( \frac{h_\xi}{\xi_1} \right)^{1/\bar{k}_1} \right\}. \quad (38)
\]

According to the analyses in Theorem 1, the proposed control law can drive the system states into the certain region, and combining with the definition of the DFOTS surface, we have $\Delta e(k) + \xi_1 \Delta^{\bar{k}_1} e(k)^{\bar{k}_1} = g(k)$. Using Grünwald–Letnikov
definition, the above expression becomes

\[
\Delta e(k) + \frac{\xi_1}{h^{\alpha-1}} \left| e(k) \right|^{\lambda_1} + \sum_{j=1}^{L} (-1)^j \binom{\alpha - 1}{j} \left| e(k - j) \right|^{\lambda_1} = g(k).
\]

By the note of Lemma 1, it follows from the above expression that

\[
\Delta e(k) + \frac{\xi_1}{h^{\alpha-1}} \left| e(k) \right|^{\lambda_1} \leq \gamma + \frac{\rho_e (K - 1)}{h^{\alpha-1}}
\]

(39)

where \( \rho_e \geq \left\{ \left| e(k - 1) \right|^{\lambda_1} \right\} \).

Based on the result in Lemma 2, we can obtain

\[
|e(k)| \leq \Lambda(\lambda_1) \cdot \max \left\{ \left( \frac{h^{\alpha-1} \gamma_0}{\xi_1} \right)^{1/\lambda_1}, \left( h^{2-\alpha} \xi_1 \right)^{1/\lambda_1} \right\}
\]

(40)

where \( \gamma_0 = \gamma + \frac{\rho_e (K - 1)}{h^{\alpha-1}} \). Compared with the precision in Lemma 2, because \( h \) is the sampling interval with \( 0 < h \ll 1 \) s for digital computer system, we have

\[
(h^{2-\alpha} \xi_1)^{1/\lambda_1} < (h\xi_1)^{1/\lambda_1},
\]

which means that the proposed tracking precision is higher than the discrete-time terminal sliding mode (DTSM) control if \( |e(k)| \) in (40) is determined by \( (h^{2-\alpha} \xi_1)^{1/\lambda_1} \), i.e.,

\[
(h^{2-\alpha} \xi_1)^{1/\lambda_1} \geq \left( \frac{h^{\alpha-1} \gamma_0}{\xi_1} \right)^{1/\lambda_1} \Rightarrow \xi_1 \geq \frac{\gamma_0^{1/\lambda_1}}{h^{1+\lambda_1-\alpha}}.
\]

(41)

IV. EXPERIMENTAL STUDIES

A. Experimental Setup

The proposed control scheme is examined on an iron-core LM, as shown in Fig. 1, and implemented in Twincat 3.0 with real-time code running in an industrial programmable logic controller (PLC). The communication protocol of the entire control system is CANopen, and the sampling interval for the performance test is decided to be 1 ms, which cover the sampling time of the position sensor and the communication period. The setup is equipped with an absolute position sensor with a resolution of 50 nm, and the velocity signal is estimated by using backward difference technique. The mass of nominal model are provided by the manufacturer \( m = 11 \) kg, and the values of parameters involving the sliding friction \( k_r = 7.847 \) N and \( k_v = 0.019 \) Ns/m are identified by using a PSO-based evaluation with 300 particles iterating 300 generations [29]. Conservatively, we set the bounds of parameters which indicate the parametric uncertainties as follows: \( \theta_d = 2, \theta_f = 5 \), and \( d_m = 10 \).

B. Controller Parameter Selections

Next, we will discuss the selections of the controller parameters. For satisfying Assumption 2, \( \xi_3 \) is conservatively set as \( \xi_3 = 450 \), and we set \( \xi_2 = 10 \) to obtain a broad selection of the initial sliding mode dynamics \( z(0) \). It is worth noting that there exists a contradiction between the selections of \( \xi_1 \) and \( \alpha \) if \( \lambda_1 \) is fixed. A small \( \alpha \) results in a higher tracking precision and, meanwhile, requires a large \( \xi_1 \) to guarantee a better performance, as shown in (40) and (41). However, too large \( \xi_1 \) may not meet the requirement of Assumption 2. Through trial and error, we set \( \xi_1 = 500, \lambda_2 = 0.6, \lambda_1 = 0.6 \), and \( \alpha = 0.5 \), and among these selections \( \lambda_1 = 0.6 \) and \( \alpha = 0.5 \) are frequently used in the fractional order or integer-order terminal sliding mode controllers [16]–[18]. These parameters can yield a theoretical tracking precision of 31.44 \( \mu \)m.

C. Experimental Results

In the previous sections, we have analyzed the tracking precision and corresponding parameter selections, and here will employ the real LM to verify the effectiveness. As a comparison, the DTSM controller presented in [24] is also tested on the experimental setup, and for a fair comparison, the controller parameters of the comparison are same to the proposed controller’s.

In this paper, we assign two experiments as follows.

1) Exp1: Controllers are applied on the experimental setup without extra payload to test the tracking performances.

2) Exp2: Extra payload of 3 kg is added on the moving stage to test the robustness against the parametric uncertainties.

In both experiments, the controllers are required to track two kinds of reference signals to inspect the corresponding tracking performances. The first reference signal is a cosinoidal function with the final frequency of 0.5 Hz and the amplitude of 1 cm: \( p_r(k) = 10^{-2} (1 - e^{-hk}) [0.5 - \cos(\pi h k)] \), and the other one is a triangle signal with varying slopes. For clarity, we denote Exp1 with the cosinoidal reference as Case1A, and denote Exp1 with the triangle reference as Case1B. Accordingly, Case2A and Case2B are denoted to represent Exp2 with different references. The DFOTSM and DTSM controller are marked as C1 and C2, respectively.

![Fig. 2. Exp1 with the cosinoidal reference. Tracking trajectories.](image-url)
The experimental results of Exp1 are shown in Figs. 2–4. It is obvious from Fig. 2 that the tracking trajectory of C1 is closer to the reference than C2’s. From the enlarged figures, one can see that there exist some overshoots when C2’s motion direction changes. Meanwhile, C1’s trajectory is smooth and almost coincides with the reference everywhere.

Case1B is assigned to test the low-speed tracking performance, and the experimental results are shown in Fig. 3. Because the amplitude of the reference is 1 mm, which is smaller than of Case1A, the slight steady-state errors can be found in C1’s trajectories, which are far smaller than C2’s.

The tracking errors of Exp1 are given out in Fig. 4. Intuitively, in Case1A, C2’s errors are about 40 μm, much larger than C1’s errors that are less than 5 μm. These actual tracking precisions are much higher than the theoretical precisions because the bounds of the physical parameters are chosen conservatively in the analyses. In addition, the C2’s errors fluctuate more widely than C1, and the enlarged figures indicate that C1’s maximum errors appear at the moment of motion direction change and fall back in no more than 0.1 s. Moreover, the maximum errors are not big enough to cause the overshoots, which coincides with the aforementioned analyses.

Exp2 is carried out to examine the tracking performance of the proposed control scheme when the moving stage is subjected to severe parametric uncertainty. The tracking errors are shown in Fig. 5. Although the fluctuation of C1’s errors become more frequent, and the maximum error also increases, C1 can still guarantee the tracking precision of approximating 5 μm.

The typical control inputs are illustrated by Fig. 6. It is clear to see that the adjustment range of C1’s inputs is around 10 N, which is similar to C2’s. Some fine adjustments can also been found in C1’s inputs and C2’s inputs are more smooth but superficial. Considering the corresponding tracking errors, this contrast means that C1 is sensitive to the reference signal change that is the control effort can be regulated in time to guarantee a high-precision tracking when the reference signal is time varying. This phenomena is more conspicuous in Fig. 6(c), the motion direction changes at 1 s and 2.4 s, and C1’s control effort responds quickly to this change and turn steady at about 1.1 s; whereas C2 regulates the input more slowly and becomes steady after 1.3 s.

D. Summary and Statistics of Experimental Results

The previous qualitative analysis has demonstrated that the DFOTSM controller owns the better performance than the
DFTSM controller, and here we carry out the quantitative comparison based on the statistics of experimental results.

We define the average tracking error as $\frac{1}{N} \sum_{k=1}^{N} |e(k)|$, and besides the average tracking error, we also calculate the root-mean-square error (RMSe) defined as $\text{RMSe} = \sqrt{\frac{1}{N} \sum_{k=1}^{N} e^2(k)}$, where $N$ is the sampling number. The results are shown in Fig. 7 and indicate that the DFOTSM controller owns the better average error and RMSe. Moreover, these indexes of the DFOTSM controller maintain stable even though the payload is varying, which coincides with the theoretical analyses and experimental results. Compared the statistic results of Exp1 and Exp2, it is apparent that the varying payload does not affect the stability of the control system.

V. CONCLUSION

In this paper, a DFOTSM control strategy has been developed for the practical LM-based high-precision motion control system. Grünwald–Letnikov fractional order difference was introduced into integer-order DTS model to synthesize a novel sliding surface. The control scheme takes into bounded parametric uncertainties, including sliding friction, varying payload, and unknown disturbance, and the corresponding reaching law guarantees the sliding mode dynamics to converge into the certain region in finite steps. The theoretical analyses proved that the proposed sliding surface can increase the tracking precision drastically compared with the DTS model. Experiments were carried out on an industrial LM-based setup with different payload, and the proposed controller performs high-precision, fast-response, and robustness to uncertainties, which coincided with the theoretical analyses.

APPENDIX

Proof of $|h\hat{\rho}_0| \leq h\xi_3\hat{\rho}^{\lambda_2} \leq (1 - h\xi_2)\hat{\rho}$ in (25): According to the definition of $\hat{\gamma}$ and Assumption 2, consider the situation that $\frac{\hat{\rho}_0}{\xi_3} \geq (\frac{h\xi_3}{1-h\xi_2})^{\frac{1}{\lambda_2}}$, and since $\hat{\rho}_0$, $\xi_3$, $h$ and $1-h\xi_2$ are all positive, we have

$$h\xi_3\hat{\rho}^{\lambda_2} \geq h\xi_3(1-h\xi_2)^{\frac{1}{\lambda_2}}.$$

(A.1)

Using the above expression, we can obtain

$$\hat{\rho} = \left(\frac{\hat{\rho}_0}{\xi_3}\right)^{\frac{1}{\lambda_2}} \Rightarrow |h\hat{\rho}_0| = h\xi_3\hat{\rho}^{\lambda_2} \leq (1 - h\xi_2)\hat{\rho}.$$  \hspace{1cm} (A.2)

On the contrary, if $\frac{\hat{\rho}_0}{\xi_3} < (\frac{h\xi_3}{1-h\xi_2})^{\frac{1}{\lambda_2}}$ established, we have

$$\hat{\rho} = \left(\frac{h\xi_3}{1-h\xi_2}\right)^{\frac{1}{\lambda_2}}$$

which implies $|h\hat{\rho}_0| < h\xi_3\hat{\rho}^{\lambda_2}$.

Hence, we have obtained $|h\hat{\rho}_0| \leq h\xi_3\hat{\rho}^{\lambda_2} \leq (1 - h\xi_2)\hat{\rho}$.  \hspace{1cm} ■

Remark 3: It is obvious from the result of this lemma that one can also get $|h\hat{\rho}_0| \leq h\xi_3\hat{\rho}^{\lambda_2} \leq (1 - h\xi_2)\hat{\rho}$.

REFERENCES


