# Fundamental Problems in Viscoplasticity

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The fundamental assumption of all theories of plasticity—that of time independence of the equations of state—makes simultaneous description of the plastic and rheologic properties of a material impossible.

It is well known that in many practical problems the actual behaviour of a material is governed by plastic as well as by rheologic effects. It can even be said that for many important structural materials rheologic effects are more pronounced after the plastic state has been reached.

Every material shows more or less pronounced viscous properties. Today this is a common place. In some problems the influence of viscous properties of the material may be negligible while in other problems it may be essential.

Both sciences—plasticity and rheology—are concerned with the description of very important mechanical properties of structural materials. Each of them has created its own methods of investigation and has developed within the framework of certain assumptions which, unfortunately, cannot always be satisfied in reality. The results of rheology are confined to cases where plastic strain is of no decisive importance. On the other hand, the results of the theory of plasticity permit correct description of only such problems where the influence of rheologic effects may be considered unessential. In other words, if methods of rheology are used we should confine our considerations to the study of those states of stress that do not produce plastic flow of the material. If methods of plasticity are used we must limit ourselves to quasi-static processes the duration of which is sufficiently short, so that creep or relaxation effects do not occur. However, recent research concerning the description of dynamic properties of materials has shown that the application of the theory of plasticity, in which rheologic effects are disregarded, leads to too large discrepancies between the theoretical and experimental results.

Thus there is no need to point out the advantages that can be gained by simultaneous description of rheologic and plastic effects, and the general problem is that of viscoplasticity. Before we proceed to a more detailed description let us emphasize the fact that the methods of viscoplasticity belong neither to rheology nor to plasticity. It will be seen that the specific character of the problem and the complicated way in which rheologic and plastic effects are interconnected require special methods, based on careful analysis of the physics of solids and special mathematical methods.

However, the difficulties of combined treatment of rheologic and plastic phenomena are enormous. The viscous properties of the material introduce a time dependence of the states of stress and strain. The plastic properties, on the other hand, make these states depend on the loading path.
Thus, as a result of simultaneous introduction of viscous and plastic properties, we obtain a dependence on the load history and on the time. A description of strain in viscoplasticity will therefore involve the history of the specimen, expressed in the type of the loading process, and the time. Different results will be obtained for different loading paths and different duration of the process.

The aim of the present survey is to discuss the new problems, in which combined treatment of rheologic and plastic phenomena is essential, and to describe the principal lines along which the methods of viscoplasticity develop. Although viscoplasticity cannot yet be regarded as a theory in its definitive shape, certain research trends can be discerned. The fundamental problem of viscoplasticity is the determination of an adequate yield criterion for a viscoelastic material. Another important problem is that of establishing suitable constitutive equations.

2. Experimental Results for Metals

It is not the aim of the present section to give a survey of experimental methods. Rather, a critical review of the results obtained by various research workers using different apparatus and different methods will be presented. However, the validity of the results obtained cannot be appraised without taking into consideration the accuracy of the method applied; therefore, in the course of our discussion, we shall try to give a brief characteristic of the method and an analysis of the validity of the interpretation.

Often the experimental results are obtained correctly by an accurate method, but their interpretation gives rise to some doubts. Indeed, the dynamic phenomena occurring in a test piece during the impact process is influenced by so many factors that the selection of the dominating one is very difficult. Often a phenomenon is determined simultaneously by several factors of equal importance. Thus, for instance, for a dynamic tension or compression test the propagation of stress waves is essential which, in turn, depends to a large extent on the length and the form of the test piece, the stress concentration, and the distribution of the strain rate. The strain rate during an impact test is not constant but varies along the bar within broad limits and is also variable in time. As a result, the actual process is extremely complicated. In addition, the entire phenomenon lasts often only a few tens or hundreds of a microsecond.

An ideal solution would be to eliminate completely the stress-wave effect, thus allowing the separation of the influence of the strain rate on the mechanical properties of a metal. This has been pointed out by E. H. Lee and H. Wolf [104], D. S. Clark [40], and J. D. Campbell and J. Duby [27]. Correct interpretation of results can also be obtained by taking into account the wave phenomenon in the test piece. This method would, of course, be more difficult.
Very great progress in the experimental investigation of dynamic properties of materials was made during the last few years; research methods and measuring apparatus were considerably improved. Thus, errors of measurement have been reduced, and a variety of causes influencing the behaviour of a metal during dynamic and static loading has been recognized. The general interest in the dynamic properties of metals is stimulated not only by the complicated character of the dynamic problems, but the differences between the static and dynamic behaviour of a metal are of such practical importance that the results of static tests can no longer be used for an appraisal of a dynamic phenomenon.

Depending on the main object of research, experimental investigations may be classed in various groups. Let the first group be tests which aim at a detailed analysis of the influence of the strain rate on the mechanical properties of a metal. Another group are tests concerned with basic research with the aim to determine the dynamic stress-strain curves of materials. The third group comprises studies of the distribution of permanent strain along a test piece. Next, let us mention investigations of the role of the transversal motion during the propagation of longitudinal stress waves. The influence of the transversal motion is the increase of the dispersion effect. It has also been observed that stresses higher than the upper yield point can be applied for a very short time without producing plastic flow. This phenomenon may be called the delay-time phenomenon. There are many investigations in which the delay-time effect is analysed quantitatively and qualitatively and which try to explain its causes. An interesting group of papers are those, in which changes in the static properties of metals due to previous dynamic loads are studied. The last two groups of experimental works show how the dynamic behaviour of metals is influenced by temperature and irradiation.

The results of the tests of G. I. Taylor and A. C. Whiffin [172, 189], P. E. Duwez and D. S. Clark [67], J. E. Johnson, D. S. Wood, and D. S. Clark [88], and J. D. Campbell [24, 25], have shown that higher stresses are needed for metals to reach plastic state for a sudden than for a slowly acting load.

Many theoretical and experimental investigations have pointed out that metals having a distinct yield limit are particularly sensitive to the strain rate (cf. for instance M. P. White [190], J. Miklowitz [115], and H. G. Hopkins [83]). Very good examples of metals behaving in a different way during static and dynamic loading and showing considerable rate sensitivity are mild steel and pure iron.

The influence of the strain-rate on the yield limit for mild steel has been examined in detail by G. I. Taylor [171], M. J. Manjoine [112], D. S. Clark and P. E. Duwez [39], F. E. Hauser, J. A. Simmons, and J. E. Dorn [77], and K. J. Marsh and J. D. Campbell [113].
G. I. Taylor [171], was able to perform tensile tests over a wide range of average strain rates by firing bullets against an anvil bar held as a simple beam between two tensile specimens. The other end of each specimen was fastened to a ballistic pendulum. Since the velocity of the anvil after impact decreased from the impact velocity to zero, he assumed a mean rate of strain equal to \( \frac{1}{2} \) (velocity of anvil) / (length of specimen). Applying the law of conservation of energy and assuming that the kinetic energy equals the plastic strain energy he determined for steel the variability of the ratio of the dynamic to the static yield limit as a function of the mean strain-rate, Fig. 1.

![Graph](image)

**Fig. 1.** Effect of strain rate on dynamic yield in steel (G. I. Taylor [171]).

The results of D. S. Clark and P. E. Duwez, [39], for mild steel (0.22% carbon, 39,000 lb/in\(^2\) upper static yield limit) have shown (cf. Fig. 2) that the proportionality limit, which can in this case be identified with the upper yield limit increases with increasing strain-rate until it coincides with the ultimate strength of the metal. The latter quantity increases until the strain rate reaches the value of about 200 sec\(^{-1}\). In these tests the specimen was loaded in circumferential tension due to the action of compressed mercury in the bore of the tube.

The properties of low-carbon structural steel of 28,000 lb/in\(^2\) lower yield point were studied by M. J. Manjoine [112]. His results are somewhat different from those obtained by D. S. Clark and P. E. Duwez.

The paper of F. E. Hauser, J. A. Simmons, and J. E. Dorn [77], presents a method by which the plastic properties of materials can be investigated at strain rates up to \( 1.5 \times 10^4 \) sec\(^{-1}\) through impulsive-loading techniques. Some of these data, for high purity aluminium at 295° K, are plotted in Fig. 3 where the stress is represented as a function of the strain for a given strain rate.

K. J. Marsh and J. D. Campbell [113] using a rapid-loading hydraulic test machine obtained interesting results of constant-stress tests on mild-steel specimens of different mean ferrite-grain sizes. These results suggested
that the behaviour of the material may be represented by a functional relationship between stress, mean strain and mean strain-rate, but this

Fig. 2. Effect of strain rate on the proportional limit and ultimate strength of a 0.22% carbon steel (D. S. Clark and P. E. Duwex [39]).

Fig. 3. Effect of stress on strain at constant strain rate of high purity aluminium (F. E. Hauser, J. A. Simmons and J. E. Dorn [77]).
cannot be considered a fundamental one when the strain distribution is non-uniform.

A direct measurement of the effect of strain rate on the character of the $\sigma, \varepsilon$-curve was carried out by T. E. Tietz and J. E. Dorn [177].

Experimental investigations of J. D. Campbell [25, 26], J. D. Campbell and J. Duby [27], J. Harding, E. O. Wood, and J. D. Campbell [75], K. J. Marsh and J. D. Campbell [113], have shown that the upper yield limit for mild steel during a dynamic loading-process may reach a value 2.5-3 times as high as during the static test. These tests show clearly that the yield limit increases with increasing strain rate. At the same time, the very important phenomenon of a definite reduction of the strain-harden-ning effect during the process of dynamic loading as compared to the strain-hardening effect observed during the static test is pointed out.

![Figure 4](image-url)

**Fig. 4.** Super-purity iron based 0.21 per cent carbon steel curves: (a) strain rate against strain for dynamic test; (b) stress against strain, A. Static, B. Dynamic; numbers denote time in microseconds; impact velocity 198 in./sec. (J. Harding, E. O. Wood and J. D. Campbell [75]).

Dynamic $\sigma, \varepsilon$-curves were obtained by J. E. Johnson, D. S. Wood, and D. S. Clark [88], and J. D. Campbell [24], for aluminium and by J. D. Campbell [26] and J. D. Campbell and J. Duby [27], for mild steel. J. Harding, E. O. Wood, and J. D. Campbell [75], determined the dynamic characteristics for a few metals, among others for mild steel and pure iron (cf. Figs. 4 and 5). H. Kolsky and L. S. Douch [95], measured the dynamic stress-strain curves of annealed specimens of pure copper, pure aluminium, and an aluminium alloy by experiments in which short bars of these materials were fired from a compressed-air gun at a steel pressure bar. It was found that for copper and aluminium the dynamic curves lay appreciably above the static ones whereas for the aluminium alloy there appeared to be no appreciable strain-rate effect.
Detailed analysis of the distribution of permanent strain along a specimen subjected to a dynamic load was performed by many scientists (cf. for instance Th. Kármán and P. Duwez [92], J. D. Campbell [26], and H. Kolsky and L. S. Douch [95]). J. D. Campbell [26], showed that if a specimen (12.7 mm diameter and 266.7 mm length) undergoes an impact with initial velocity of 400 in/sec, yielding occurs only in the neighbourhood of end of the test piece under impact and is limited to about 10 mm (Fig. 6). These tests showed that there is a strong dispersion due to rheologic effects.

The influence of the transversal motion on the phenomenon of propagation of stress waves in a bar is discussed in a paper by H. Kolsky and L. S. Douch [95].
The phenomenon of the delay time was studied in detail for mild steel by J. E. Johnson, D. S. Wood, and D. S. Clark [87]. The delay time which varies within the limits of 40 μsec to 1.5 msec is a function of the initial impact stress varying from 80,000 lb/in² to 5000 lb/in², respectively. It was also shown that for low carbon steel the delay effect in the dynamic compressive test is of the same nature as in the static tensile test (Fig. 7; cf. also [28, 32, 38, 40]).

The results of tests of D. B. C. Taylor [173], showed that the delay phenomenon can be observed not only for metals in the non-plastic state but also in a partially plastic state.

In a paper by J. D. Campbell and C. J. Maiden [29], the influence of previous dynamic loadings on the static properties of the material was studied. In these experiments annealed medium-carbon steel specimens were subjected to rapidly applied compressive loads maintained for periods of the order of $10^{-4}$ sec. The applied stresses were between two and three times the static upper yield stress of the steel and the permanent deformation varied from 1.2 percent down to very small amounts. Static stress-strain curves were obtained by reloading the specimens in compression immediately after impact. It appears that the upper yield stress can be considerably reduced by the application of an impact stress of magnitude

![Fig. 7. Stress versus log delay time, 73 °F (J. E. Johnson, D. S. Wood and D. S. Clark [87]).](image)
and duration insufficient to cause appreciable permanent deformation (see Fig. 8).

The influence of temperature on the dynamic behaviour of metals was the object of tests of, among others, D. D. Sokolov [159, 160], M. Manjoine [112], J. F. Alder and V. A. Philips [1], J. M. Krafft, A. M. Sullivan, and

![Diagram](image_url)

**Fig. 8.** Static stress-strain curves obtained after dynamic loading. Curve A: annealed specimen (not pulsed). Curves B, C, D: specimens pulsed at 277, 293 and 308 in./sec., respectively. Curves E, F, G, H: specimens impacted at 326, 340, 352 and 365 in./sec., respectively (J. D. Campbell and C. J. Maiden [29]).

C. F. Tipper [96], R. J. Mac Donald, R. L. Carlson, and W. T. Lankford [60], C. J. Maiden and J. D. Campbell [109], and J. L. Chiddister and L. E. Malvern [34]. It was observed that if the temperature is decreased the rate-sensitivity of the material increases. This is illustrated by some results taken from the work of C. J. Maiden and J. D. Campbell [109], and collected in Table 1.

The influence of neutron irradiation on the dynamic properties of metals has not yet been investigated in a satisfactory manner. Some results are contained in the paper by J. D. Campbell and J. Harding, [31]. The influence of neutron irradiation on the dynamic properties seems to be of a more complex nature, and more detailed conclusions are so far not possible. The paper by J. D. Campbell and J. Harding has shown, however, that these changes are very pronounced, and decisive for the behaviour of the materials.
### Table 1. The Properties of a Carbon Steel at Low Temperature (C. J. Maiden and J. D. Campbell [109])

<table>
<thead>
<tr>
<th>Temperature of test °C</th>
<th>Impact velocity in/sec</th>
<th>Time to yield µsec</th>
<th>Dynamic upper yield stress $10^3$lb/in²</th>
<th>Dynamic lower yield stress $10^3$lb/in²</th>
<th>Ratio of dynamic to static upper yield stress</th>
<th>Maximum strain rate sec⁻¹</th>
<th>Average strain rate sec⁻¹</th>
<th>Permanent deformation %</th>
</tr>
</thead>
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<tr>
<td>+15</td>
<td>503</td>
<td>25</td>
<td>113</td>
<td>94</td>
<td>2.57</td>
<td>1220</td>
<td>870</td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td>470</td>
<td>28</td>
<td>108</td>
<td>93</td>
<td>2.45</td>
<td>970</td>
<td>570</td>
<td>4.8</td>
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<td></td>
<td>430</td>
<td>30</td>
<td>105</td>
<td>92</td>
<td>2.38</td>
<td>800</td>
<td>440</td>
<td>3.3</td>
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<tr>
<td>-41</td>
<td>503</td>
<td>30</td>
<td>125</td>
<td>107</td>
<td>2.49</td>
<td>950</td>
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<td>116</td>
<td>104</td>
<td>2.38</td>
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<td>135</td>
<td>2.13</td>
<td>470</td>
<td>200</td>
<td>0.9</td>
</tr>
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</table>

### 3. Experimental Results for Soils

Only a limited number of measurements of plastic constants of soil have been made under dynamic conditions, but recent experimental data show clearly that soils exhibit rheological effects and are sensitive to the change of the strain rate. Thus the characteristic feature of the behaviour of soils, especially in their dynamic response, is the time-dependence of the deformation process.

Of particular value are data discussed by P. Chadwick, A. D. Cox and H. G. Hopkins [33] (see also A. W. Skempton and A. W. Bishop [157]). These data are presented in Fig. 9 and show that, as the rate of strain increases from zero to about 1.5 sec⁻¹, the strengths of clays and sands increase by factors of about 2 and 1.2, respectively. The authors of paper [33] note that strain rate effects in soils are not well understood, but in clays the resistance to the flow of pore water is probably one reason for their existence.

The influence of the strain rate effects on the behaviour of soils have also been investigated by N. A. Alexiev, Ch. A. Rakhmatulin and A. Y. Sagamonian [2].
It is clear that in real soil the inelastic strain-rate tensor depends on the time-history as well as on the path-history of the stress.

![Graph showing strength ratio vs. rate of strain for clay and sand](image)

**Fig. 9.** Relation between strength and rate of strain for (a) clay and (b) sand (data of D. W. Taylor, A. Casagrande and W. L. Shannon, from A. W. Skempton and A. W. Bishop [157]).

4. Some Theoretical Ideas for the One-Dimensional Case

The difference in behaviour of a material during static and dynamic loading is also of interest to theoreticians. They have tried, above all, to describe the influence of the strain-rate on the yield limit of the material.

The general physical relation for the one-dimensional problem may be represented in the form (cf. the papers of H. G. Hopkins [83–86])

\[ \sigma = \phi(\varepsilon^p, \dot{\varepsilon}^p) \]

where \( \sigma \) is the nominal tensile stress and \( \varepsilon^p \) and \( \dot{\varepsilon}^p \) are the nominal plastic strain and strain rate, respectively. P. Ludwik [107], and H. Deutler [58], on the basis of experimental investigations, and L. Prandtl [147], on the basis of some theoretical considerations, proposed a logarithmic expression for the function \( \phi \) (see also A. L. Nadai [116]).

In the analysis of the problem of propagation of plastic stress waves in bars, the effect of strain rate was first taken into consideration by V. V. Sokolovsky [161, 162], and L. E. Malvern [110, 111]. The physical law of L. E. Malvern can be expressed thus

\[ \dot{\varepsilon} = \dot{\varepsilon}/E + \langle \Phi(\sigma - f(\varepsilon)) \rangle \]

where \( \varepsilon \) denotes total nominal strain, \( E \) denotes Young's modulus, and \( \sigma = f(\varepsilon) \) is the static curve for simple tension or compression.
The symbol $\langle \Phi \rangle$ is defined thus

$$
\langle \Phi \rangle = \begin{cases} 
\Phi & \text{if } \sigma > f(e), \\
0 & \text{if } \sigma \leq f(e).
\end{cases}
$$

L. E. Malvern discussed in detail two cases of the function $\Phi$: the exponential function

$$
\Phi = a\left\{\exp b[\sigma - f(e)] - 1\right\}
$$

and the linear function

$$
\Phi = c[\sigma - f(e)],
$$

where $a$, $b$ and $c$ are constants depending on the material. Equation (1.5) had been introduced earlier by V. V. Sokolovsky [161, 162].

For the solution of the problem of propagation of plastic-stress waves in a bar a more general physical relation was used by L. E. Malvern:

$$
\dot{\varepsilon} = \dot{\sigma}/E + \langle g(\sigma,e) \rangle.
$$

An analysis of the physical relation (1.2) with (1.5) reveals easily that an increase of the plastic strain rate proportional to the difference between the actual stress and the stress computed from the static curve is the assumption of L. E. Malvern [110]. This difference produces the strain rate in agreement with the viscosity law. The elastic strain component is considered to be independent of the strain rate. In agreement with these assumptions

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![Graph](image-url)

**Fig. 10.** Strain distribution in a long bar (S. Rajnak, F. Hauser and J. E. Dorn [150]).
and in contrast with the results of the theory assuming the static characteristic as a basis for dynamic considerations, the front of a plastic wave propagates with the same velocity as the front of an elastic wave.

The theoretical results obtained by L. E. Malvern show that in this strain-rate dependent theory there is no region of constant strain in the neighbourhood of the end subjected to the impact load. It is worthwhile to observe that in many experimental investigations indeed no such region was found (see for instance the tests by J. D. Campbell [26], Fig. 6). A detailed discussion of this problem can be found in the paper by H. G. Hopkins [83]. S. Rajnak, F. E. Hauser, and J. E. Dorn [150], examined theoretically the strain distribution in a long bar subjected to dynamic load. As a basis for their study they took the physical relation of L. E. Malvern (1.6), the function $g(\sigma, e)$ being determined from experimental investigations with aluminium. Their results confirm the validity of the assumptions of the theory that accounts for the influence of the strain rate. Fig. 10 represents the strain distribution along the bar, computed for various times. Similarly, Fig. 11 represents the distribution of the strain rate. Fig. 12 represents a comparison of theoretical and experimental results.
The experimental results obtained by H. Kolsky and L. S. Douch [95], show, in comparison with the results of the theory disregarding the strain rate, that permanent deformation at the end of the bar is in agreement with the theoretically expected results. The theoretical predictions of strain distribution were less satisfactory, and there was some indication in the results for copper at low impact velocities that the simple theory needed modification in the manner suggested by L. E. Malvern.

![Final strain distribution in a long bar](image)

**Fig. 12.** Final strain distribution in a long bar (S. Rajnak, F. Hauser and J. E. Dorn [150]).

B. E. K. Alter and C. W. Curtis [3], carried out a number of tests with the aim to determine how impulses producing plastic strain propagate along cylindrical lead bars. These tests have shown considerable influence of the strain rate on the propagation of plastic strain. They are also in agreement with the assumptions of L. E. Malvern's nonlinear theory.

E. J. Sternglas and D. A. Stuart [165], who tested copper specimens, showed that stress waves produced by an increase of dynamic load superposed on a static load at the yield limit propagate with the elastic velocity. A similar phenomenon was observed by J. F. Bell [10] with steel specimens.
These observations are contradicted, however, by recent tests carried out by H. Kolsky and L. S. Douch [95], who point out the fact (cf. also H. G. Hopkins [83–85]) that the load conditions in [10] and [165] were not purely dynamic but partially static. Most experimental results show clearly that the adoption of the static tension curve as a basis for the description of dynamic phenomena leads, in general, to erroneous results. This does not mean that the influence of the strain rate is equally important for every material. There are many materials for which this phenomenon has not been observed to be essential, and the assumptions of Kármán, Taylor, and Rakhmatulin are thus justified. Many special tests concerned with this problem could be quoted. In a series of tests for heat-treated aluminium it has been shown by J. F. Bell [11–17], who used an original test method based on diffraction, that the results obtained are in good agreement with the assumptions of the strain-rate independent theory. A similar conclusion has been obtained by H. Kolsky and L. S. Douch [95], for an aluminium alloy.

The physical relations (1.1)–(1.6) have purely phenomenological character. It will be of interest to analyse them in the light of the dislocation theory of crystalline materials. Much recent theoretical and experimental work is concerned with the detailed investigation of this problem.

The idea of explaining the phenomenon of plastic flow of metals on the basis of dislocation theory has been communicated by A. M. Cottrell and B. A. Bilby [48]. General foundations for the explanation of many phenomena during dynamic loading processes of metals are given in a later paper by A. M. Cottrell, [49]. Generalizing this idea J. D. Campbell [25, 97] has shown that the phenomenon of delay before yielding can be explained satisfactorily by dislocation theory.

Although this theory has not yet been developed sufficiently to ensure an adequate basis for the derivation of physical relations describing the dynamic behaviour of a material, it has been shown by J. A. Simmons, F. Hauser, and J. E. Dorn [156], that from the viewpoint of dislocation theory the physical relation (1.1) may be written as

\[
\dot{\varepsilon} = f(\sigma, \varepsilon, \dot{\varepsilon})d\sigma + g(\sigma, \varepsilon, \dot{\varepsilon})d\varepsilon + h(\sigma, \varepsilon, \dot{\varepsilon})dt,
\]

where the influence of strain acceleration is taken into account in addition to that of strain rate.

Another interesting question is: what is the interpretation of the existing physical relations in the light of the dislocation theory. This is the subject matter of a paper by J. A. Simmons, F. Hauser, and J. E. Dorn [156]. A study by means of the dislocation scheme of the classical theory of Kármán, Taylor, and Rakhmatulin and the viscoplastic theory in which the influence of the strain rate is taken into consideration has shown the advantages of the viscoplastic theory. It is hoped that some of the experimental tests
now being carried out and a tentative interpretation of their results by means of the dislocation theory will lay the foundations for the derivation of general physical relations which describe the dynamic behaviour of plastic materials.

The behaviour of Frank-Read dislocation sources under the action of impact stress was studied by J. D. Campbell, J. A. Simmons, and J. E. Dorn [30]. This analysis has shown that if the initial accelerating part of the dynamic process of dislocation can be disregarded, the nonlinear theory of L. E. Malvern can be adopted, at least formally, as a basis for the description of the behaviour of crystalline materials.

A particular form of the relations (1.7) has recently been analysed by J. Lubliner, [106]. Taking as a basis the equation

\[
\frac{\partial e}{\partial t} = f(\sigma, e) \frac{\partial \sigma}{\partial t} + g(\sigma, e)
\]

he proposed the following modification

\[
\frac{\partial e}{\partial t} = \langle g(\sigma, e) \rangle + \begin{cases} 
(f(\sigma, e) \frac{\partial \sigma}{\partial t}, \quad \frac{\partial \sigma}{\partial t} > 0, \\
(1/E) \frac{\partial \sigma}{\partial t}, \quad \frac{\partial \sigma}{\partial t} < 0.
\end{cases}
\]

It is evident that these modified physical relations become, in particular cases, those of the Kármán-Taylor-Rakhmatulin or of the Sokolovsky-Malvern theory.

Physical relations similar to (1.9) were also the object of the investigations by N. Cristescu [56, 57] who used them in problems of propagation of stress waves in thin strings.

5. Assumptions and Definitions

We shall make a distinction between an elastic-viscoplastic material and an elastic/viscoplastic material. Elastic-viscoplastic we shall call a material showing viscous properties in both the elastic and plastic regions. The term elastic/viscoplastic will be reserved for materials showing viscous properties in the plastic region only.

A similar distinction was introduced by P. M. Naghdi and S. A. Murch in [118]. The notion of an elastic/viscoplastic material is evidently an idealization that simplifies the argument considerably. To see this consider the problem of choice of an adequate yield criterion. The determination of the yield condition for an elastic-viscoplastic material is very difficult and has not yet been done. The assumption of the elastic/viscoplastic scheme solves this problem at once, because the initial yield condition can remain
the same as in flow theory. Difficulties are encountered only if the variability of flow surfaces in the course of a deformation process of an elastic/viscoplastic body is considered.

The distinction just introduced requires that the considerations concerning the constitutive equations should be subdivided into two parts. Thus, methods of description of elastic-viscoplastic materials will be discussed first in Secs. 2 to 6. Sections 7 to 13 will be concerned with elastic/viscoplastic materials and with the application of that scheme to the description of the dynamic properties of plastic materials.

It will be assumed throughout the entire work that the displacement gradients in the straining processes under consideration are infinitely small (cf. the definition in the monograph by C. Truesdell and R. Toupin [183]).

II. Constitutive Equations

1. The Behaviour of a Material in the Viscoelastic Region

Let us denote by $\varepsilon_{ij}$ the strain tensor and by $\sigma_{ij}$ the stress tensor in Cartesian coordinates. For an elastic-viscoplastic body it will be assumed that the strain tensor can be represented in the form of the sum

\begin{equation}
\varepsilon_{ij} = \varepsilon''_{ij} + \varepsilon^v_{ij} + \varepsilon^p_{ij},
\end{equation}

where $\varepsilon'', \varepsilon^v, \varepsilon^p$ denote the elastic, viscous, and plastic strain components, respectively.

Let us emphasize the fact that the equation (2.1) is not satisfied in general. Its introduction will be treated as a fundamental simplifying assumption.

The stress and strain deviator tensors used in what follows are defined thus:

\begin{equation}
\varepsilon_{ij} = \varepsilon_{ij} - \delta_{ij}, \quad \varepsilon = \frac{1}{2} \varepsilon_{ii},
\end{equation}

\begin{equation}
\sigma_{ij} = \sigma_{ij} - \delta_{ij}, \quad s = \frac{1}{3} \sigma_{ii},
\end{equation}

where $\delta_{ij}$ is Kronecker's delta.

The components of the strain-rate and stress-rate tensor will be denoted by $\dot{\varepsilon}_{ij}$ and $\dot{\sigma}_{ij}$, respectively.

Problems of linear theory of viscoelasticity have been extensively treated in numerous monographs and survey papers. Let us quote several recent fundamental investigations giving a systematic analysis of physical relations between the stress tensor and the strain tensor for viscoelastic bodies, within the frame of the linear theory. These are the papers by D. R. Bland [18], B. D. Coleman and W. Noll [42], M. E. Gurtin and E. Sternberg [73],
A. C. Eringen [68], and W. Noll and C. Truesdell [122]. They also contain extensive lists of literature.

With the notations of [73] the physical relations between the tensors of strain and stress in the viscoelastic region read

\begin{equation}
\epsilon^{(e)}_{ij} = \hat{s}_{ij}(x) \mathcal{I}_1(t) + \int_0^t \mathcal{I}_1(t - \tau) \frac{\partial s_{ij}(x, \tau)}{\partial \tau} d\tau,
\end{equation}

\begin{equation}
\epsilon^{(v)} = \hat{s}(x) \mathcal{I}_2(t) + \int_0^t \mathcal{I}_2(t - \tau) \frac{\partial s(x, \tau)}{\partial \tau} d\tau,
\end{equation}

where \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) denote the distortion and the bulk expansion creep-functions, respectively. The notations \( s_{ij}(x, t) \) and \( \hat{s}_{ij}(x) \) stand for \( s_{ij}(x, t) \), \( s_{ij}(x, 0 +) \), respectively.

The physical relations (2.4) and (2.5) express the dependence of the state of stress at the time \( t \) on the history of that state in the entire time interval in which the phenomenon takes place, that is from 0 to \( t \).

The authors of [73] have shown that the initial elastic state of a viscoelastic body obeying (2.4) and (2.5) is described by the generalized Hooke's law

\begin{equation}
\epsilon_i^e = \frac{1}{2\mu} s_{ij}, \quad \sigma^e = \frac{1}{3K} s,
\end{equation}

where \( \mu \) and \( K \) are the elastic constants of the material (shear and bulk modulus).

Making use of this observation, the viscous strain components can be separated off in a simple manner. From (2.4), (2.5) we find (cf. P. M. Naghdi and S. A. Murch [118])

\begin{equation}
\epsilon_i^v = \left\{ \hat{s}_{ij}(x) \mathcal{I}_1(t) + \int_0^t \mathcal{I}_1(t - \tau) \frac{\partial s_{ij}(x, \tau)}{\partial \tau} d\tau \right\}
\end{equation}

\begin{equation}
- \left\{ \mathcal{I}_1 \left[ \hat{s}_{ij}(x) + \int_0^t \frac{\partial s_{ij}(x, \tau)}{\partial \tau} d\tau \right] \right\}
\end{equation}

\begin{equation}
= \hat{s}_{ij}(x) \left[ \mathcal{I}_1(t) - \mathcal{I}_1 \right] + \int_0^t \left[ \mathcal{I}_1(t - \tau) - \mathcal{I}_1 \right] \frac{\partial s_{ij}(x, \tau)}{\partial \tau} d\tau,
\end{equation}
2. The Yield Criterion for a Viscoelastic Material

In order to explain the differences between the establishment of the plastic state in an elastic body and in a viscoelastic body, let us consider the load path in the nine-dimensional space of stresses (Fig. 13).

![Figure 13. Load path and yield surface for viscoelastic material.](image)

In the elastic case the plastic state produced will be represented by the same point independently of the time in which the state represented by A is reached, provided that the load path is the same. If the material is viscoelastic the plastic state may be reached at different points A, or A', say, depending on the time in which the load path is through. This difference is caused by the viscosity of the material and the dependence of the load history on time. It is obvious that by passing through the path OA in the same overall time but with different strain-rates at the same points of the path, different yield limits will be obtained.

In order to describe the complicated problem of a viscoelastic material becoming plastic the notion of flow surface has been introduced by P. M. Naghdi and S. A. Murch [118]:

\[(2.9) \quad f = f(\sigma_{ij}, \varepsilon_{ki}, \kappa, \beta).\]

The elastic-viscoplastic state is determined by the condition \( f = 0 \), while the viscoelastic states correspond to the condition \( f < 0 \). The function \( f \) depends on the state of stress \( \sigma_{ij} \), the state of plastic strain \( \varepsilon_{ki} \), the parameter...
$$\beta = \beta(\varepsilon^{pl}_{li})$$ which represents viscous effects and the strain-hardening parameter $\kappa$. The latter depends, in turn, on the plastic strain and is defined in the same manner as in the theory of plastic flow describing isotropic strain-hardening of the material (cf. the definition (2.44)).

Let us now consider the time-variability of a flow surface. The time-derivative of the function $f$ is

$$f = \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} + \frac{\partial f}{\partial \dot{\varepsilon}^{pl}_{li}} \dot{\varepsilon}^{pl}_{li} + \frac{\partial f}{\partial \kappa} \kappa + \frac{\partial f}{\partial \beta} \beta.$$

If the state under consideration is elastic-viscoplastic and undergoes a change such that $f < 0$, this change leads to a viscoelastic state because $f + jdt$ gives a new value of $f$ which lies below zero. Such a change of the state of stress will be called an unloading process. During such a process there is no increase of plastic strain, therefore $\dot{\varepsilon}^{pl}_{ij} = 0$ which involves $\dot{\kappa} = 0$ because no change of the strain-hardening parameter can take place if $\dot{\varepsilon}^{pl}_{ij} = 0$.

Since $\dot{\beta}$ can be expressed by means of the physical relations (2.7) and (2.8) as a function of the stress rate $\delta_{ij}$, we can introduce

$$L(\delta_{ij}) = \frac{\partial f}{\partial \sigma_{ij}} \sigma_{ij} + \frac{\partial f}{\partial \beta} \beta.$$

The mathematical condition for unloading can now be written as

$$f = 0, \quad L(\delta_{ij}) < 0.$$

A change of state of stress from one elastic-viscoplastic state to another elastic-viscoplastic state accompanied by no increase of plastic strain will be called a neutral process. A neutral state is characterized by

$$f = 0, \quad L(\delta_{ij}) = 0.$$

An active loading process, which is accompanied by an increase of plastic strain, takes place if

$$f = 0, \quad L(\delta_{ij}) > 0.$$

By considering the flow surface in the stress space we observe easily that a neutral process does not correspond to the direction of the stress increment $\delta_{ij}dt$ which is tangential to the flow surface at the point considered. This is different from flow theory.

A neutral state will now be realized if the vector of stress increase $\delta_{ij}dt$ deviates from the direction normal to the flow surface by the angle $\theta$, where

$$\theta = \arccos \left[ - \frac{\partial f}{\partial \beta} \beta \left/ \left| \frac{\partial f}{\partial \sigma_{ij}} \right| |\delta_{ij}| \right. \right].$$
The expression (2.16) shows why the load criteria for the viscoplastic case are changed, if compared to those known from the classical theory of plasticity; the influence of the viscous properties of the material on the yield condition is here responsible. There is also an influence of the strain-rate on the yield condition, of which a detailed discussion will be given later.

M. Reiner [152] discussed the problem of attainment of the plastic state with a constant load, that is, the possibility of a material becoming plastic in a creep process.

An original analysis of yield criteria or, more generally, criteria of obtaining a critical state was proposed by W. Olszak and Z. Bychawski [124, 125]. This idea is based on the observation that the fracture of some viscoelastic bodies depends not only on the elastic energy but also on the rate at which the dissipation energy varies. Examples of bodies were analysed where destruction occurs if the rate of variability of the dissipation energy reaches a certain value. For such bodies the strain rate is decisive. For an analysis of a sufficiently large class of real bodies the authors of [125] propose the following condition for reaching the critical state:

\[
W_E + W_D = k^2 \quad \text{or} \quad (\omega_E + \omega_D)W = k^2,
\]

where \(W_E\) is the reduced elastic distortion energy and \(W_D\) the rate of change of the dissipation energy expressed in the dimension of energy. The quantities \(\omega_D, \omega_E\) and \(k\) are certain constants characterizing the material.

In [125] several models are considered, the form of the criterion (2.16) being analysed in detail.

\section*{3. A Definition of a Stable Inelastic Material}

It appears that a general approach to describing the properties of an elastic-viscoplastic material is given by the ideas of D. C. Drucker set forth in [62, 64].

D. C. Drucker uses certain observations in the domain of the theory of plasticity and points out: If we want to give a uniform and unambiguous description of the viscous and plastic properties of a material, we must introduce certain additional postulates which actually are limitations introduced in a judicious manner.

\* It should be observed, however, that the attainment of the critical state is understood in [125] in a more general manner than the attainment of the plastic state. The plastic state may constitute a particular case of the critical state.
By introducing the postulate of stability of the material we can obtain fundamental conditions the satisfaction of which permits a consistent derivation of the physical relations. These conditions restrict the considerations to a certain class of materials but, at the same time, allow a consistent mathematical description of that class.

Consider a body of volume \( V \) bounded by a regular surface \( S \) subject to surface tractions \( T_i \) and mass forces \( P_i \) which are functions of time. The states of displacement \( u_i \), strain \( \varepsilon_{ij} \), and stress \( \sigma_{ij} \) produced by these boundary conditions are also functions of time. Assume now that the boundary conditions undergo certain variations, determined by the surface tractions \( T_i + \Delta T_i \) and the mass forces \( P_i + \Delta P_i \) to which correspond: the state of displacement \( u_i + \Delta u_i \), the state of strain \( \varepsilon_{ij} + \Delta \varepsilon_{ij} \), and the state of stress \( \sigma_{ij} + \Delta \sigma_{ij} \). The definition of a stable inelastic material (elastic-viscoplastic) is furnished by the following postulate of Drucker:

The work performed by the increments of the external forces on the corresponding increments of the components of the displacement vector must be non-negative.

This may be expressed as follows:

\[
(2.18) \quad \int_{t=0}^{t_k} \left\{ \int_{S} \Delta T_i \Delta u_i \, dS + \int_{V} \Delta P_i \Delta u_i \, dV \right\} \, dt \geq 0,
\]

where \( t = 0 \) is the instant at which the external load increments are applied.

It is often more convenient to introduce two loading paths \( T_i^{(1)}, P_i^{(1)} \) and \( T_i^{(2)}, P_i^{(2)} \), which become different after \( t = 0 \) (Fig. 14). Then (2.18) can be written in the form

\[
(2.19) \quad \int_{t=0}^{t_k} \left\{ \int_{S} \left[ T_i^{(2)} - T_i^{(1)} \right] [\dot{u}_i^{(2)} - \dot{u}_i^{(1)}] \, dS + \int_{V} \left[ P_i^{(2)} - P_i^{(1)} \right] [\dot{u}_i^{(2)} - \dot{u}_i^{(1)}] \, dV \right\} \, dt \geq 0.
\]

If in the expressions (2.18), (2.19) the choice of \( t_k \) is limited by no additional condition and can be assumed to be sufficiently large, we are concerned with "stability in large." If \( t_k \) must be close to \( t = 0 \) we have the mathematical expression of "stability in small."

Using the principle of virtual work, the surface and mass forces and the velocity can be replaced by the stresses and the strain rates. The principle of virtual work asserts that for any continuous velocity \( \dot{u}_i \) we have

\[
(2.20) \quad \int_{S} T_i \dot{u}_i \, dS + \int_{V} P_i \dot{u}_i \, dV = \int_{V} \sigma_{ij} \dot{\varepsilon}_{ij} \, dV,
\]
where \( T_i, P_i, \sigma_{ij} \) represent a set of mechanical quantities in equilibrium and \( \dot{u}_{ij}, \dot{\varepsilon}_{ij} \) constitute a consistent set of kinematic quantities. It should be stressed that the two systems of mechanical and kinematic quantities are not interdependent.

Assuming that only homogeneous states of stress and strain are considered, we obtain from (2.19), on the basis of (2.20), the condition

\[
\int_0^{t_k} \left[ \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)} \right] \left[ \dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)} \right] \, dt \geq 0.
\]

4. Convexity of the Flow Surface

Convexity of the flow surface should be assumed in the theory of elastic-viscoplastic bodies, since it is required by condition (2.21) which follows from the postulate of Drucker.

Assume that the state \( \sigma_{ij}^{(1)} \) is a steady state \( \sigma_{ij}^* \) at the instant \( t = 0 \) and that the state \( \sigma_{ij}^{(2)} \) is time-variable; denote it by \( \sigma_{ij} \). Now consider the following closed load cycle. At \( t = 0 \) the state of load \( \sigma_{ij} \) coincides with the steady state \( \sigma_{ij}^* \). Next, \( \sigma_{ij} \) varies along the path \( M_0M_1 \) (Fig. 15) reaching at \( t = t_1 \) the point \( M_1 \) which represents a plastic state. Increments of plastic strain occur on the path \( M_1M_2 \). The state \( M_2 \) is reached at \( t = t_2 \). Starting from \( t_2 \) unloading takes place along the path \( M_2M_0 \). At \( t = t_k \) the state again coincides with the initial state, thus \( \sigma_{ij} = \sigma_{ij}^* \).

The condition (2.21) for the closed cycle \( M_0M_1M_2M_0 \) in the time interval \( 0 \leq t \leq t_k \) yields

\[
\int_0^{t_k} (\sigma_{ij} - \sigma_{ij}^*)(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^*) \, dt \geq 0,
\]

where \( \dot{\varepsilon}_{ij} \) and \( \dot{\varepsilon}_{ij}^* \) replace \( \dot{\varepsilon}_{ij}^{(2)} \) and \( \dot{\varepsilon}_{ij}^{(1)} \), respectively.
Bearing in mind that $\dot{e}_{ij}^* = 0$ and $\dot{e}_{ij} = \dot{e}_{ij}^{(e+v)} + \dot{e}_{ij}^p$, the expression (2.22) can be written in the form (cf. P. M. Naghdi and S. A. Murch [118]):

$$
\int_{t_1}^{t_2} (\sigma_{ij} - \sigma_{ij}^*) \dot{e}_{ij}^p \, dt + \{\Psi[\sigma_{ij}, \dot{e}_{ij}^{(e+v)}]\}_{t_1}^{t_2} \geq 0,
$$

where

$$
\{\Psi\}_{t_1}^{t_2} = \int_{0}^{t_2} (\sigma_{ij} - \sigma_{ij}^*) (\dot{e}_{ij}^{(e+v)} - \dot{e}_{ij}^{(e+v)}^*) \, dt.
$$

Expanding the first term of (2.23) in a Taylor series at the point $t = t_1$ we obtain in $O(\Delta t)$

$$
[(\sigma_{ij} - \sigma_{ij}^*) \dot{e}_{ij}^p]_{t_1}^{t_2} \Delta t + \{\Psi\}_{t_1}^{t_2} \geq 0.
$$

If we assume $\Delta t = t_2 - t_1$ sufficiently small (i.e. restricting ourselves to "stability in the small") the inequality (2.25) implies the inequality (2.23).

Fig. 15. Closed loading cycle in stress space.

In order to discuss the problem of flow-surface convexity completely we shall introduce, in agreement with P. M. Naghdi and S. A. Murch [118], the notion of rapid load path and instantaneous loading surface. Rapid load path in the nine-dimensional stress space is a finite process of stress change realized in an infinitely short time, with the parameter $\beta(\varepsilon_{kl}^v)$ remaining unchanged, i.e.

$$
\lim_{\Delta t \to 0} \Delta \beta(\varepsilon_{kl}^v) = 0.
$$

For a rapid load path the behaviour of the flow surface is exactly the same as in the inviscid theory of flow for an ordinary path.
Instantaneous loading surface will be defined now. Let us assume that at a given time $t_0 \geq 0$ we have a state characterized by the variables $\sigma_{ij}^{(a)}$, $\varepsilon_{ij}^{(a)}, \kappa^{(a)}, \beta^{(a)}$ such that the body is viscoelastic, i.e.

$$f(\sigma_{ij}^{(a)}, \varepsilon_{ij}^{(a)}, \kappa^{(a)}, \beta^{(a)}) < 0.$$  

The equation

$$f_a = f(\sigma_{ij}, \varepsilon_{ij}^{(a)}, \kappa^{(a)}) = 0$$

is the instantaneous loading surface corresponding to the state which in the stress space is represented by point $A$ (Fig. 16). It should be observed that the state $A$ not only depends on the components of the stress tensor but also on the quantities $\varepsilon_{ij}^{(a)}, \beta^{(a)}, \kappa^{(a)}$, which describe the history of the loading process. The latter is characterized by the load path and is followed by the varying state of the body at the time at which the point $A$ is reached.

Let us now assume that a loading process takes place at $t > t_a$ accompanied for $t_1 > t_a$ by an increase of plastic strain $\dot{\varepsilon}_{ij}^{(p)} \, dt$; unloading then starts so that the stress state returns to the initial state at $t = t_k$. The application of the inequality (2.25) to the load cycle at the instant $t$, $t_a \leq t \leq t_k$, gives

$$[\sigma_{ij} - \sigma_{ij}^{(a)}] \dot{\varepsilon}_{ij}^{(p)} + (\mathbf{Y})_{ij}^{(a)} \dot{\varepsilon}_{ij}^{(p)} / \Delta t \geq 0.$$  

For any rapid load cycle the flow surface is on the basis of (2.26) identical with the instantaneous flow surface.
If it is assumed that the quantity \( \{Y\}_{i}^{t} \) is a continuous function at the point \( t = t_{a} \), then, with the notation

\[
(2.31) \quad \{Y\}_{i}^{t_{a}} = \int_{t_{a}}^{t} \psi(\dot{\varepsilon}_{ij}) \, dt,
\]

it can easily be shown that in the limit \( t_{k} \to t_{a}, \Delta t \to 0, \Delta \beta \to 0 \)

\[
(2.32) \quad \lim_{t_{k} \to t_{a}} \frac{\{Y\}_{i}^{t_{k}}}{\Delta t} = \psi(t_{a}) = 0.
\]

Thus the inequality \((2.29)\) for a rapid load path can be assumed in the same form as in the inviscid theory of plastic flow. In particular, if \( t_{a} \) is assumed to be equal to zero and \( \sigma_{ij}^{(a)} \) is treated as an arbitrary point in the viscoelastic region, identical with \( \sigma_{ij}^{*} \), we have

\[
(2.33) \quad (\sigma_{ij} - \sigma_{ij}^{*}) \dot{\varepsilon}_{ij}^{p} \geq 0.
\]

For rapid load paths the direction of \( \dot{\varepsilon}_{ij}^{p} \) is fixed and independent of the time. By considering every possible rapid load path from an arbitrary point \( \sigma_{ij}^{*} \) in the viscoelastic region to the point \( \sigma_{ij} \) at the instantaneous flow surface \( f_{a} = 0 \) we can prove, on the basis of the inequalities \((2.33)\), the convexity of the instantaneous flow surface.

The direction of \( \dot{\varepsilon}_{ij}^{p} \) is, for every rapid load path, orthogonal to the instantaneous flow surface.

The above proof of convexity of the instantaneous flow surface is valid for every point and every path in the stress space, hence the flow surface \( f = 0 \) must also be convex. This leads to the conclusion that the convexity of the flow surface depends on neither the load path nor the time. But there is no such conclusion for the orthogonality of the vector \( \dot{\varepsilon}_{ij}^{p} \) which is orthogonal to the flow surface for rapid load paths only. For a real loading process the direction of the vector \( \dot{\varepsilon}_{ij}^{p} \) remains an open question.

5. The Direction of the Vector of Plastic Strain Rate

Let us again consider the load path in the stress space and assume that the instantaneous stress point moves along the same path with various velocities. The duration of each loading process will be different, and each will involve different rheologic effects. The plastic state will therefore be
realized at various points of the path and determined by different flow surfaces. As a result of the various rheologic effects for the same load path we obtain a family of flow surfaces depending on the parameter $\beta$.

In order to simplify the considerations assume that all flow surfaces of the family under consideration pass through the same point, for instance $A$ (Fig. 16). Then it is easily observed that the actual direction of the vector of plastic strain rate may deviate from the direction normal to the instantaneous flow surface and is contained within a cone whose apex angle depends on $\{\Psi\}^0$.

Making use of (2.25) in the case of Fig. 16 and assuming that we have $\sigma_{ij} = \sigma_{ij}^*$ for $t = 0$, we infer that the maximum apex angle of the cone is a function of $\{\Psi\}^0$.

The influence of rheologic effects on the variability of the flow surface leads to a certain indeterminacy, the position of the instantaneous flow surface in the stress space being unknown as well as the point, at which the plastic state is attained. The direction of the hyperplane tangential to the instantaneous flow surface at the point considered is not uniquely determined. These problems are fully discussed in a paper by P. N. Naghdi and S. A. Murch [118].

Certain conclusions have also been drawn by these authors in the case of inviscid plasticity which admits, however, the existence of irregular flow surfaces. In such a theory the vector of the plastic strain rate $\dot{e}_{ij}^p$ lies in the fan formed by the normals to the smooth surfaces which constitute the "plastic corner." If an experimenter observes that the vector of plastic strain rate has a direction different from the direction normal to the expected flow surface or that this direction varies within certain limits, he will conclude, on the basis of inviscid plasticity, that a "plastic corner" exists since by viscoplasticity the flow surface should be regular.

6. Constitutive Equations for an Elastic-Viscoplastic Material

Assuming that the vector of plastic strain rate $\dot{e}_{ij}^p$ is directed along the outer normal to the instantaneous flow surface $f = 0$, P. M. Naghdi and S. A. Murch [118] proposed the relation

$$\dot{e}_{ij}^p = \Lambda \frac{\partial f}{\partial \sigma_{ij}}.$$  

Determining $\Lambda$ from the condition $\dot{f} = 0$, we obtain

$$\Lambda = -\left( \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{kl}} + \frac{\partial f}{\partial \beta} \right) \left[ \frac{\partial f}{\partial e_{mm}} \frac{\partial f}{\partial e_{nn}} + \frac{\partial f}{\partial k} \kappa \left( \frac{\partial f}{\partial \sigma_{kl}} \right) \right]^{-1},$$  

(2.35)
where
\[ \dot{\varepsilon}_{ij}^p = A \tilde{\kappa} \left( \frac{\partial f}{\partial \sigma_{pq}} \right). \]

On introducing the notation
\[ h = - \left[ \frac{\partial f}{\partial \varepsilon_m \partial \sigma_{mn}} + \frac{\partial f}{\partial \kappa} \frac{\partial f}{\partial \sigma_{pq}} \right]^{-1} \]
and bearing in mind the load criteria we obtain
\[ \dot{\varepsilon}_{ij}^p = \begin{cases} 0 & \text{if } f < 0, \\ h \langle \mathcal{L}^{(\varepsilon_{kl})} \rangle \frac{\partial f}{\partial \sigma_{ij}} & \text{if } f = 0, \end{cases} \]
where \( \langle x \rangle \) is defined thus:
\[ \langle x \rangle = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x > 0. \end{cases} \]

In the equations (2.38) just derived the difficulty mentioned earlier, which results from the indeterminacy of direction of \( \varepsilon_{ij}^p \) in relation to the actual flow surface is not accounted for. To correct this we introduce the following artifice. The loading surface \( f \) is first replaced by a characteristic yield condition \( f_0 \), for which hardening is assumed to take place isotropically. A second loading function \( g \) is then introduced, which differs from \( f_0 \) in such a way that bounds on the direction of \( \varepsilon_{ij}^p \) are implied by the constitutive equations. The constitutive equation (2.38) is then replaced by
\[ \dot{\varepsilon}_{ij}^p = \left[ L \frac{\partial f_0}{\partial \sigma_{ij}} + M \frac{\partial g}{\partial \sigma_{ij}} \right] \langle \mathcal{L}^{(\varepsilon_{kl})} \rangle, \]
where \( L \) and \( M \) are scalars dependent on \( \{\mathcal{Y}_j\}_0 \).

We shall generalize the constitutive equations (2.38) in a different way by introducing two loading functions \( h \) and \( f \) and the following relations:
\[ \dot{\varepsilon}_{ij}^p = N \frac{\partial h}{\partial \sigma_{ij}} \left( \frac{\partial f}{\partial \varepsilon_{kl}} \sigma_{kl} + \frac{\partial f}{\partial \beta} \beta \right). \]
The function \( h \) plays now the role of the viscoplastic potential, and the function \( f \) represents the yield criterion for the elastic-viscoplastic material. In order to determine the coefficient \( N \) and the connection between functions \( h \) and \( f \) certain conditions must be imposed on the constitutive equations (2.41).

A possibility seems to exist to develop the theory of viscoplasticity in a broader sense than that for a stable inelastic material. To do that, an
idea given first by B. D. Coleman and W. Noll [41] may be used. It leads to constitutive inequalities or thermostatic and thermodynamic inequalities which ensure a physically natural response. This idea was extended by C. Truesdell [184], B. D. Coleman [43], R. A. Toupin and B. Bernstein [182], C. Truesdell and R. A. Toupin [185], L. E. Bragg and B. D. Coleman [21] and B. D. Coleman and W. Noll [44] (see also W. Noll and C. Truesdell [122]).

Methods describing the behaviour of elastic-viscoplastic bodies are not a means by which we could solve practically important problems. This is a very real general shortcoming and its correction will require considerable work. Many questions remain at present unanswered; some of them have only been touched. The merit of the general studies lies in their considerable cognitive value. Essential difficulties are indicated and a procedure that is most likely generally correct and consistent is developed. These studies unquestionably succeeded in the establishment of general foundations for the solution of problems of viscoplasticity in a broad sense.

Among the basic problems to be solved we mention, above all, detailed discussion and study of the yield criterion for viscoelastic bodies, the establishment of the function representing the yield condition, and the determination of how it depends on rheologic effects. The latter problem is connected with the variability of the flow surface, depending on the duration of the loading process. Its solution will remove the indeterminacy mentioned above. Other problems are: the full explanation and discussion of the orthogonality of the vector of plastic strain rate, and the study of the character of the constitutive equations in order to approximate better real materials.

One of the fundamental problems is of course whether the fundamental assumption expressed by (1.1) is at all valid.

Let us now mention the problems of generalization of considerations and results already obtained.

Generalization to—or a new study assuming—finite strain and also extension to the case of non-isothermal processes is desirable. No full fundamental study of the thermodynamics of elastic-viscoplastic deformations has as yet been done. This would enable us to base all phenomenological considerations on more accurate physical foundations. The question of uniqueness of the solution of the fundamental boundary-value problem of viscoplasticity remains also open. No variational theorems that would permit systematic application of approximate methods have as yet been demonstrated.

7. Constitutive Equations for Rate-Sensitive Plastic Materials

Studies in this domain were initiated earlier than those described in the foregoing section. The general foundations for the study of viscoplastic problems were laid by K. Hohenemser and W. Prager [80] already in 1932 (see also W. Prager [142, 146]).
This work was not fully appreciated for a long time. It was only in the years 1948–1950 that it attracted attention subsequent to the observations of V. V. Sokolovsky [162], and later of L. E. Malvern [110], who showed that the assumption of K. Hohenemser and W. Prager may be used as a basis for description of certain dynamic properties of rate-sensitive materials. This concerned, however, one-dimensional problems only (see Section I).

Further development of the idea of K. Hohenemser and W. Prager is contained in the references [132–134].

Owing to the assumption that the viscous properties of the materials become manifest only after the passage to the plastic state and that these properties are not essential in elastic regions, the basic concepts of the description of viscoplastic properties differ from the methods explained before. It will be assumed that the strain rate can be resolved into an elastic and inelastic part

\[(2.42) \quad \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p.\]

The inelastic part of the strain rate, which is denoted by \(\dot{\varepsilon}_{ij}^p\), represents combined viscous and plastic effects.

Since the material has no viscous properties in the elastic region, the choice of an adequate yield criterion will be much simpler than in the case of an elastic-viscoplastic material. The initial yield condition, which will be called the static yield criterion, will not differ from the known condition of the inviscid theory of plasticity.

In order to keep our considerations sufficiently general we introduce a static yield function in the form

\[(2.43) \quad F(\sigma_{ij}, \varepsilon_{kl}^p) = \frac{f(\sigma_{ij}, \varepsilon_{kl}^p)}{\kappa} - 1,\]

where the function \(f(\sigma_{ij}, \varepsilon_{kl}^p)\) depends on the state of stress \(\sigma_{ij}\) and the state of plastic strain \(\varepsilon_{kl}^p\). The parameter \(\kappa\) is defined by the expression

\[(2.44) \quad \kappa = \kappa(W_p) = \kappa \left( \int_0^{\varepsilon_{kl}^p} \sigma_{ij} \, d\varepsilon_{ij}^p \right).\]

This quantity is called the strain hardening parameter (detailed discussion of the two fundamental definitions of that parameter may be found in R. Hill's book [79], or in a paper by P. M. Naghdi [117]).

The flow surface \(F = 0\), which is again in the nine-dimensional stress space, is assumed regular and convex, and we propose the constitutive equations for work-hardening and rate-sensitive plastic materials in the following form [134]:

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This page contains a continuation of the discussion on viscoelasticity, focusing on the work of V. V. Sokolovsky and L. E. Malvern, who contributed to the understanding of rate-sensitive materials. The text elaborates on the resolution of strain rate into elastic and inelastic parts and introduces the concept of a static yield criterion, which is crucial for describing the behavior of viscoplastic materials. The equations and definitions provided are foundational for further studies in this field.
(2.45) \[ \dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{\sigma}_{ij} + \frac{1 - 2\nu}{E} \dot{\varepsilon}_{ij} + \gamma_0 \langle \Phi(F) \rangle \frac{\partial F}{\partial \sigma_{ij}}, \]

where the symbol \( \langle \Phi(F) \rangle \) is defined as follows:

(2.46) \( \langle \Phi(F) \rangle = \begin{cases} 0 & \text{for } F \leq 0, \\ \Phi(F) & \text{for } F > 0. \end{cases} \)

The function \( \Phi(F) \) may be chosen to represent the results of tests on the behaviour of metals under dynamic loading.

It will be much more convenient to write the constitutive equations (2.45) in the slightly different form

(2.47) \[ \dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{\sigma}_{ij} + \frac{1 - 2\nu}{E} \dot{\varepsilon}_{ij} + \gamma \langle \Phi(F) \rangle \frac{\partial f}{\partial \sigma_{ij}}, \]

where \( \gamma = \gamma_0/\kappa \) denotes a viscosity constant of the material.

The relations (2.47) involve the assumption that the rate of increase of the inelastic components of the strain tensor is a function of the excess stresses above the static yield criterion.

This function of stresses above the static yield criterion generates the inelastic strain rate according to a viscosity law of the Maxwell type. The elastic components of the strain tensor are considered to be independent of the strain rate.

The constitutive equations (2.47) describe the work-hardening effect of the material. Owing to the introduction of the function \( F \), it is possible to study the anisotropic and the isotropic work-hardening for a broad class of compressible materials.

To discuss the constitutive equations more accurately let us consider the inelastic part of the relations (2.47)

(2.48) \[ \dot{\varepsilon}_{ij}^e = \gamma \Phi(F) \frac{\partial f}{\partial \sigma_{ij}}. \]

Squaring both sides of (2.48), and denoting by \( I_8^p = (1/2) \dot{\varepsilon}_{ij}^e \dot{\varepsilon}_{ij}^e \) the invariant of the inelastic strain-rate tensor, we obtain

(2.49) \[ (I_8^p)^{1/2} = \gamma \Phi(F) \left( \frac{1}{2} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \right)^{1/2}. \]

According to (2.49) we have

(2.50) \[ f(\sigma_{ij}, \varepsilon_{kl}) = \kappa(W_p) \left( 1 + \Phi^{-1} \left[ (I_8^p)^{1/2} \cdot \left( \frac{1}{2} \frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{pq}} \right)^{-1/2} \right] \right). \]
This expression implicitly represents the dynamical yield condition for elastic/viscoplastic, work-hardening materials, and describes the dependence of the yield criterion on the strain rate.

It can be seen from (2.48) and (2.50) that the inelastic strain-rate tensor considered as a vector in the nine-dimensional stress space is always directed along the normal to the subsequent dynamic loading surface (Fig. 17).

![Fig. 17. Dynamic loading surface and strain-rate vector.](image)

The change of the yield surface during the deformation process is caused by isotropic and anisotropic work-hardening effects, and by the influence of the strain-rate effect.

8. Special Cases of Constitutive Equations

We shall first study the elastic/viscoplastic material with isotropic work-hardening. Let us assume the function $F$ in the form:

$$F = \frac{f(\sigma_i)}{\kappa} - 1,$$

where the function $f(\sigma_i)$ depends now on the state of stress only. Thus

$$f(\sigma_i) = f(J_1', J_2, J_3),$$

where $J_1'$ denotes the first invariant of the stress tensor $\sigma_i$; $J_2$ and $J_3$ are the second and third invariants of the stress deviator $s_{ij}$, respectively. We can write

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial J_1'} \delta_{ij} + \frac{\partial f}{\partial J_2} s_{ij} + \frac{\partial f}{\partial J_3} t_{ij}.$$
where

\( t_{ij} = s_{ik} s_{kj} - \frac{1}{3} J \delta_{ij} \). \( \tag{2.54} \)

In this case the constitutive equations (2.48) take form

\( \varepsilon_{ij} = A(J_1^1, J_2, J_3, \kappa) \delta_{ij} + B(J_1^1, J_2, J_3, \kappa)s_{ij} + C(J_1^1, J_2, J_3, \kappa)t_{ij}, \) \( \tag{2.55} \)

where the following notations have been introduced

\[ A(J_1^1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{f}{\kappa} - 1 \right] \frac{\partial f}{\partial J_1'}, \]

\[ B(J_1^1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{f}{\kappa} - 1 \right] \frac{\partial f}{\partial J_3}, \]

\[ C(J_1^1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{f}{\kappa} - 1 \right] \frac{\partial f}{\partial J_3} \]. \( \tag{2.56} \)

A. Application to Metals. Assuming plastic incompressibility, i.e.,

\[ A(J_1^1, J_2, J_3, \kappa) = 0 \quad \text{or} \quad \frac{\partial f}{\partial J_1'} = 0, \]

the general constitutive equations for strain-rate sensitive metals may be written as follows:

\( \varepsilon_{ij}^p = B^*(J_2, J_3, \kappa)s_{ij} + C^*(J_2, J_3, \kappa)t_{ij}. \) \( \tag{2.57} \)

If we then assume the Huber-Mises yield condition, i.e., that the function

\[ f(\sigma_{ij}) = (J_2)^{1/2}, \]

we obtain by (2.47)*:

\[ \sigma_{ij} = \frac{1}{2} \mu \delta_{ij} + \gamma \left\langle \Phi \left( \frac{\sqrt{J_2}}{\kappa} - 1 \right) \right\rangle \frac{s_{ij}}{\sqrt{J_2}}, \quad \dot{\varepsilon}_{ij} = \frac{1}{3 \kappa} \delta_{ij}. \] \( \tag{2.58} \)

According to (2.50), the dynamical yield criterion has the form:

\[ \sqrt{J_2} = \kappa (W_p) \left[ 1 + \Phi^{-1} \left( \frac{\sqrt{J_2}}{\gamma} \right) \right]. \] \( \tag{2.59} \)

For one-dimensional states the relations (2.58) furnish the strain-rate law

\[ \dot{\varepsilon} = \dot{\sigma} / E + \gamma \left\langle \Phi \left[ \frac{\sigma}{\phi(\dot{\varepsilon})} - 1 \right] \right\rangle, \] \( \tag{2.60} \)

* A similar case of constitutive equations has been recently studied by S. Kaliski [89]. The difference between Eqs. (2.58) and those of S. Kaliski lies in the definition of the work-hardening parameter \( \kappa \).
where

\[(2.61)\quad \gamma^* = (2/\sqrt{3})\gamma, \quad \phi(\varepsilon^p) = \sqrt{3}\kappa(W_p).\]

Relation (2.60) was first introduced by Malvern [110].

The expression (2.59) now gives

\[(2.62)\quad \sigma = \phi(\varepsilon^p) \left[ 1 + \Phi^{-1} \left( \frac{\dot{\varepsilon}}{\gamma^*} \right) \right],\]

where \(\sigma = \phi(\varepsilon^p)\) represents the static stress-strain relation for simple tension.

---

**Fig. 18.** Dynamic stress-strain curves for work-hardening and strain-rate sensitive material, \(\dot{\varepsilon} = \text{const}.\)

**Fig. 19.** Dynamic stress-strain curve for work-hardening and strain-rate sensitive material, \(\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon).\)

Equation (2.62) describes the dynamic stress-strain relation in the plastic range for simple tension. The result of this relation is plotted for \(\dot{\varepsilon} = \text{const}\) in Fig. 18, and for the more realistic case \(\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon)\) in Fig. 19.

As a second case, we shall examine the elastic/visco-(perfectly plastic) material. We obtain this case by introducing the assumption that the function \(F\) does not depend on the strain, i.e.

\[(2.63)\quad F = \frac{f(U_s, J_b)}{c} - 1, \quad c = \text{const}.\]
The constitutive equations (2.47) can now be written as:

\[
\dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{s}_{ij} + \gamma \left( \Phi \left[ \frac{f(J_2, J_3)}{c} - 1 \right] \right) \frac{\partial f}{\partial \sigma_{ij}}, \quad \dot{\varepsilon}_i = \frac{1}{3K} \sigma_{ii},
\]

where \( \gamma = \gamma^0/c \), as first introduced in [132].

According to (2.50), we can write:

\[
f(J_2, J_3) = c \left( 1 + \Phi^{-1} \left[ \frac{\sqrt{J_3}}{\gamma} \left( \frac{1}{2} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \right)^{-1/2} \right] \right).
\]

By assuming

\[
F = \frac{\sqrt{J_3}}{k} - 1,
\]

we have the constitutive equations

\[
\dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{s}_{ij} + \gamma \left( \Phi \left[ \frac{\sqrt{J_3}}{k} - 1 \right] \right) \frac{\dot{s}_{ij}}{\sqrt{J_3}}, \quad \dot{\varepsilon}_i = \frac{1}{3K} \sigma_{ii},
\]

and the dynamical yield criterion

\[
\sqrt{J_3} = k \left[ 1 + \Phi^{-1} \left( \frac{\sqrt{J_3}}{\gamma} \right) \right].
\]

The dependence of \( \sqrt{J_3} \) on \( \sqrt{J_3}^p \) as given by (2.68) is plotted for some function \( \Phi \) in Fig. 20.

It can be seen from (2.68) that the constitutive equations (2.67) lead to a similar result as the inviscid theory of plasticity for isotropic work-hardening material. But in the inviscid theory of plasticity for isotropic
work-hardening, the radius $R$ of the cylindrical yield-locus in stress space depends on the plastic strain, whereas in the case considered here $R$ depends on the inelastic strain rate, according to

$$R = R_0 \left[ 1 + \Phi^{-1} \left( \frac{\sqrt{I_2'}}{\gamma} \right) \right],$$

where $R_0 = k\sqrt{2}$ denotes the radius of the static yield cylinder (Fig. 21).

In the inviscid theory of plasticity for isotropic work-hardening material we have three possibilities according to whether $\dot{J}_2 > 0$ (loading), $\dot{J}_2 = 0$ (neutral loading), or $\dot{J}_2 < 0$ (unloading).

![Fig. 21. The cylindrical yield locus for elastic/visco-(perfectly plastic) material.](image)

Since in viscoplasticity $J_2$ is a function of the strain rate, the plastic flow (relaxation effect) occurs if $J_2 > k^2$, independent of $\dot{J}_2 \leq 0$. Owing to the relations (2.67) the material will be elastic on the path $OP_0$, but on the path $P_0P_1P_1'P_2$ it will be elastic/viscoplastic (Fig. 21).

For one-dimensional states, (2.67) and (2.68) lead to the relations

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \gamma^* \langle \Phi(\sigma/\sigma_0 - 1) \rangle,$$

$$\sigma = \sigma_0 \left[ 1 + \Phi^{-1} \left( \frac{\dot{\varepsilon}}{\gamma^*} \right) \right],$$

respectively. Law (2.70) describes the dynamic stress-strain relation in simple tension for an elastic/visco-(perfectly plastic) material. The result of (2.71) is plotted for $\dot{\varepsilon} = \text{const}$ in Fig. 22 and for $\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon)$ in Fig. 23.
When the elastic deformations are negligible in comparison with the plastic deformations, then for a linear function \(\Phi(F)\) (2.67) furnishes the constitutive equations first given by Hohenemser and Prager [80].

Prager [146] showed that the constitutive equations of the viscoplastic and perfectly plastic materials have the same relations to each other as the constitutive equations of the viscous and perfect fluids.

It may be interesting to see that the constitutive equations of the flow theory result as a limiting case from the general constitutive equations of the elastic/viscoplastic material. With this object let us assume that \(\gamma \to \infty\). According to (2.50) this means that

\[
(2.72) \quad f(\sigma_{ij},\varepsilon_{kl}) = \kappa \quad \text{or} \quad F = 0.
\]
From (2.72) and by definition (2.46) of the function \( \Phi(F) \), \( \Phi(F) \to 0 \) in the limiting case \( \gamma \to \infty \); thus \( \gamma \Phi(F) = \Lambda \) becomes indeterminate. In this case, we have

\[
\varepsilon^\Phi_{ij} = \Lambda \frac{\partial F}{\partial \sigma_{ij}}. \tag{2.73}
\]

It is possible to determine the parameter \( \Lambda \) by satisfying the static yield criterion

\[
F(\sigma_{ij}, \varepsilon^\Phi_{kl}) = 0. \tag{2.74}
\]

Since the subsequent static loading surface (2.74) passes through the loading point, and since during loading

\[
\dot{F} = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial F}{\partial \varepsilon^\Phi_{kl}} \dot{\varepsilon}^\Phi_{kl} = 0, \tag{2.75}
\]

we have (see, e.g. [117]) by (2.73)

\[
\Lambda = -\begin{vmatrix} \frac{\partial F}{\partial \sigma_{ij}} & \frac{\partial F}{\partial \varepsilon^\Phi_{kl}} \end{vmatrix}. \tag{2.76}
\]

In the special case of elastic/visco-(perfectly plastic) material we obtain by putting the result (2.68) into the relations (2.67)

\[
\varepsilon^\phi_{ij} = \sqrt{I^g_{ij}} \left\{ k \left[ 1 + \Phi^{-1} \left( \frac{\sqrt{I^g_{ijkl}}}{} \right) \right] \right\}^{-1} s_{ij}. \tag{2.77}
\]

By setting \( \gamma = \infty \) in (2.77), we obtain the constitutive equations

\[
\varepsilon^\phi_{ij} = \sqrt{I^g_{ij}} k s_{ij} \tag{2.78}
\]

of the incompressible perfectly plastic material [146]. These constitutive equations hold only when the rate of deformation does not vanish.

B. Application to Soils. As a particular case of the constitutive equations (2.55), we shall study the elastic/visco-(perfectly plastic) soil for the following static yield function*:

\[
F = \frac{\alpha J_1 + J_2^{1/2}}{k} - 1, \tag{2.79}
\]

where \( \alpha \) is a constant describing the dilatation rate of the soil.

* For the discussion of the other cases see papers [127, 128].
Here the constitutive equations (2.47) give

(2.80)
\[ \dot{e}_{ij} = \frac{1}{2\mu} \dot{s}_{ij} + \frac{1 - 2\nu}{E} \dot{s} \delta_{ij} + \gamma \left( \Phi \left( \frac{\alpha J_1' + J_2^{1/2}}{k} - 1 \right) \right) \left( \alpha \delta_{ij} + \frac{s_{ij}}{2J_2^{1/2}} \right). \]

The rate of cubical dilatation takes the following form

(2.81)
\[ \dot{e}_{ii} = \frac{1 - 2\nu}{E} + 3\alpha \gamma \left( \Phi \left( \frac{\alpha J_1' + J_2^{1/2}}{k} - 1 \right) \right). \]

Equation (2.81) shows that the inelastic deformation is accompanied by an increase of volume, if \( \alpha \neq 0 \). This property is known as dilatancy.

The dynamic yield condition following from (2.80) has the form

(2.82)
\[ \alpha J_1' + J_2^{1/2} = k \left( 1 + \Phi^{-1} \left[ \frac{(I_2^p)^{1/2}}{\gamma} \left( \frac{3}{2} \alpha^2 + \frac{1}{4} \right)^{-1/2} \right] \right). \]

In the limiting case, \( \gamma \to \infty \), we obtain from (2.80) the known constitutive equations for an elastic-perfectly plastic soil (see D. C. Drucker and W. Prager [201]):

(2.83)
\[ \dot{\epsilon}_{ij}^p = \lambda \left[ \alpha \delta_{ij} + \frac{s_{ij}}{2J_2^{1/2}} \right], \]

where

(2.84)
\[ \lambda = \left[ I_2^p \left( \frac{3}{2} \alpha^2 + \frac{1}{4} \right) \right]^{1/2}. \]

The plastic rate of cubical dilatation is then expressed by the relation

(2.86)
\[ \dot{e}_{ii}^p = 3\alpha \lambda. \]

9. The Dynamical Condition for a Stable Elastic/Viscoplastic Material

A definition of stable inelastic materials with inclusion of dynamic terms was introduced by Drucker, [82]; it leads to the following condition:

(2.86)
\[ \int_0^t \left\{ \int_V \left[ \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)} \right] \dot{\epsilon}_{ij}^{(1)} dV \right\} dt + \left\{ \int_V \int_0^t \frac{1}{2} \rho \left[ \dot{\nu}_i^{(2)} - \dot{\nu}_i^{(1)} \right]^2 dV \right\}_t = 0. \]
where the stress tensors $\sigma^{(1)}_{ij}$ and $\sigma^{(2)}_{ij}$ and the corresponding strain-rate tensors $\dot{\varepsilon}^{(1)}_{ij}$ and $\dot{\varepsilon}^{(2)}_{ij}$ refer to two distinct mechanical paths. The first integral is extended over a finite interval of time $0 \leq t \leq t_k$, the second term is the value of the volume integral at $t_k$ minus the value at the instant of the divergence of paths, $t = 0$. Necessarily, in all cases the velocity $u_i^{(2)}$ throughout the volume is the same as $u_i^{(1)}$ at $t = 0$, hence the second term of (2.86) is positive or zero at all times $t_k$. Therefore, the inequality

\[
\int_{t=0}^{t_k} \left\{ \left[ \sigma^{(2)}_{ij} - \sigma^{(1)}_{ij} \right] \left[ \dot{\varepsilon}^{(2)}_{ij} - \dot{\varepsilon}^{(1)}_{ij} \right] dV \right\} dt \geq 0
\]

is much more significant (compare with (2.19)). It is obvious that the inclusion of dynamic terms does not affect the conclusions concerning the stress-strain relations to be drawn as a result of neglecting these terms.

From the inequality (2.87), we reobtain the condition of orthogonality of the inelastic strain-rate vector to the yield surface and that of convexity of the subsequent dynamic loading surface. Both were assumed as basis for introducing the general constitutive equations (2.47).

10. Comparison with Experimental Data

Let us introduce the following special types of function $\Phi(F)$ (see [132, 133]):

\[
\Phi(F) = F^a, \quad \Phi(F) = F, \quad \Phi(F) = \exp F - 1,
\]

\[
\Phi(F) = \sum_{a=1}^{N} A_a [\exp F^a - 1], \quad \Phi(F) = \sum_{a=1}^{N} B_a F^a.
\]

When the elastic deformations are negligible in comparison with the plastic deformations, then we obtain by (2.70) and (2.88) for a one-dimensional state of stress the equation

\[
\dot{\varepsilon} = \gamma^* \left( \frac{\sigma}{\sigma_0} - 1 \right)^\delta.
\]

The relation (2.89) is equivalent to the Cowper-Symonds-Bodner strain-rate law [19, 55]. In the case (2.88b) the constitutive equations (2.87) are equivalent to Freudenthal's relations [69].

For the one-dimensional case we have by (2.70) and (2.88a)

\[
\dot{\varepsilon} = \dot{\varepsilon}/E + \gamma^* \left[ \exp \left( \frac{\sigma}{\sigma_0} - 1 \right) - 1 \right],
\]

which, in slightly different form, was introduced by Malvern [110].
We shall examine the types of function $\Phi(F)$ (2.88) in the light of experimental data. At first we shall try to determine the constants $\gamma^*$, $\delta$, $A^*_\alpha$ and $B^*_\alpha$ ($\alpha = 1, 2, \ldots, N$) according to the results of D. S. Clark and P. E. Duwez [39], so as to describe best the dependence of yield stress upon strain rate, but we must remember that the strain rate in impact problems accompanied by the wave propagation effect may change from zero to about 1000 sec$^{-1}$.

![Figure 24](image)

**Fig. 24.** Comparison of the experimental data with the prediction of the power strain-rate law.

The experimental investigations of D. S. Clark and P. E. Duwez [39], gave the curve $\sigma$ versus $\dot{\varepsilon}$ in the range of strain rate from zero to 200 sec$^{-1}$ and it was pointed out that no further increase of yield stress was observed when the strain rate exceeded 200 sec$^{-1}$ (see Fig. 2).

According to the first three types of function $\Phi(F)$ (2.88$_1$)–(2.88$_3$) and by Eq. (2.71) we can write

\begin{equation}
\sigma = \sigma_0 \left[1 + \left(\frac{\dot{\varepsilon}}{\gamma^*}\right)^{1/\delta}\right], \\
\sigma = \sigma_0 \left(1 + \frac{\dot{\varepsilon}}{\gamma^*}\right), \\
\sigma = \sigma_0 \left[1 + \log\left(1 + \frac{\dot{\varepsilon}}{\gamma^*}\right)\right].
\end{equation}

In Fig. 24 the experimental data of Clark and Duwez for mild steel are compared with the predictions of the power law (2.91$_1$) for $\delta = 5$, $\gamma^* = 40.4$.
sec\(^{-1}\) (see S. R. Bodner and P. S. Symonds assumption \cite{19, 20}), \( \delta = 3, \gamma^* = 180 \text{ sec}^{-1}; \delta = 3, \gamma^* = 240 \text{ sec}^{-1} \).

**Fig. 25.** Comparison of the experimental data with the prediction of the linear strain-rate law.

**Fig. 26.** Comparison of the experimental data with the prediction of the exponential strain-rate law.
The analogous comparisons with the linear law (2.91a) and the exponential law (2.91b) are made in Fig. 25 for $\gamma^* = 180-300$ sec\(^{-1}\) and in Fig. 26 for $\gamma^* = 80-240$ sec\(^{-1}\).

To determine the constants $A_\alpha^*$ and $B_\alpha^*$ ($\alpha = 1, 2, \ldots, N$) we may take the data of D. S. Clark and P. E. Duwez [39], given $\dot{\varepsilon}$ as a function of $\sigma/\sigma_0 - 1$. Hence we can write two systems of linear equations for $A_\alpha^*$ and $B_\alpha^*$

\[
\sum_{\alpha=1}^{N} A_\alpha^* [\exp(X_i^\alpha) - 1] = \dot{\varepsilon}(X_i), \quad i = 1, 2, \ldots, N,
\]

\[
\sum_{\alpha=1}^{N} B_\alpha^* (X_i)^\alpha = \dot{\varepsilon}(X_i), \quad i = 1, 2, \ldots, N,
\]

where $X_i = \sigma_i/\sigma_0 - 1$ for some points of $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_N$. and $\dot{\varepsilon}(X_i)$ ($i = 1, 2, \ldots, N$) are known from experimental data.

The approximations of the experimental curve $\sigma$ versus $\dot{\varepsilon}$ by the functions (2.88a) and (2.88b) for $N = 5$ are plotted in Fig. 27. The values of constants $A_\alpha^*$ and $B_\alpha^*$ ($\alpha = 1, 2, \ldots, N$) are given in Table 2.

We assume here that the influence of the elastic part of the strain rate is very small, and that the curve $\sqrt{\dot{J}_g}$ versus $\sqrt{\dot{J}_g}$ is the same as the curve $\sigma$ versus $\dot{\varepsilon}$. In the light of the recent experimental results the first assumption seems acceptable.

Since all experimental investigations concerning the dynamic behaviour of the materials have been performed under one-dimensional conditions, we cannot discuss the validity of the second assumption and must consider...
it as a hypothesis. This hypothesis will permit the use of the same constants in the general constitutive equations (e.g. in (2.58) or in (2.67)) as those determined for one-dimensional states of stress.

\[ \begin{array}{cccccc}
\alpha & 1 & 2 & 3 & 4 & 5 \\
\hline
A_\alpha \,(\text{sec}^{-1}) & 217.56 & -654.11 & 874.52 & -484.15 & 93.56 \\
B_\alpha \,(\text{sec}^{-1}) & 337.53 & -1470.56 & 3271.71 & -3339.98 & 1280.06 \\
\end{array} \]

From experimental results of J. D. Campbell and J. Duby [27], in which a mild-steel specimen was subjected to compressive impact load, we have

\[ \delta = 3, \gamma = 180 \]
\[ \delta = 5, \gamma = 404 \]

**Fig. 28.** Comparison of the experimental data for strain rate with the predictions of the power law.

the average stress-time curve and the average (strain-rate)-time curve. Taking the stress-time curve, we propose to compute the strain rate as a function of time according to the constitutive equations already introduced:
\[ \dot{\varepsilon} = \frac{\dot{\varepsilon}}{E} + \gamma^* \left( \frac{\sigma}{\sigma_0} - 1 \right), \quad \dot{\varepsilon} = \frac{\dot{\varepsilon}}{E} + \gamma^* \left( \frac{\sigma}{\sigma_0} - 1 \right), \]

(2.93)

\[ \dot{\varepsilon} = \frac{\dot{\varepsilon}}{E} + \gamma^* \left[ \exp \left( \frac{\sigma}{\sigma_0} - 1 \right) - 1 \right], \]

\[ \dot{\varepsilon} = \frac{\dot{\varepsilon}}{E} + \sum_{\alpha=1}^{N} A_{\alpha}^* \left[ \exp \left( \frac{\sigma}{\sigma_0} - 1 \right) - 1 \right] \]

\[ \dot{\varepsilon} = \frac{\dot{\varepsilon}}{E} + \sum_{\alpha=1}^{N} B_{\alpha}^* \left( \frac{\sigma}{\sigma_0} - 1 \right)^{\alpha} \]

Fig. 29. Comparison of the experimental data for strain rate with the predictions of the exponential law.

The best agreement with experimental data can be observed for the power function \( \Phi(F) \), (2.93a) and for the exponential function \( \Phi(F) \), (2.93b). The results for these functions are plotted in Figs. 28 and 29, respectively (cf. [133]).

All theoretical curves show that in comparison with experimental data the maximum value for the strain rate is shifted in time by about 15 \( \mu \)sec, which may be due to the delay time. Indeed these shifts have the order of magnitude of the delay times observed in mild steel under stress of up to 100,000 lb/in\(^2\) (see J. E. Johnson, D. S. Wood, and D. S. Clark [87]).
In Fig. 30 the experimental data of J. Harding, E. O. Wood, and J. D. Campbell [75], for mild steel are compared with the theoretical predictions concerning elastic/visco-perfectly plastic material (curve $A$, $\Phi(F) = \exp F - 1$, $\gamma = 240$ sec$^{-1}$) and concerning elastic/viscoplastic, work-hardening material (curve $B$, $\Phi(F) = \exp F - 1$, $\gamma = 350$ sec$^{-1}$). The analogous comparisons for pure iron are given in Fig. 31 (curve $A$, $\Phi(F) = \exp F - 1$, $\gamma = 300$ sec$^{-1}$; curve $B$, $\Phi(F) = \exp F - 1$, $\gamma = 500$ sec$^{-1}$).
In Fig. 32 the experimental data of F. E. Hauser, J. A. Simmons, and J. E. Dorn [77], for pure aluminium are compared with the predictions of elastic/viscoplastic work-hardening theory.

![Figure 32. Comparison between the experimental and theoretical stress-strain curves at \( \dot{\varepsilon} = \text{const} \) for pure aluminium.](image)

11. Temperature-Dependent and Strain-Rate Sensitive Plastic Materials

The influence of the strain rate on the mechanical properties of metals depends markedly on the value of absolute temperature. It has been demonstrated experimentally [149], that at \(-180^\circ\text{C}\) the low yield point of pure iron is equal for all strain rates. A similar result was obtained by J. M. Krafft, A. M. Sullivan, and C. F. Tipper, [90]. On the other hand, even a slight change of the strain rate at high temperatures causes a considerable rise or drop in the yield point.

Recent years have brought a number of improvements in measuring the dynamic properties of metals at various temperatures. Papers [34, 109, 149] may be quoted here. Unfortunately, these investigations do not cover the entire range of variation of strain rate, strain and temperature. Therefore the effect of strain-hardening of materials cannot be studied and all results derived are only certain in the range of strain rate \( \dot{\varepsilon} \) and temperature \( \theta \) already examined.

The constitutive equations describing the temperature-dependent and strain-rate sensitive plastic materials have been discussed in paper [126]. A detailed analysis of some particular cases of the constitutive equations and a comparison of theoretical with experimental results can be found in
[138] together with a complete discussion of the problem of an appropriate selection of the temperature-dependent coefficients. But it should be noted that the considerations given in [138] are purely phenomenological, without regard to thermodynamics of irreversible processes. Irreversible thermodynamics for viscoplastic materials has been presented by H. Ziegler [197] and by C. Wehrli and H. Ziegler [188]. A review of thermodynamic considerations in continuum mechanics has been given by H. Ziegler [198, 199], in which also information concerning the original papers on thermodynamics of irreversible processes can be found.

It appears that the simultaneous influence of strain rate and temperature can be described by (2.47) if only the quantities appearing there are taken in their dependence on temperature. In general, temperature-dependent quantities can be $\gamma$, $F$, as well as the function $\Phi$ itself. With $\bar{a}$ denoting the coefficient of thermal expansion, the constitutive equations for temperature-dependent and strain-rate sensitive materials take the form

$$
\dot{\epsilon}_{ij} = \frac{1}{2\mu} \dot{\epsilon}_{ij} + \gamma(\theta) \left< \Phi \left[ \frac{f(\sigma_{kl}, \dot{\epsilon}_{kl})}{\kappa(\theta)} - 1 \right] \right> \frac{\partial f}{\partial \sigma_{ij}},
$$

$$
\dot{\epsilon}_u = \frac{1}{3K} \dot{\sigma}_u + \bar{a} \dot{\theta}.
$$

Consider now the inelastic part of the strain-rate tensor. For perfectly plastic materials with Huber-Mises yield condition (2.94) gives

$$
\dot{\epsilon}_{ij}^{pl} = \gamma(\theta) \Phi \left[ \frac{\sqrt{J_2}}{k(\theta)} - 1 \right] \frac{s_{ij}}{\sqrt{J_2}} \quad \text{for} \quad \sqrt{J_2} > k(\theta),
$$

or equivalently

$$
\sqrt{J_2} = k(\theta) \left\{ 1 + \Phi^{-1} \left[ \frac{\sqrt{J_2}}{\gamma(\theta)} \right] \right\}.
$$

For a one-dimensional state of stress (2.95) and (2.96) transform to (compare with (2.70)–(2.71))

$$
\dot{\epsilon} = \frac{2}{\sqrt{3} \gamma(\theta)} \Phi \left[ \frac{\sigma}{\sigma_0(\theta)} - 1 \right],
$$

$$
\sigma(\dot{\epsilon}, \theta) = \sigma_0(\theta) \left[ 1 + \Phi^{-1} \left[ \frac{\sqrt{3} \dot{\epsilon}}{2\gamma(\theta)} \right] \right].
$$

Since all experiments of interest were carried out under tension or compression, (2.97) and (2.98) will be useful in order to give experimental evidence of the present approach.
The most general case in which the quantities \( y \) and \( \sigma_{0} \) and the function \( \Phi \) are temperature-variable has not been studied in the literature. The main reason for this is that the next two cases considered, where only two quantities vary with temperature, were found to give satisfactory correspondence with experimental data over the entire range of temperature. Let us consider in detail the following three particular cases:

A. \( \sigma_{0} = \sigma_{0}(\theta) \) and \( y = y(\theta) \) and \( \Phi \) is independent of temperature. It was shown that the viscosity of the plastic material proposed by Hohenemser and Prager can be used to describe the strain-rate sensitivity of metals. Since the viscosity constant \( \gamma \), similarly as the yield point \( \sigma_{0} \), depends on temperature, the case A is a further generalization of the above-mentioned phenomenological approach. This assumption allows, as will be indicated later, for the description of the dynamic properties of a number of metals over the whole investigated range of \( \dot{\varepsilon} \) and \( \theta \) (Figs. 33–37). A particular case of this assumption is the hypothesis [187] that

\[
\sigma(\dot{\varepsilon},\theta) = \sigma_{1}(\dot{\varepsilon}) + \sigma_{0}(\theta),
\]

which can be derived from (2.98) if we assume that the product
Mild steel

\[ a_0(\theta) = 73000 - 7440(\theta - 125)^{1/3} \]
\[ n = 10 \]
\[ \dot{\varepsilon} = 300 \text{ sec}^{-1} \]

Pure iron

\[ a_0(\theta) = 45900 - 3100(\theta - 175)^{1/2} \]
\[ n = 10 \]
\[ \dot{\varepsilon} = 300 \text{ sec}^{-1} \]

Fig. 34. Comparison of the theoretical predictions with the experimental data for mild steel.

Fig. 35. Comparison of the theoretical predictions with the experimental data for pure iron.
FIG. 36. Comparison of the theoretical predictions with the experimental data for pure iron.

\[ a_0(\theta) = 14040 \exp\left\{ 0.982 \left( \frac{\theta}{86} - 1 \right) \right\} \]
\[ n = 15, \gamma = 2 \]

Pure iron

FIG. 37. Comparison of the theoretical predictions with the experimental data for aluminium.

\[ a_0 = -1.55 \left( \frac{\theta}{100} \right)^4 + 31.25 \left( \frac{\theta}{100} \right)^3 + 87 \left( \frac{\theta}{100} \right)^2 - 3744 \left( \frac{\theta}{100} \right) + 43435 \]
\[ \gamma = 0.375 \left( \frac{\theta}{100} - 4.85 \right)^2 + 0.18 \]
\[ n = 10 \]

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\[ \sigma_0(\theta) \Phi^{-1} \left[ \frac{1/3 \dot{\varepsilon}^p}{2\gamma(\theta)} \right] \]

is independent of temperature throughout the whole range of \( \dot{\varepsilon}^p \). This assumption can never be true for real materials, and the results of the tests quoted here do not conform with the foregoing hypothesis.

B. \( \sigma_0 = \sigma_0(\theta) \), \( \Phi \) depends on temperature, and \( \gamma \) is independent of temperature. This is the conception of L. D. Sokolov [159], J. F. Alder and V. A. Phillips [1] and many others. These authors studied the power function

\[(2.100) \quad \sigma(\dot{\varepsilon}, \theta) = \sigma_0(\theta)(\dot{\varepsilon})^{n(\theta)}, \]

where the exponent \( n \) is also a function of temperature. Such a form of a state equation has a certain thermodynamical justification and therefore attracted the attention of a number of investigators, but no consistent results have as yet been obtained. L. D. Sokolov [159] investigated four kinds of steel and found the exponent \( n \) as a linear function of the homologous temperature with the extrapolated line coming through the origin. Experimental data [1] on aluminium, copper and steel show that \( n(0) \) can be approximated by two straight lines. Recent experiments [34] indicated the non-linear relation between \( n \) and \( \theta \) for aluminium in the temperature range 0–550°C. No algebraic representation of the function \( n(\theta) \) has been proposed.

C. \( \sigma_0 = \sigma_0(\theta) \), \( \gamma \) and \( \Phi \) are independent of temperature. This is a generalization of W. Prager's [144] idea for materials exhibiting strain-rate sensitivity. Eq. (2.98) yields

\[(2.101) \quad \sigma(\dot{\varepsilon}, \theta) = \sigma_0(\theta) \left[ 1 + \Phi^{-1} \left( \frac{1/3 \dot{\varepsilon}^p}{2\gamma} \right) \right]. \]

This hypothesis suggested in [186] has been criticized in [187]. The analysis of experimental data presented later shows that neither (2.99) nor (2.101) holds over the whole range of \( \dot{\varepsilon} \) and \( \theta \). However, for narrow ranges of temperature (2.101) can give a fairly adequate description. From the point of view of practical applications (2.101) leads to the simplest results but, as already mentioned, its range of validity is very limited. Although cases A and B both give a good description of the mechanical properties of metals, it is more convenient to use relations where not the function but only the coefficients of the material depend on temperature. Thus, the case A seems here more appropriate.

In the following the available experimental data for aluminium, pure iron and mild steel are discussed in order to show the validity of A.
The results of tests on aluminium, iron, and steel taken from [1, 34, 96, 109, 112] and [149], have been elaborated for our purpose. Only the range of temperature is considered, in which the material is metallurgically "stable." In order to use the results of tests the form of the function $\Phi$ has to be assumed. The power function has been found to be preferable (see (2.88)), and the introduction of more involved function is unnecessary, since the power function $\Phi$ gives a very good agreement with all tests considered.

Circles and triangles in Figs. 33–37 are experimental points both from static and dynamic tests. All static tests of $\sigma_0$ were approximated by either power or exponential functions and the lower solid lines are plots of $\sigma_0(\theta)$. The upper solid lines represent $\sigma(\dot{\varepsilon}, \theta)$ for one or several values of strain rate. It is seen, that by a suitable choice of $\gamma(\theta)$ very good agreement with experimental data can be achieved. This is evidence of the correctness of the formula (2.95), case A. In the same figures the dash and dash-dot lines correspond to the hypotheses of Vishman, Zlatin, and Joffe [186] and Volo- shenko [187], respectively. In both cases a definite deviation from the experimental points is evident.

It should be pointed out here that the comparison with experiments has been carried out under the assumption that the material is perfectly plastic, hence the strain-hardening effect is disregarded according to (2.95). If we include strain-hardening, (2.94) yields

$$\dot{\varepsilon}^p = \frac{2}{\sqrt{3}} \gamma(\theta) \Phi \left( \frac{\sigma}{\varphi(\varepsilon^p, \theta)} - 1 \right)$$

or

$$\sigma(\dot{\varepsilon}^p, \varepsilon^p, \theta) = \varphi(\varepsilon^p, \theta) \left[ 1 + \Phi^{-1} \left( \frac{\sqrt{3} \dot{\varepsilon}^p}{2\gamma(\theta)} \right) \right]$$

for a one-dimensional state of stress. Eqs. (2.102) and (2.103) could not be compared with experimental data because of lack of suitable tests. On the other hand, an excellent agreement of (2.95) with all tests here presented by no means denies the existence of strain-hardening properties when the rate of strain is high. This is because of the arbitrariness in the choice of the function $\gamma(\theta)$ which was taken to fit the experimental data. If we had included the strain-hardening in the analysis, we would also have changed the function $\gamma(\theta)$. It seems most probable that (2.102) and (2.103) might give a better description of the dynamic response of certain metals in larger ranges of variables.

Recent investigations indicate the strong interaction between yield stress, strain rate and temperature in dynamic tests of various metals. The present approach shows how a simple temperature modification of the
constitutive equation for strain-rate sensitive materials developed in [132–134], can describe the effect of simultaneous change in $\dot{\varepsilon}$ and $\theta$.

Once more, our conclusions refer only to ranges of $\dot{\varepsilon}$ and $\theta$ already examined; particular attention should be paid to extending these conclusions to other ranges.

12. Linearization

In the theory of perfectly plastic solids, the quadratic yield condition of Huber-Mises is often approximated by a piecewise linear yield condition to simplify the analysis of boundary value problems. W. Prager in [145] discussed an analogous linearization in the theory of viscoplastic solids.

Assuming $\Phi(F) = F$ and neglecting the elastic part of the strain rate in (2.45) we obtain

\begin{equation}
\dot{\varepsilon}_{ij} = \gamma^0 \langle F \rangle \frac{\partial F}{\partial \sigma_{ij}}.
\end{equation}

Piecewise linear approximations to the constitutive equation (2.104) can be obtained as follows. According to (2.104), viscoplastic flow can only occur if

\begin{equation}
F > 0.
\end{equation}

This flow condition, which has the form of a single nonlinear inequality, is now approximated by the set of $m$ linear inequalities

\begin{equation}
L(v) = \alpha^{(v)}_{pq} s_{pq} - \beta^{(v)} > 0, \quad (v = 1, 2, \ldots, m),
\end{equation}

in which the symmetric deviators $\alpha^{(v)}_{pq}$ and the scalars $\beta^{(v)}$ are independent of the state of stress. Through a suitable choice of these deviators and scalars and of the number $m$, the flow condition (2.106) can be made to approximate (2.105) with any desired degree of accuracy. Finally, the constitutive equations (2.104) are replaced by

\begin{equation}
\dot{\varepsilon}_{ij} = \gamma^0 \sum_{v=1}^{m} \langle L(v) \rangle \frac{\partial L(v)}{\partial \sigma_{ij}}.
\end{equation}

Since $\alpha^{(v)}_{pp}$ is a deviator, $\alpha^{(v)}_{pp} = 0$, and it follows from (2.106) that

\begin{equation}
\frac{\partial L}{\partial \sigma_{ij}} = \alpha^{(v)}_{ij}.
\end{equation}
The constitutive equation (2.107) can therefore be written in the form

\[ \dot{\varepsilon}_{ij} = \gamma^0 \sum_{r=1}^{m} \langle L_{(r)} \rangle \alpha_{ij}^{(r)}. \]

13. Relaxation Processes

To discuss a relaxation process for general states of stress, let us consider an elastic/viscoplastic body, occupying the three-dimensional region \( V \) with the regular surface \( S \), and investigate the following boundary value problem. Consider first the loading process in which the surface tractions \( T_i \) are prescribed on the part \( S_1 \) of \( S \) and vanish on the remainder \( S_2 \). This loading is to be followed by a relaxation process in which the surface velocities \( v_i \) vanish on \( S_1 \), while the surface tractions \( T_i \) continue to vanish on \( S_2 \). In a relaxation test that is to furnish useful information about the constitutive equations, the states of stress and strain must be homogeneous. A relaxation process defined in this manner will be called an A-process (see [132]).

First consider the tensor \( \Gamma_{ij} \) defined by

\[ \Gamma_{ij} = \frac{1}{2} \int_S (T_i v_j + T_j v_i) dS. \]

During an A-process the tensor \( \Gamma_{ij} \) vanishes. Thus,

\[ \Gamma_{ij} = \frac{1}{2} \int_S (T_i v_j + T_j v_i) dS = \frac{1}{2} \int_S (\sigma_{ik} n_k v_i + \sigma_{jk} n_k v_i) dS \]

\[ = \frac{1}{2} \int_V \partial_i (\sigma_{ik} v_j + \sigma_{jk} v_i) dV = \frac{1}{2} \int_V (\sigma_{ik} \partial_k v_j + \sigma_{jk} \partial_k v_i) dV \]

\[ = \frac{1}{2} \int_V (\sigma_{ik} \dot{\varepsilon}_{kj} + \sigma_{jk} \dot{\varepsilon}_{ki}) dV = 0. \]

The states of stress and strain are homogeneous during this type of relaxation test; we therefore have

\[ \frac{1}{2} (\sigma_{ik} \dot{\varepsilon}_{kj} + \sigma_{jk} \dot{\varepsilon}_{ki}) = 0. \]

These conditions enable us to determine the state of stress as a function of time during the relaxation process.

Assume now that a certain state of stress characterized by \( \sigma_{ij}^{(0)} \) (or \( s_{ij}^{(0)} \) and \( \sigma_{kk}^{(0)} \)) and \( J_a^{(0)} = s_{ij}^{(0)} s_{ij}^{(0)}/2 > k^2 \) has been reached at the time \( t = 0 \) and
the relaxation process follows. Then from (2.67) and (2.112) we obtain a system of six differential equations with respect to $s_{ij}$ and $\sigma_{kk}$

\[
\frac{1}{2} \left\{ \left( s_{ik} + \frac{1}{3} \sigma_{kl} \delta_{ik} \right) \left[ \frac{1}{2\mu} \dot{s}_{ki} + \frac{1}{9K} \dot{\sigma}_{kl} \delta_{ki} + \gamma \Phi(F) \frac{s_{ki}}{J_a^{1/2}} \right] + \left( s_{jk} + \frac{1}{3} \sigma_{kl} \delta_{jk} \right) \left[ \frac{1}{2\mu} \dot{s}_{kj} + \frac{1}{9K} \dot{\sigma}_{kl} \delta_{kj} + \gamma \Phi(F) \frac{s_{kj}}{J_a^{1/2}} \right] \right\} = 0.
\]

(2.113)

By letting $i = j$ in (2.112) we have the condition

(2.114) \[ \sigma_{ij} \dot{\varepsilon}_{ij} = 0. \]

Multiplying (2.112) by $\sigma_{ij}$ we obtain

(2.115) \[ \sigma_{ij} \sigma_{jk} \dot{\varepsilon}_{kl} = 0. \]

Consider now the simpler boundary value problem in which the loading process is characterized by the surface tractions $T_i$ being prescribed on the entire surface $S$, while during a relaxation process the surface velocities $v_i$ vanish on the entire surface $S$. This kind of relaxation process we shall call a B-process.

Because the states of stress and strain are homogeneous during a relaxation test, we now have conditions

(2.116) \[ \dot{\varepsilon}_{ij} = 0. \]

From (2.67) and (2.116) we obtain a system of five differential equations for $s_{ij}$ in the form

(2.117) \[ \dot{s}_{ij} + 2\mu \gamma \Phi(F) \frac{s_{ij}}{J_a^{1/2}} = 0. \]

The conditions (2.116) lead to two useful scalar conditions

(2.118) \[ s_{ij} \dot{\varepsilon}_{ij} = 0, \]

(2.118) \[ s_{ij} s_{jk} \dot{\varepsilon}_{kl} = 0, \]

and it is worth noting that these conditions are valid for A-processes in incompressible materials because then

(2.118) \[ \dot{\varepsilon}_{ii} = 0. \]

The validity of (2.118) for an A-process under condition (2.119) follows readily from (2.114) and (2.115).
According to (2.67) and (2.118.1) we have during A-processes in incompressible materials and during B-processes in any materials a relaxation equation for $J_2$ in the form

$$J_2 + 4\mu y \Phi \left( \frac{J_2^{1/2}}{k} - 1 \right) J_2^{1/2} = 0.$$  

(2.120)

This can be written as a nonlinear Volterra integral equation of the second kind

$$J_2 = J_2^{(0)} - 4\mu y \int_{0}^{t} J_2^{1/2}(\xi) \Phi \left( \frac{J_2^{1/2}(\xi)}{k} - 1 \right) \, d\xi.$$  

(2.121)

Under the assumption that the integrand $J_2^{1/2}\Phi(J_2^{1/2}/k - 1)$ satisfies a Lipschitz condition, that is that

$$\left| (J_2')^{1/2} \Phi \left[ \frac{(J_2')^{1/2}}{k} - 1 \right] - (J_2'')^{1/2} \Phi \left[ \frac{(J_2'')^{1/2}}{k} - 1 \right] \right| < N_0 |J_2' - J_2''|,$$

(2.122)

where $N_0$ is a positive constant, the solution of (2.121) can be obtained by iteration based on the recurrence formula

$$J_2(n+1) = J_2^{(0)} - 4\mu y \int_{0}^{t} (J_2^{(n)}(\xi))^{1/2} \Phi \left[ \frac{(J_2^{(n)}(\xi))^{1/2}}{k} - 1 \right] \, d\xi.$$  

(2.123)

It is easily verified that the series

$$J_2^{(0)} + \sum_{n=0}^{\infty} [J_2(n+1)(t) - J_2(n)(t)]$$

(2.124)

is absolutely and uniformly convergent, and its sum

$$J_2(t) = \lim_{n \to \infty} J_2(n)(t)$$

(2.125)

is the solution of the integral equation (2.121), and hence of the differential equation (2.120). Of course, the solution (2.125) is valid only in the nonelastic region $J_2 > k^2$.

Full discussions of the solutions of the relaxation equations have been presented in the papers [132, 128, 69].

14. Fracture of Time-Dependent and Temperature-Dependent Materials

The detailed discussion of the continuum approach to fracture of metals has been presented by D. C. Drucker in papers [65, 66]. Fracture is a very
complex phenomenon which is preceded by some viscous flow or plastic deformation or both in almost all brittle as well as ductile materials. Both flow and corresponding fracture values depend upon the temperature, the strain rate and, in fact, on the entire time and temperature history of stress and strain. It has been observed that an increase in the strain rate raises the flow curve and promotes fracture, Fig. 38. A decrease of temperature has an effect analogous to an increase in strain rate, Fig. 38.

![Diagram](image)

**Fig. 38.** Increasing strain rate \( \dot{\varepsilon} \), or decreasing temperature \( \theta \) promotes fracture (D. C. Drucker [83]).

It seems, however, that the phenomenological prediction of fracture must be based on microscopic and atomic results. More elaborate models are desirable, in particular, models which contain the essential features of crystal dislocation (cf. [83, 65, 66]).
III. STRESS WAVE PROPAGATION IN AN ELASTIC/VISCOPLASTIC MEDIUM

1. Mathematical Preliminaries

A. General Considerations. We shall consider problems with initial and boundary conditions for a quasi-linear system of partial differential equations of the form

\[ U_t + AU_z + B = 0, \]

where \( U \) is a column vector with the \( n \) components \( U_1, U_2, \ldots, U_n \), \( A \) is an \( n \times n \) matrix and \( B \) is an \( n \) element column vector; \( A \) and \( B \) depend on the spatial coordinate \( z \), on the time \( t \) and on the components of the vector \( U \) (in the case of semi-linear system the matrix \( A \) is independent of the components of vector \( U \)).

The system (3.1) is assumed hyperbolic, that is, the matrix \( A \) has \( n \) real eigenvalues \( \lambda_{i(i)} \) \( (i = 1, 2, \ldots, n) \), and possesses a full set of linearly independent eigenvectors (see R. Courant [54] and A. Jeffrey [86]). The left eigenvectors of \( A \), \( l(i,k) \) with \( k = 1, 2, \ldots, s \) corresponding to the eigenvalue \( \lambda_{i(i)} \) with multiplicity \( s \) satisfy the equations

\[ l(i,k)A = \lambda_{i(i)}l(i,k), \quad k = 1, 2, \ldots, s. \]

They may be used to display the equations (3.1) in characteristic form and to introduce the \( n \) characteristic curves \( C_{(i)} \) as follows. Pre-multiply equations (3.1) by \( l(i) \) and assume for the moment that the \( n \) eigenvalues of \( A \) are distinct; the \( n \) equations are now in the characteristic form

\[ l(i)(U_t + \lambda_{i(i)}U_z) + l(i)B = 0. \]

The operator \( \partial/\partial t + \lambda_{i(i)}(\partial/\partial z) \) in the \( i \)th equation represents differentiation along the \( i \)th characteristic curve determined by

\[ C_{(i)}: \frac{dz}{dt} = \lambda_{i(i)}. \]

If \( \lambda_{i(i)} = \text{const} \) the conditions (3.3) along the characteristic lines (3.4) may be written as follows:

\[ a_{(i)}dt + b_{(i)}dz + \sum_{k=1}^{n} c_{(i)k}dU_k = 0, \]

where \( a_{(i)}, b_{(i)} \) and \( c_{(i)k} \) depend on \( t, z, \) and \( U_k \).
The solution of the initial value problem for the system (3.1) has been studied by numerous authors. First existence and uniqueness were discussed in full (see K. O. Friedrichs [70], R. Courant and P. Lax [51], P. Hartman and A. Wintner [76], P. Lax [103], A. Douglis [61] and R. Courant [54]).

Some theorems regarding existence and uniqueness may be obtained as special cases of the same theorems proved for the system of first order partial differential hyperbolic equations involving $n$ independent variables (see J. Schauder [154, 155], M. Cinquini-Cibrario [36, 37] and R. Courant [54]).

In practice, to obtain the solution of the initial and boundary-value problems for the system (3.1) the method of finite differences is used (cf. R. Courant and K. O. Friedrichs [50]). For a study of convergence of this method and the conditions of its application see R. Courant, W. Isaacson and M. Rees [52] and G. Prouse [148].

Some modifications of the finite difference method, and its application to the solution of the initial and boundary-value problems for quasi-linear and semi-linear systems (3.1) were proposed by H. B. Keller and V. Thomée [93, 94, 174–176]. R. Courant [53, 54] investigated the possibility of application of successive approximations to the solution of initial and boundary-value problems for the system (3.1).

B. A non-linear boundary-value problem for a semi-linear hyperbolic partial differential equation. Let $Q$ be the closed rectangle $\{0 \leq x \leq x_0, 0 \leq y \leq y_0\}$ [opposite vertices at $(0,0)$ and $(x_0,y_0)$], and consider the normal form of the semi-linear partial differential equation of hyperbolic type in two independent variables

$$u_{xy} = f(x,y,u,u_x,u_y),$$

where $u$ is the unknown function and $f(x,y,u,u_x,u_y)$ is a given continuous function in $Q$ for arbitrary $u$, $u_x$ and $u_y$. A solution of (3.6) is a function $u(x,y)$, which is continuous in $Q$ together with its partial derivatives $u_x$, $u_y$ and $u_{xy}$ and satisfies (3.6).

We require that $u(x,y)$ satisfy the boundary conditions

$$u_x(x,\varphi(x)) = g(x,u(x,\varphi(x)),u_y(x,\varphi(x))),$$

$$u_y(\psi(y),y) = h(y,u(\psi(y),y),u_x(\psi(y),y))),$$

$$u(x^*,y^*) = u^*,$$

where $(x^*,y^*)$ is a point in $Q$, $u^*$ is a given constant, $\varphi(x)$ and $\psi(y)$ are arbitrary curves within $Q$, and $g$ and $h$ are given continuous functions in $Q$ for arbitrary $u$, $u_x$ and $u_y$.

The boundary-value problem (3.6)-(3.7) was first stated by Z. Szmydt [167], who proved it as follows:
Theorem 1. If

1. the functions \( \varphi(x) \) and \( \psi(y) \) are continuous in \( Q \), and \( 0 \leq \varphi(x) \leq y_0, 0 \leq \psi(y) \leq x_0 \);

2. \( f, g \) and \( h \) are continuous real-valued functions of \( x, y, u, u_x, u_y \) in \( Q \), and satisfy the following Lipschitz conditions:

\[
\begin{align*}
|f(x,y,\tilde{u},\tilde{u}_x,\tilde{u}_y) - f(x,y,u,u_x,u_y)| & \leq L(\tilde{u}_x - u_x + |\tilde{u}_y - u_y|), \\
|g(x,\tilde{u},\tilde{u}_y) - g(x,u,u_y)| & \leq L|\tilde{u}_x - u_x| + M|\tilde{u}_y - u_y|, \\
|h(y,\tilde{u},\tilde{u}_x) - h(y,u,u_x)| & \leq L|\tilde{u}_x - u_x| + N|\tilde{u}_y - u_y|;
\end{align*}
\]  

(3.8)

3. the constants \( L, M, N \) satisfy the inequalities

\[
2L(X_0^2 + X_0 + 1) < 1,
\]

(3.9)

\[
2L(X_0^2 + X_0) + N(X_0 + 1) < 1,
\]

\[
2L(X_0^2 + X_0) + M(X_0 + 1) < 1,
\]

where \( X_0 = \max(x_0, y_0) \);

there exists a unique solution \( u(x,y) \in C_Q^1 \) to the boundary value problem (3.6), (3.7).

Proof. We shall discuss here only the basic elements of the proof, which we shall use to construct the solution of the nonlinear problem (3.6)–(3.7) by means of the iteration method.

Consider the linear space \( C_Q^1 \) of the functions \( u(x,y) \). The norm in this space we shall define in the following form:

\[
(3.10) \quad ||u(x,y)|| = \sup_{Q}|u(x,y)| + \sup_{Q}|u_x(x,y)| + \sup_{Q}|u_y(x,y)|.
\]

Convergence in the space \( C_Q^1 \) stands here for uniform convergence; this should hold for sequences of functions \( u(x,y) \), together with their first derivatives, in \( Q \). Due to the fact that for uniform convergence the Cauchy criterion is fulfilled, the space \( C_Q^1 \) is a complete metric space.

Let us now introduce in the space \( C_Q^1 \) the integral transformation \( \mathcal{R} \), mapping any function \( u(x,y) \in C_Q^1 \) into the function

\[
(3.11) \quad \tilde{u}(x,y) = \mathcal{R}[u(x,y)].
\]

* For a complete proof see Z. Szmydt [167].
where

\[ \mathcal{R}[u(x,y)] = u^* + \int_{x^*}^{x} g(s,u(s,\varphi(s)),u_y(s,\varphi(s)))\,ds \]

(3.12)

\[ + \int_{y^*}^{y} h(s,u(\psi(z),z),u_x(\psi(z),z))\,dz + \int_{y^*}^{y} \int_{\mathbb{R}^n} f(s,z,u(s,z),u_x(s,z),u_y(s,z))\,ds\,dz \]

\[ + \int_{\mathbb{R}^n} f(s,z,u(s,z),u_x(s,z),u_y(s,z))\,dz\,ds. \]

The functions \( \tilde{u}(x,y) \) determined by (3.11), (3.12) and their first derivatives are continuous in \( Q \). Thus, the transformation \( \mathcal{R} \) maps the space \( C_0^1 \) into itself.

This shows that every function which is a fixed point of the mapping \( \mathcal{R} \), i.e.,

(3.13)

\[ \tilde{u}(x,y) = u(x,y) \]

satisfies (3.6) and the boundary conditions (3.7). To complete the proof, it is sufficient to show that there exists only one fixed point of the mapping \( \mathcal{R} \).

By Banach's theorem,* every contraction mapping defined in a complete metric space has in this space exactly one fixed point. Since the metric space \( C_0^1 \) is complete, we have to prove that \( \mathcal{R} \) is a contraction mapping.

Using the definition of the norm (3.10), we may easily obtain the following result:

(3.14)

\[ ||\tilde{u}_1 - \tilde{u}_2|| = k||u_1 - u_2||, \]

where

\[ k = \max[2L(X_0^2 + 2X_0 + 1), 2L(X_0^2 + X_0) + N(X_0 + 1), 2L(X_0^2 + X_0) + M(X_0 + 1)], \]

(3.16)

and from assumptions (3.9) we have \( k < 1 \); hence the mapping \( \mathcal{R} \) is contractive.

Taking advantage of the mapping \( \mathcal{R} \), we may apply the method of successive approximations to obtain the solution of the non-linear problem (3.6)–(3.7). The iterative scheme is as follows:

* For a discussion of the Banach theorem see Ref. [91].
In view of our assumptions, the sequence \( \{u_n(x,y)\} \) given by (3.16) is a Cauchy sequence. Hence the iterations (3.16) converge uniformly to the solution

\[
u(x,y) = \lim_{n \to \infty} u_n(x,y).
\]

In practice we have the following iterative scheme:

\[
u_{n+1}(x,y) = \mathcal{R}[u_n(x,y)],
\]

(3.17)

\[
u_{x(n+1)}(x,y) = \mathcal{R}_x[u_n(x,y)],
\]

\[
u_{y(n+1)}(x,y) = \mathcal{R}_y[u_n(x,y)].
\]

Note also that by assuming the Lipschitz condition (3.8) in the weaker form

\[
|f(x,y,u,\bar{u}_x,\bar{u}_y) - f(x,y,u,u_x,u_y)| \leq L|\bar{u}_x - u_x| + |\bar{u}_y - u_y|,
\]

(3.18)

\[
|g(x,u,\bar{u}_y) - g(x,u,u_y)| \leq M|\bar{u}_y - u_y|,
\]

\[
|h(y,u,\bar{u}_x) - h(y,u,u_x)| \leq N|\bar{u}_x - u_x|
\]

and using the Schauder theorem we may prove the existence of the solution for our boundary-value problem.*

Consider now the special case of the boundary-value problem (3.7), where we assume

\[
h(y,u,u_x) = b(y,u,u_x) - u_x\psi(y) \quad \text{on} \quad x = \psi(y).
\]

Then the boundary conditions have the form

\[
u_x(x,\varphi(x)) = g(x,u(x,\varphi(x)),u_y(x,\varphi(x))),
\]

(3.19)

\[
u(\psi(y)),y) = u^* + \int_{\psi(y)}^{y} b(\eta,u(\psi(\eta)),\eta,u_x(\psi(\eta),\eta))d\eta,
\]

(3.20)

\[
u(x^*,y^*) = u^*
\]

and the mapping is

---

* For details see Z. Szmydt [169, 170].
This special case of the non-linear boundary-value problem has great practical significance for the solution of the stress-wave propagation problem in an inelastic medium.

Another special case of the boundary-value problem (3.7) is the linear relation

\[
\sigma_1(x) = \alpha_1(x) + \beta_1(x),
\]

where \(\sigma_1, \alpha_1, \beta_1 \in C^0\) for \(i = 1, 2\). The same problem as (3.6), (3.22) was posed by J. Conlan [46] who solved it by a polygon method.

The mapping for the problem (3.6), (3.22) takes the form

\[
R[u(x, y)] = u^* + \int_0^y \left[ \sigma_1(\xi) + \alpha_1(\xi) u_x(\xi, \eta) + \beta_1(\xi) u(\xi, \eta) \right] d\eta
\]

\[
+ \int_0^y \left[ \sigma_2(\eta) + \alpha_2(\eta) u_x(\psi(\eta), \eta) + \beta_2(\eta) u(\psi(\eta), \eta) \right] d\eta
\]

\[
+ \int_{\psi(x)}^{\psi(y)} \left[ \int_{\xi(x)}^{\xi(y)} f(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta)) d\xi \right] d\eta
\]

\[
(3.23)
\]

* The Euler-Cauchy polygon method for proving the existence of a solution for

\[
\frac{dy}{dx} = f(x, y), \quad y(0) = y_0
\]

was extended by J. B. Diaz [69] to the characteristic boundary-value problem for a hyperbolic partial differential equation [3.6] and further extended by J. Conlan [45] to the Cauchy problem, and to the mixed boundary-value problem (Picard's problem).
It has been shown (see [166-168, 99]) that all classical problems (i.e. the Cauchy, Darboux, Picard and Goursat problems) can be analogically treated as special cases of the nonlinear problem (3.6)-(3.7). In such a procedure, particular note must be taken of the restrictions on the functions which describe the boundary conditions.

Note that all the results discussed here are valid for a system of second-order semi-linear hyperbolic partial differential equations.*

C. A non-linear boundary-value problem for a linear hyperbolic equation. Consider the normal form of the second order linear partial differential equation of hyperbolic type in two independent variables:

\[(3.24)\quad \mathcal{L}(u) \equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y).\]

The coefficients \(a, b, c, d\) are given as real-valued functions of \(x\) and \(y\) defined in \(Q\).

The purpose of the present section is to construct a solution \(u(x,y)\) of Eq. (3.24), which is continuous in \(Q\) together with its partial derivatives \(u_x, u_y\) and \(u_{xy}\), and satisfies the boundary-value conditions (3.7).

The problem (3.24), (3.7) was discussed in [137]. It has been shown that the solution of this problem may be obtained by means of the procedure proposed by S. C. Chu and J. B. Diaz [35] for solving the linear problem.

Let \(V(\xi, \eta; x,y)\) be the Riemann function\(^\dagger\) for the homogeneous equation \(\mathcal{L}(u) = 0\). Any solution of Eq. (3.24) in \(Q\) may be given in the form:

\[(3.25)\quad u(x,y) = \mathcal{F}(x,y) + \int_{\xi}^{x} \Omega(\xi)V(\xi, y^*; x,y)d\xi + \int_{\eta}^{y} \Pi(\eta)V(x^*, \eta; x,y)d\eta,\]

where

\[(3.26)\quad \mathcal{F}(x,y) = u^*V(x^*, y^*; x,y) + \int_{\xi}^{x} \int_{\eta}^{y} d(\xi, \eta)V(\xi, \eta; x,y)d\eta d\xi.\]

The functions \(\Omega(x)\) and \(\Pi(y)\) should be determined so that the solution (3.25) satisfies the boundary conditions (3.7). Then the boundary-value problem (3.24), (3.7), reduces to the equivalent system of non-linear functional-integral equations:

\(^*\) For the discussion of these general cases see Z. Szmydt [167-170].
\(^\dagger\) For a review of methods for obtaining the Riemann function see E. T. Copson [47]. He has also made a survey of equations for which the Riemann function is known in closed form. See also R. Courant [64] and A. Wintner [198].
\[ \Omega(x) = G(x) + \int \Omega(\xi) \kappa_1(\xi,x) d\xi + \int \Pi(\eta) \kappa_2(\eta,x) d\eta \]
\[ + \left[ V(x,y^*;x,\varphi(x)) \right]^{-1} g \left( x, \mathcal{F}(x,\varphi(x)) \right) + \int \Omega(\xi) \kappa_3(\xi,x) d\xi \]
\[ + \int \Pi(\eta) \kappa_4(\eta,x) d\eta, \mathcal{F}_y(x,\varphi(x)) + \int \Omega(\xi) \kappa_5(\xi,x) d\xi \]
\[ + \int \Pi(\eta) \kappa_6(\eta,x) d\eta + \Pi(\varphi(x)) V(x^*,\varphi(x);x,\varphi(x)) \right), \]
(3.27)
\[ \Pi(y) = H(y) + \int \Omega(\xi) \kappa_7(\xi,y) d\xi + \int \Pi(\eta) \kappa_8(\eta,y) d\eta \]
\[ + \left[ V(x^*,y;\psi(y),y) \right]^{-1} h \left( y, \mathcal{F}(\psi(y),y) \right) + \int \Omega(\xi) \kappa_9(\xi,y) d\xi \]
\[ + \int \Pi(\eta) \kappa_{10}(\eta,y) d\eta, \mathcal{F}_x(\psi(y),y) + \int \Omega(\xi) \kappa_{11}(\xi,y) d\xi \]
\[ + \Omega(\psi(y)) V(\psi(y),y^*;\psi(y),y) + \int \Pi(\eta) \kappa_{12}(\eta,y) d\eta \right), \]

where
\[ G(x) = - \frac{\mathcal{F}_x(x,\varphi(x))}{V(x,y^*;x,\varphi(x))}, \quad H(y) = - \frac{\mathcal{F}_y(\psi(y),y)}{V(x^*,y;\psi(y),y)}, \]
\[ \kappa_1(\xi,x) = - \frac{V_x(\xi,y^*;x,\varphi(x))}{V(x,y^*;x,\varphi(x))}, \quad \kappa_2(\eta,x) = - \frac{V_x(x^*,\eta;x,\varphi(x))}{V(x,y^*;x,\varphi(x))}, \]
(3.28) \[ \kappa_3(\xi,x) = V(\xi,y^*;x,\varphi(x)), \quad \kappa_4(\eta,x) = V(x^*,\eta;x,\varphi(x)), \]
\[ \kappa_5(\xi,x) = V_y(\xi,y^*;x,\varphi(x)), \quad \kappa_6(\eta,x) = V_y(x^*,\eta;x,\varphi(x)), \]
\[ \kappa_7(\xi,y) = - \frac{V_y(\xi,y^*;\psi(y),y)}{V(x^*,\psi(y),y)}, \quad \kappa_8(\eta,y) = - \frac{V_{y^*}(x^*,\eta;\psi(y),y)}{V(x^*,y;\psi(y),y)}, \]
\[ \kappa_9(\xi,y) = V(\xi,y^*;\psi(y),y), \quad \kappa_{10}(\eta,y) = V(x^*,\eta;\psi(y),y), \]
\[ \kappa_{11}(\xi,y) = V_z(\xi,y^*;\psi(y),y), \quad \kappa_{12}(\eta,y) = V_z(x^*,\eta;\psi(y),y). \]
In what follows we shall use the notations:

\[ K_0 = \max_Q |\kappa_i(\xi, x)|, |\kappa_j(\eta, x)|, |\kappa_k(\xi, y)|, |\kappa_l(\eta, y)|, \]

\[ (3.29) \quad i = 1,3,5; \quad j = 2,4,6; \quad k = 7,9,11; \quad l = 8,10,12; \]

\[ K^* = \max_Q |V(x,y^*;x,\varphi(x))|, |V(x^*,\varphi(x);x,\varphi(x))|, |V(x^*,y^*;\psi(y),y)|, \]

\[ |V(\psi(y),y^*;\psi(y),y)|. \]

The precise formulation of the problem to be solved is given in the following

**Theorem 2.** If

1. the real-valued functions \( a(x,y), b(x,y), c(x,y) \) and \( d(x,y) \) are continuous in \( Q \);

2. the function \( \varphi(x) \) is continuous on \( 0 \leq x \leq x_0 \) and the function \( \psi(y) \) is continuous on \( 0 \leq y \leq y_0 \), with \( 0 \leq \varphi(x) \leq y_0; \quad 0 \leq \psi(y) \leq x_0; \)

3. the functions \( g \) and \( h \) are continuous in \( Q \) and satisfy the Lipschitz conditions

\[ |g(x,\tilde{u},\tilde{u}_y) - g(x,u,u_y)| \leq M^*|\tilde{u} - u| + N^*|\tilde{u}_y - u_y|. \]

\[ |h(y,\tilde{u},\tilde{u}_x) - h(y,u,u_x)| \leq M^*|\tilde{u} - u| + N^*|\tilde{u}_x - u_x|. \]

4. the constants \( K_0, K^*, X_0, M^* \) and \( N^* \) satisfy the inequality

\[ 2K_0X_0 \left( 1 + \frac{M^* + N^*}{K^*} \right) + N^* < 1; \]

there exists, throughout \( Q \), a unique solution \( u(x,y) \in C_0^1 \) to the boundary-value problem (3.24), (3.7).

**Proof.** We shall show by successive approximations that the system (3.27) possesses a unique continuous solution set \( \Omega(x) \) and \( \Pi(y) \). We define the sequences \( \{\Omega_{(n)}(x)\} \) and \( \{\Pi_{(n)}(x)\} \) as follows

\[ \Omega_{(n)}(x) = G(x) + \int_{\xi}^{x} \Omega_{(n-1)}(\xi) \kappa_1(\xi,x) d\xi + \int_{\eta}^{x} \Pi_{(n-1)}(\eta) \kappa_2(\eta,x) d\eta \]

\[ + [V(x,y^*;x,\varphi(x))]^{-1} G \left( x, \mathcal{F}(x,\varphi(x)) + \int_{\xi}^{x} \Omega_{(n-1)}(\xi) \kappa_3(\xi,x) d\xi \right. \]

\[ \left. \int_{\eta}^{x} \Pi_{(n-1)}(\eta) \kappa_4(\eta,x) d\eta \right]^{-1} \]

\[ \int_{\xi}^{x} \Omega_{(n-1)}(\xi) \kappa_5(\xi,x) d\xi + \int_{\eta}^{x} \Pi_{(n-1)}(\eta) \kappa_6(\eta,x) d\eta \]
for \( n = 1, 2, 3, \ldots \), and for arbitrary continuous functions \( \Omega_{(0)}(x) \) and \( \Pi_{(0)}(y) \). Equations (3.30) for \( n = 1, 2, 3, \ldots \), yield:

\[
|\Omega_{(n+1)}(x) - \Omega_{(n)}(x)|
\leq K^0 \left[ \int_0^{x^*} |\Omega_{(n)}(\xi) - \Omega_{(n-1)}(\xi)| d\xi + \int_0^{\gamma_n^*} |\Pi_{(n)}(\eta) - \Pi_{(n-1)}(\eta)| d\eta \right]
\]

\[
+ \frac{K^0(M^* + N^*)}{K^*} \left[ \int_0^{x^*} |\Omega_{(n)}(\xi) - \Omega_{(n-1)}(\xi)| d\xi + \int_0^{\gamma_n^*} |\Pi_{(n)}(\eta) - \Pi_{(n-1)}(\eta)| d\eta \right]
\]

\[
+ N^*|\Pi_{(n)}(y) - \Pi_{(n-1)}(y)|,
\]

(3.31)
With the notation

\[
\frac{K^0(M^* + N^*)}{K^*} \left[ \int_0^{\gamma} |\Pi_{(n)}(\eta) - \Pi_{(n-1)}(\eta)| \, d\eta + \int_0^{x} |\Omega_{(n)}(\xi) - \Omega_{(n-1)}(\xi)| \, d\xi \right]
\]

\[+ N^*|\Omega_{(n)}(y) - \Omega_{(n-1)}(y)|.
\]

we obtain by adding the inequalities (3.31)

\[
Z_n + Z_{n+1} \leq \left[ 2K^0X_0 \left( 1 + \frac{M^* + N^*}{K^*} \right) + N^* \right] (Z_n + Z_n).
\]

The inequality

\[
2K^0X_0 \left( 1 + \frac{M^* + N^*}{K^*} \right) + N^* < 1
\]

implies that \{\Omega_{(n)}(x)\} and \{\Pi_{(n)}(y)\} converge uniformly to the continuous limit functions \(\Omega(x)\) and \(\Pi(y)\), which constitute a solution to the system (3.27).

The above procedure shows simultaneously that this system has a unique solution*.

We now turn to the following particular case of the boundary-value problem (3.7):

\[
\begin{align*}
&u_x(x, \varphi(x)) = g(x, u(x, \varphi(x))), \\
&u_y(\psi(y), y) = h(y, u(\psi(y), y)), \\
&u(x^*, y^*) = u^*.
\end{align*}
\]

In this case, using the same procedure as above, we can prove

Theorem 2a. If

1. and 2. of Theorem 2 hold;

3. the function \(g\) and \(h\) are continuous in \(Q\) and satisfy the Lipschitz conditions

* That the problem (3.24), (3.7) has a unique solution can be shown easily.
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\[ |g(x,\hat{u}) - g(x,u)| \leq M^0|\hat{u} - u|, \]
\[ |h(y,\hat{u}) - h(y,u)| \leq M^0|\hat{u} - u|; \]

4. the constants \( K^0, K^*, X_0 \) and \( M^0 \) satisfy the inequality

\[ 2K^0X_0\left(1 + \frac{M^0}{K^*}\right) < 1; \]

there exists, throughout \( Q \), a unique solution to the boundary-value problem (3.24), (3.35).

The non-linear generalized Picard problem stated first by G. Majcher [108] may be treated in a similar way, as a special case of the problem (3.24), (3.7). In that case the boundary-value conditions have the form:

\[ u(x,0) = g(x), \]
\[ u_y(\psi(y),y) = h(y,u(\psi(y),y),u_x(\psi(y),y)). \]

We shall now discuss in detail a linear mixed boundary-value problem for a linear hyperbolic partial differential equation. This problem can be stated in the following form*:

\[ d(x,y), \]
\[ \alpha_0(x)u(x,\varphi(x)) + \alpha_1(x)u_x(x,\varphi(x)) + \alpha_2(x)u_y(x,\varphi(x)) = \sigma_1(x), \]
\[ \beta_0(y)u(\psi(y),y) + \beta_1(y)u_x(\psi(y),y) + \beta_2(y)u_y(\psi(y),y) = \sigma_2(y), \]
\[ u(x^*,y^*) = u^*. \]

The solution of the problem (3.39) has been given by C. Chu and J. B. Diaz [36]. The case \((x^*,y^*) = (0,0)\) was treated in the paper [5] by A. K. Aziz and J. B. Diaz.

Using the solution (3.2)-(3.3), we can reduce the boundary-value problem (3.16) to an equivalent system of linear functional-integral equations:

\[ \Omega(x) = G^*(x) + A(x)\Pi(\varphi(x)) + \int_{x^*}^{x} k_1(\xi,y^*;x)\Omega(\xi)d\xi + \int_{y^*}^{y} k_1(x^*,\eta;x)\Pi(\eta)d\eta, \]
\[ \Pi(y) = H^*(y) + B(y)\Omega(\psi(y)) + \int_{y^*}^{y} k_2(\xi,y^*;y)\Omega(\xi)d\xi + \int_{x^*}^{x} k_2(x^*,\eta;y)\Pi(\eta)d\eta, \]

* It is obvious that the problem (3.39) can be treated as a linearized form of the problem (3.24), (3.7).
where

\[ G^*(x) = [\alpha_1(x)V(x,y^*;x,\varphi(x))]^{-1}[\sigma_1(x) - \alpha_0(x)F(x,\varphi(x)) - \alpha_1(x)F_x(x,\varphi(x)) \]
\[ - \alpha_9(x)F_y(x,\varphi(x))], \]
\[ H^*(y) = [\beta_3(y)V(x^*,y;\psi(y),y)]^{-1}[\sigma_2(y) - \beta_0(y)F_\psi(y),y) - \beta_1(y)F_y(\psi(y),y) \]
\[ - \beta_2(y)F_{\psi y}(\psi(y),y)], \]
\[ A(x) = - [\alpha_1(x)V(x,y^*;x,\varphi(x))]^{-1}[\alpha_2(x)V(x^*,y;x,\varphi(x))], \]
\[ B(y) = - [\beta_3(y)V(x^*,y;\psi(y),y)]^{-1}[\beta_1(y)V(x,y^*;\psi(y),y)]. \]

(3.41)

\[ k_1(\xi,\eta;x) = - [\alpha_1(x)V(x,y^*;x,\varphi(x))]^{-1}[\alpha_0(x)V(\xi,\eta;x,\varphi(x)) \]
\[ + \alpha_1(x)V_x(\xi,\eta;x,\varphi(x)) + \alpha_2(x)V_y(\xi,\eta;x,\varphi(x))], \]
\[ k_2(\xi,\eta;y) = - [\beta_3(y)V(x^*,y;\psi(y),y)]^{-1}[\beta_0(y)V(\xi,\eta;\psi(y),y) \]
\[ + \beta_1(y)V_x(\xi,\eta;\psi(y),y) + \beta_2(y)V_y(\xi,\eta;\psi(y),y)]. \]

We shall discuss the principal results obtained by C. Chu and J. B. Diaz in the paper [35]. There they proved:

**Theorem 2b.** If

1. and 2. of Theorem 2 hold;
2. \( \sigma_i, \alpha_i, \beta_j \) are continuous in \( Q \) for \( i = 1,2 \) and \( j = 0,1,2 \);
3. \( \alpha_1(x) \neq 0 \) for all \( x \) in \( 0 \leq x \leq x_0 \), \( \beta_2(y) \neq 0 \) for all \( y \) in \( 0 \leq y \leq y_0 \);
4. the following inequalities hold:
\[ |\alpha_2(x^*)/\alpha_1(x^*)| < 1, \quad |\beta_1(y^*)/\beta_2(y^*)| < 1; \]
5. \( \varphi(x^*) = y^*, \psi(y^*) = x^*; \)
6. \( x_0, y_0 \) are positive constants such that \( A_0 + 2N X_0 < 1 \), where \( A_0 \)
\[ = \max \max |A(x)| \text{ and } |B(y)| \text{ over all } x \text{ in } 0 \leq x \leq x_0 \]
\[ \text{and } y \text{ in } 0 \leq y \leq y_0, \text{ and } \]
\[ N = \max \max |k_1(\xi,\eta;x)|,|k_2(\xi,\eta;y)|; \]

there exists, throughout \( Q \), a unique solution \( u(x,y) \in C^1 \) to the boundary-value problem (3.39).
Proof. By the definitions of \( A(x) \) and \( B(y) \), the conditions 5. and 6. imply that \( |A(x^*)| < 1 \) and \( |B(y^*)| < 1 \). Since \( A(x) \) and \( B(y) \) are continuous this means that, if \( X_0 \) is sufficiently small, the inequality \( A_0 + 2NX_0 < 1 \) of the condition 7. holds.

Define the sequences \( \{\Omega_{(n)}(x)\} \) and \( \{\Pi_{(n)}(y)\} \) as follows

\[
\Omega_{(n)}(x) = G^*(x) + A(x)\Pi_{(n-1)}(\varphi(x)) + \int_{x^*}^{x} h_1(\xi, y^*; x)\Omega_{(n-1)}(\xi)d\xi
\]

\[
+ \int_{x^*}^{y} h_1(x^*, \eta; x)\Pi_{(n-1)}(\eta)d\eta,
\]

(3.42)

\[
\Pi_{(n)}(y) = H^*(y) + B(y)\Omega_{(n-1)}(\psi(y)) + \int_{y^*}^{y} h_2(\xi, y^*; y)\Omega_{(n-1)}(\xi)d\xi
\]

\[
+ \int_{y^*}^{x^*} h_2(x^*, \eta; y)\Pi_{(n-1)}(\eta)d\eta,
\]

for \( n = 1, 2, 3, \ldots \), and for arbitrary continuous functions \( \Omega_{(0)}(x) \) and \( \Pi_{(0)}(y) \). Proceeding as in the proof of Theorem 2, we easily obtain from (3.42):

(3.43)

\[
Z_{n+1} + \dot{Z}_{n+1} \leq (A_0 + 2NX_0)(Z_n + \dot{Z}_n),
\]

for \( n = 1, 2, 3, \ldots \), from which the conclusion of the theorem follows immediately.

For the case

(3.44)

\[
\alpha_1(x) \neq 0, \quad \alpha_2(x) = 0 \quad \beta_1(y) = 0, \quad \beta_2(y) \neq 0 \quad \text{in } Q,
\]

Theorem 2b is valid (see C. S. Chu and J. B. Diaz [35]). In that case, the constants \( N \) and \( X_0 \) should satisfy the following inequality: \( 2NX_0 < 1 \).

G. Majcher [108] has considered the boundary-value problem consisting of the partial differential equation (3.24) subject to the boundary conditions (linear generalized Picard problem)

(3.45)

\[
\begin{align*}
\varphi(x, 0) &= \sigma_1(x), \\
\beta_0(y)u(\psi(y), y) + \beta_1(y)u_x(\psi(y), y) + \beta_2(y)u_y(\psi(y), y) &= \sigma_2(y).
\end{align*}
\]

A. K. Aziz and J. B. Diaz [5] have shown that the problem (3.45) stated by G. Majcher [108] may be treated as a particular case of the linear problem (3.39).
Full discussion of the other particular cases of the linear problem (3.39) has been presented in paper [5], where the history of the linear problem and an extensive bibliography is also given.

All theorems discussed in sections B and C are valid "in the small." The inequalities (3.9), (3.34), (3.37), and (3.43) required to prove the existence and uniqueness of the solutions are very strong and impose important restrictions on the size of the rectangle \( Q \) and on the boundary conditions.

2. Formulation of the Problem

The object of this chapter is to discuss solutions of certain boundary-value problems for a work-hardening and rate-sensitive plastic material. Four types of waves will be considered; spherical waves, cylindrical radial waves, cylindrical shear waves, and plane waves.

We assume that the rate-sensitive plastic material can be treated as elastic/viscoplastic according to the constitutive equations (2.58).

In papers [130, 131, 135, 136] it has been shown that the solution of the propagation problem of stress waves of all types in an infinite elastic/viscoplastic medium may be reduced to one mathematical problem. By changing the coefficients of the differential equations and the boundary conditions, we may obtain the mathematical description of the four wave types. This treatment has been applied in both the elastic and inelastic regions. In each case the problem in the range of elastic/viscoplastic deformations reduces to the solution of the corresponding boundary-value problem for the quasi-linear or semi-linear first-order hyperbolic system (3.1).

Similarly, all problems of one-directional propagation of a stress wave in a work-hardening elastic-plastic medium may be described by the system (3.1) (see for instance G. Hopkins [82]).

Full particulars concerning the stress-wave propagation in elastic/viscoplastic soil have been discussed in [127].

Refs. [130, 131] are concerned with the analysis of wave propagation in a non-homogeneous elastic/viscoplastic body. The solution of the problem of propagation of stress waves in a work-hardening and rate-sensitive plastic material has been given in [90, 136]. Paper [90] also discusses the reflection of an elastic/viscoplastic wave from a plane.

In Section 3 we shall follow the solutions obtained in Refs. [135, 136].

3. Application of the Method of Finite Differences

A. Spherical Waves. Consider an infinite, elastic/viscoplastic medium with a spherical cavity of radius \( r_0 \). To the spherical surface of the cavity the radial uniform, time-dependent pressure \( p(t) \) is applied. With reference to the spherical coordinates \( r, \varphi, \theta \) we thus have
where $u_r$, $u_\varphi$, $u_\theta$ are the spherical components of the displacement vector. Assumption (3.46) (spherical symmetry) yields the following components of the strain and stress tensors:

\begin{align}
(3.47) & \quad \varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta} = \frac{u}{r}, \\
(3.48) & \quad \sigma_{rr}(r,t), \quad \sigma_{\varphi\varphi}(r,t) = \sigma_{\theta\theta}(r,t).
\end{align}

Denote by $v = \partial u / \partial t$ the radial velocity and by $\rho$ the density of the material. In that case the system of differential equations take the form (3.1), where

\[
U = \begin{bmatrix}
    v \\
    \sigma_{rr} \\
    \sigma_{\varphi\varphi} \\
    \varepsilon_{rr} \\
    \varepsilon_{\varphi\varphi}
\end{bmatrix}, \quad A = \begin{bmatrix}
    0 & -\frac{1}{\rho} & 0 & 0 & 0 \\
    -(K + \frac{2}{3}\mu) & 0 & 0 & 0 & 0 \\
    \frac{2}{3}\mu - K & 0 & 0 & 0 & 0 \\
    -1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
    \frac{2}{\sqrt{3}} \mu \gamma \Phi \left( \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\sqrt{3} \kappa} - 1 \right) - \left( 2K - \frac{4}{3} \mu \right) \frac{v}{r} \\
    - \left( 2K + \frac{2}{3} \mu \right) \frac{v}{r} + \frac{2}{\sqrt{3}} \mu \gamma \Phi \left( \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\sqrt{3} \kappa} - 1 \right) \\
    0 \\
    - \frac{v}{r}
\end{bmatrix}.
\]

The characteristics (3.4) have the form

\begin{align}
(3.50) & \quad r = \text{const}, \quad r = r_0 \pm \lambda t + \text{const}, \quad \text{where } \lambda = \left( \frac{4\mu + 3K}{3\rho} \right)^{1/2},
\end{align}

and $r = \text{const}$ is a triple characteristic.
According to (3.5), along these lines the following conditions must be satisfied:

\[ 3K \left( \sqrt{3} \gamma \Phi + \frac{3}{r} v \right) dt + [3K/(2\mu) - 1]d\sigma_{rr} = [3K/(2\mu) + 2]d\sigma_{\phi\phi}. \]

\[ d\varepsilon_{rr} = -\frac{1}{2\mu} d\sigma_{rr} + \frac{1}{2\mu} d\sigma_{\phi\phi} - \left( \frac{v}{r} + \sqrt{3} \gamma \Phi \right) dt = 0, \]

(3.51)

\[ d\varepsilon_{\phi\phi} = \frac{v}{r} dt, \]

\[ \left[ \pm \frac{6\lambda}{r} (\sigma_{rr} - \sigma_{\phi\phi}) + 4 \sqrt{3} \mu \gamma \Phi + \frac{v}{r} (4\mu - 6K) \right] dr \]

\[ - (4\mu + 3K) dv \pm 3\lambda d\sigma_{rr} = 0, \]

respectively.

If a pressure exceeding the plasticity limit, \( p > p_0 \), is suddenly applied to the surface of the cavity, then the characteristic line \( r - r_0 - \lambda t = 0 \) will be a strong discontinuity. Along the discontinuity additional conditions must be satisfied, which will be called the conditions of kinematic and dynamic continuity. For the spherical discontinuity these conditions have the form (see for instance G. Hopkins [82]):

(3.52) \( v + \lambda \varepsilon_{rr} = 0, \)

(3.53) \( \rho \lambda v + \sigma_{rr} = 0, \)

respectively.

Since the strain rate \( \dot{\varepsilon}_{rr} \) at the discontinuity must be regarded as infinite, the constitutive equations give the relation (see [119])

(3.54) \( \sigma_{\phi\phi} = \sigma_{rr} \left( 1 - \frac{2\mu}{\rho \lambda^2} \right). \)

Using the relation which is satisfied along the characteristic \( r - r_0 - \lambda t = 0 \), and (3.52) through (3.54), we obtain

(3.55) \( \frac{d\sigma_{rr}}{dr} = -\Psi(r, \sigma_{rr}), \)

with the condition

(3.56) \( \sigma_{rr} \bigg|_{r = r_0} = \dot{p}_0. \)
where

\begin{equation}
\Psi(r, \sigma_{rr}) = \frac{1}{r} \sigma_{rr} + \frac{2\mu\gamma}{\lambda^{\frac{1}{3}}} \Phi \left( \frac{2\mu\sigma_{rr}}{\sqrt[3]{3\rho\lambda^2}} - 1 \right).
\end{equation}

To solve Eq. (3.55) one has to show that on the front of the shock \( r = r_0 + \lambda t \) the hardening parameter \( \kappa = \kappa(W_p) \) appearing in (3.57) depends only on the variables \( r \) and \( \sigma_{rr} \). By definition (2.44) we have for spherical waves

\begin{equation}
W_p = \int_0^{r_r} \sigma_{rr} \, d\varepsilon_{rr}^p + 2 \int_0^{r_{qq}} \sigma_{qq} \, d\varepsilon_{qq}^p,
\end{equation}

but the last term vanishes, since

\begin{equation}
[e_{qq}] = \begin{bmatrix} \frac{u}{r} \\ v \\ \end{bmatrix} = 0;
\end{equation}

here \([ \ ]\) denotes the jump of the quantity in brackets across the discontinuity. Separating the component of the strain tensor \( \varepsilon_{rr} \) into the elastic part \( \varepsilon_{rr}^e \) and the plastic part \( \varepsilon_{rr}^p \) and using (3.52) and (3.53) we can write

\begin{equation}
\varepsilon_{rr}^p = \sigma_{rr} / (\lambda^2 \rho) - \varepsilon_{rr}^e.
\end{equation}

By Hooke's law and (3.54) we obtain

\begin{equation}
\varepsilon_{rr}^e = \sigma_{rr} / (\lambda^2 \rho),
\end{equation}

and after a straightforward calculation

\begin{equation}
W_p = 0.
\end{equation}

Since on the front of the wave the plastic work equals zero, the problem of the work-hardening plastic material is reduced to the perfectly plastic material, the hardening parameter \( \kappa \) being equal to the yield stress \( k \).

In the case of cylindrical waves and plane waves the proof that along the shock \( \kappa(W_p) = k \) is fully analogical and will be omitted.

Equation (3.55) with the condition (3.56) leads to the nonlinear Volterra integral equation of the second kind

\begin{equation}
\sigma_{rr} = \rho_0 - \int_{r_s}^{r} \Psi[\xi, \sigma_{rr}(\xi)] \, d\xi,
\end{equation}

where \( \Psi(r, \sigma_{rr}) \) has now the form

\begin{equation}
\Psi(r, \sigma_{rr}) = \frac{1}{r} \sigma_{rr} + \frac{2\mu\gamma}{\lambda^{\frac{1}{3}}} \Phi \left( \frac{2\mu\sigma_{rr}}{\rho\lambda^2\kappa^{\frac{1}{3}}} - 1 \right).
\end{equation}
If $\Psi$ satisfies the Lipschitz condition then the solution of (3.63) may be obtained by

$$\sigma_{rr} = \lim_{n \to \infty} \sigma_{rr}^{(n)}$$

with the recurrent relation

$$\sigma_{rr}^{(n)} = \rho_0 - \int_{r_*}^{r} \Psi[\xi, \sigma_{rr}^{(n-1)}(\xi)] d\xi,$$

and

$$\nu^{(n)} = -\sigma_{rr}^{(n)}/(\rho \lambda), \quad \sigma_{\varphi \varphi}^{(n)} = \left(1 - \frac{2\mu}{\rho \lambda^2}\right) \sigma_{rr}^{(n)},$$

$$\varepsilon_{rr}^{(n)} = \sigma_{rr}^{(n)}/(\rho \lambda^2); \quad \varepsilon_{\varphi \varphi} = 0.$$

The solution given by (3.66)-(3.67) is valid for $r < r^*$ (Fig. 39) where $r^*$ satisfies the condition

$$\sqrt{J_2(r^*)} = k.$$

The solution along the discontinuity for $r > r^*$ has the closed form

$$\sigma_{rr} = \frac{1}{R^*} \frac{r^*}{r}, \quad \nu = -\frac{1}{\rho \lambda R^*} \frac{r^*}{r}, \quad \sigma_{\varphi \varphi} = \left(1 - \frac{2\mu}{\rho \lambda^2}\right) \frac{1}{R^*} \frac{r^*}{r},$$

$$\varepsilon_{rr} = (\rho \lambda^2 R^*)^{-1} \frac{r^*}{r}, \quad \varepsilon_{\varphi \varphi} = 0.$$

Fig. 39. Elastic and plastic regions in the $t, r$-plane for the case of discontinuous front wave.
where

\[(3.70) \quad R^* = 2\mu(\rho \lambda^3 k/3)^{-1}.\]

Consider now the solution in the elastic/viscoplastic region, i.e., region \(P^*\) in Fig. 39 which is bounded by the discontinuity \(r = r_0 - \lambda t = 0\) and the straight line \(r = r_0\). On the discontinuity the quantities \(\sigma_{rr}, \sigma_{qq}, \nu, \epsilon_{rr}\) and \(\epsilon_{qq}\) are determined by means of the solution (3.66) and (3.67), while \(\sigma_{rr}\) is known on the line \(r = r_0\) from the boundary condition.

Denoting the intersection points of the characteristic net as shown in Fig. 41 we approximate Eqs. (3.61) by the following difference equations:
In every point of the characteristic net we have a system of five algebraic

\[ \text{equations with respect to five unknowns } \sigma_{rr}, \sigma_{\varphi\varphi}, \nu, \epsilon_{rr}, \text{ and } \epsilon_{\varphi\varphi}. \]

The procedure of finite differences can be applied to the elastic/viscoplastic region in the case shown in Fig. 40. In this case we know the quantities \( \sigma_{rr}, \sigma_{\varphi\varphi}, \nu, \epsilon_{rr}, \text{ and } \epsilon_{\varphi\varphi} \) at the point \( t = t_1 \) on the line \( r = r_0 \) from the elastic solution in the region \( E^* \).

The boundary \( I^* \) of the elastic/viscoplastic region (see Figs. 39 and 40) can be obtained by an approximate method using the condition \( \sqrt{f_r(r)} = \kappa(r) \).

B. Cylindrical Radial Waves. Consider an infinite cylindrical cavity of radius \( r_0 \) in an infinite elastic/viscoplastic medium. To the surface of this cavity let there be applied the radial pressure \( p(t) \), variable in time and independent of \( \varphi \) and \( z \). In cylindrical coordinates \( r, \varphi, z \), we have

\[ u_r = u(r,t), \quad u_\varphi = u_z = 0, \]

where \( u_r, u_\varphi, u_z \) are the cylindrical components of displacement. The components of the strain tensor and the components of the stress tensor are

\[ \epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\varphi\varphi} = \frac{u}{r}, \quad \epsilon_{zz} = 0. \]
\[(3.74) \quad \sigma_{rr} = \sigma_{rr}(r,t), \quad \sigma_{pp} = \sigma_{pp}(r,t), \quad \sigma_{ss} = \sigma_{ss}(r,t).\]

The system of differential equations which describes the problem has the form (3.1) with

\[
U = \begin{bmatrix} u \\ \sigma_{rr} \\ \sigma_{pp} \\ \sigma_{ss} \\ \varepsilon_{rr} \\ \varepsilon_{pp} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 \\ -\left(\frac{3}{2}\mu + K\right) & 0 & 0 & 0 & 0 \\ \frac{3}{2}\mu - K & 0 & 0 & 0 & 0 \\ \frac{3}{2}\mu - K & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[(3.75) \quad B = \begin{bmatrix} -\frac{\sigma_{rr} - \sigma_{pp}}{r} \\ \left(\frac{2}{3}\mu - K\right) \frac{v}{r} + \frac{2}{3} \mu \gamma \Phi \left(J_2^{1/2} \frac{1}{\kappa} - 1\right) \\ -\left(\frac{4}{3}\mu + K\right) \frac{v}{r} - \frac{4}{3} \mu \gamma \Phi \left(J_2^{1/2} \frac{1}{\kappa} - 1\right) \\ \left(\frac{2}{3}\mu - K\right) \frac{v}{r} + \frac{2}{3} \mu \gamma \Phi \left(J_2^{1/2} \frac{1}{\kappa} - 1\right) \\ 0 \\ -\frac{v}{r} \end{bmatrix},\]

where

\[(3.76) \quad J_2 = \frac{1}{2} \left[ (\sigma_{rr}^2 + \sigma_{pp}^2 + \sigma_{ss}^2) - (\varepsilon_{rr}\sigma_{pp} + \sigma_{pp}\sigma_{ss} + \sigma_{ss}\varepsilon_{rr}) \right].\]

The characteristic lines (3.4) now have form

\[(3.77) \quad r = \text{const}, \quad r = r_0 \pm \lambda t + \text{const},\]

where \(r = \text{const}\) is a fourfold characteristic. Along these lines the following conditions hold by (3.5):
\[ \left( 1 - \frac{3K}{2\mu} \right) d\sigma_{rr} + \left( 1 + \frac{3K}{2\mu} \right) d\sigma_{\varphi\varphi} + d\sigma_{ss} - 3K \left[ \gamma \Phi \left( \frac{J_2^{1/2}}{\kappa} - 1 \right) + \frac{2\nu}{r} \right] dt = 0, \]

\[ 2\mu \left[ \gamma \Phi \left( \frac{J_2^{1/2}}{\kappa} - 1 \right) + \frac{\nu}{r} \right] dt + d\sigma_{rr} - d\sigma_{\varphi\varphi} = 0, \]

\[ d\varepsilon_{\varphi\varphi} = \frac{\nu}{r} dt, \]

(3.78)

\[ d\varepsilon_{rr} - \frac{1}{3K} (d\sigma_{rr} + d\sigma_{\varphi\varphi} + d\sigma_{ss}) + \frac{\nu}{r} dt = 0, \]

\[ \frac{3K + 4\mu}{\mu} d\nu + \frac{3\lambda}{\mu} d\sigma_{rr} + \left[ \frac{\nu}{r} \left( 2 - \frac{3K}{\mu} \right) + 2\nu \Phi \left( \frac{J_2^{1/2}}{\kappa} - 1 \right) \right] \frac{3\lambda}{\mu} \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} dr = 0. \]

On the cylindrical discontinuity the conditions of kinematic and dynamic continuity have the same form as in the spherical case (cf. (3.52) and (3.53)).

The property that \( \varepsilon_{rr} \) tends to infinity on the cylindrical shock wave leads to the relations

\[ \sigma_{\varphi\varphi} = \sigma_{rr} - 2\mu\varepsilon_{rr}, \]

(3.79)

\[ \sigma_{ss} = \sigma_{rr} - 2\mu\varepsilon_{rr}. \]

Using the relation satisfied along the characteristic line \( r - r_0 - \lambda t = 0 \), and (3.52) and (3.53), and relations (3.79) we can reduce the problem to the same Volterra integral equation as in spherical case (cf. (3.63)), where

\[ \psi(r,\sigma_{rr}) = \frac{1}{2} \frac{\sigma_{rr}}{r} + \frac{\nu}{3\lambda} \Phi \left( \frac{2\mu\sigma_{rr}}{\rho\lambda^2 k \sqrt{3}} - 1 \right), \]

(3.80)

and

\[ v^{(n)} = -\frac{1}{\rho} \sigma_{rr}^{(n)}, \quad \sigma_{\varphi\varphi}^{(n)} = \sigma_{ss}^{(n)} = \left( 1 - \frac{2\mu}{\rho\lambda^2} \right) \sigma_{rr}^{(n)} \]

(3.81)

\[ \varepsilon_{\varphi\varphi}^{(n)} = 0, \quad \varepsilon_{rr}^{(n)} = \frac{1}{\lambda^2 \rho} \sigma_{rr}^{(n)}. \]

The solution on the cylindrical discontinuity for \( r > r^* \) has the following closed form:
where $R^*$ is determined by (3.70).

The problem in inelastic regions is the same as in the spherical case and the difference equations are now (see Fig. 41)

\[
\left(1 - \frac{3K}{2\mu}\right) [\sigma_{rr(l,m,n)} - \sigma_{rr(l-1,m-1,n)}] + \left(1 + \frac{3K}{2\mu}\right) [\sigma_{qq(l,m,n)} - \sigma_{qq(l-1,m-1,n)}]
\]

\[+ \left[\sigma_{zz(l,m,n)} - \sigma_{zz(l-1,m-1,n)}\right] - 3K \left[\gamma \Phi \left(\frac{J_{2.1/2}^2}{\kappa} - 1\right) + \frac{2u}{r}\right] (l-1,m-1,n) \Delta t = 0,
\]

\[
\varepsilon_{qq(l,m,n)} - \varepsilon_{qq(l-1,m-1,n)} = \left(\frac{v}{r}\right) (l-1,m-1,n) \Delta t,
\]

\[
2\mu \left[\gamma \Phi \left(\frac{J_{2.1/2}^2}{\kappa} - 1\right) + \frac{v}{r}\right] (l-1,m-1,n) \Delta t
\]

\[+ \left[\sigma_{zz(l,m,n)} - \sigma_{zz(l-1,m-1,n)}\right] - \left[\sigma_{qq(l,m,n)} - \sigma_{qq(l-1,m-1,n)}\right] = 0,
\]

\[(3.83)\]

\[
\varepsilon_{rr(l,m,n)} - \varepsilon_{rr(l-1,m-1,n)} = \frac{1}{3K} \left[\left[\sigma_{rr(l,m,n)} - \sigma_{rr(l-1,m-1,n)}\right]
\]

\[+ \left[\sigma_{qq(l,m,n)} - \sigma_{qq(l-1,m-1,n)}\right] + \left[\sigma_{zz(l,m,n)} - \sigma_{zz(l-1,m-1,n)}\right]\}

\[+ \left(\frac{v}{r}\right) (l-1,m-1,n) \Delta t = 0,
\]

\[
\left[\frac{v}{r} \left(2 - \frac{3K}{\mu}\right) + 2\gamma \Phi \left(\frac{J_{2.1/2}^2}{\kappa} - 1\right) + \frac{3\lambda}{\mu} \frac{\sigma_{rr} - \sigma_{qq}}{r}\right] (l,m-1,n-1) \Delta r
\]

\[+ \frac{3\lambda}{\mu} \left[\sigma_{rr(l,m,n)} - \sigma_{rr(l-1,n-1)}\right] - \frac{3K + 4\mu}{\mu} [v(l,m,n) - v(l,m-1,n-1)] = 0,
\]

\[
\left[\frac{v}{r} \left(2 - \frac{3K}{\mu}\right) + 2\gamma \Phi \left(\frac{J_{2.1/2}^2}{\kappa} - 1\right) - \frac{3\lambda}{\mu} \frac{\sigma_{rr} - \sigma_{qq}}{r}\right] (l-1,m,n+1) \Delta r
\]

\[- \frac{3\lambda}{\mu} \left[\sigma_{rr(l,m,n)} - \sigma_{rr(l-1,m,n+1)}\right] - \frac{3K + 4\mu}{\mu} [v(l,m,n) - v(l-1,m,n+1)] = 0.
\]

C. *Cylindrical Shear Waves.* Assume now that shearing tractions $p(t)$ are uniformly distributed on the cylindrical surface with radius $r_0$ of the
cavity in an infinite elastic/viscoplastic medium. In cylindrical coordinates \( r, \varphi, z \) we have

\[
(3.84) \quad u_\varphi = u(r, t), \quad u_r = u_z = 0.
\]

The component of the shear stress is

\[
(3.85) \quad \tau_{r\varphi} = \tau_{r\varphi}(r, t),
\]

whereas all remaining cylindrical components of stress vanish identically.

For the system (3.1) we now have

\[
U = \begin{bmatrix} v \\ \tau_{r\varphi} \\ \varepsilon_{r\varphi} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\frac{1}{\rho} & 0 \\ -\mu & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{2}{\rho} \frac{\tau_{r\varphi}}{r} \\ \mu \frac{v}{r} + 2\mu \gamma \Phi \left( \frac{\tau_{r\varphi}}{\kappa} - 1 \right) \\ -\frac{1}{2} \frac{v}{r} \end{bmatrix}.
\]

(3.86)

The characteristic lines (3.4) are

\[
(3.87) \quad r = \text{const}, \quad r = r_0 \pm \lambda t + \text{const}, \quad \text{where} \quad \lambda = \left( \frac{\mu}{\rho} \right)^{1/2},
\]

along them, by (3.5), we have the relations

\[
(3.88) \quad \frac{1}{2\mu} \frac{d\tau_{r\varphi}}{dt} - d\varepsilon_{r\varphi} + \left[ \frac{v}{r} + 2\gamma \Phi \left( \frac{\tau_{r\varphi}}{\kappa} - 1 \right) \right] dt = 0,
\]

\[
- dv \pm \frac{2\lambda}{\mu} \frac{\tau_{r\varphi}}{r} dr + \left[ 2\gamma \Phi \left( \frac{\tau_{r\varphi}}{\kappa} - 1 \right) + \frac{v}{r} \right] dr \pm \frac{\lambda}{\mu} d\varepsilon_{r\varphi} = 0.
\]

Using the conditions of kinematic and dynamic continuity

\[
(3.89) \quad v + \lambda \varepsilon_{r\varphi} = 0,
\]

\[
(3.90) \quad \lambda \rho v + \tau_{r\varphi} = 0,
\]

and the first relation (3.88), we obtain on the shear discontinuity the equation

\[
(3.91) \quad \tau_{r\varphi} = \rho_0 - \left[ \frac{1}{2\gamma} \tau_{r\varphi} + \rho \lambda \gamma \Phi \left( \frac{\tau_{r\varphi}}{k} - 1 \right) \right] dr.
\]
The iteration of the order \( n \) is

\[
\tau_{rp}^{(n)} = \rho_0 - \int_0^r \Psi(\xi, \tau_{rp}^{(n-1)}(\xi)) d\xi.
\]

where

\[
\Psi(r, \tau_{rp}) = \frac{1}{2} \frac{\tau_{rp}}{r} + \rho \lambda \gamma \Phi \left( \frac{\tau_{rp}}{k} - 1 \right),
\]

and

\[
v^{(n)} = -\frac{1}{\rho \lambda} \tau_{rp}^{(n)}, \quad \varepsilon_{rp}^{(n)} = \frac{1}{\rho \lambda^2} \tau_{rp}^{(n)}.
\]

The condition for \( r^* \) is

\[
\tau_{rp}(r^*) = k.
\]

The solution on the shear discontinuity for \( r > r^* \) is

\[
\tau_{rp} = k \sqrt{\frac{r^*}{r}}, \quad v = -\frac{k}{\rho \lambda} \sqrt{\frac{r^*}{r}}, \quad \varepsilon_{rp} = \frac{k}{\rho \lambda^2} \sqrt{\frac{r^*}{r}}.
\]

In the inelastic regions we can again apply the method of finite differences along the characteristic lines (see Fig. 41) by using

\[
\left[ \frac{v}{r} + 2 \gamma \Phi \left( \frac{\tau_{rp}}{\kappa} - 1 \right) \right]_{(l, m - 1, n - 1)} \Delta t - \left[ \varepsilon_{rp(l, m, n)} - \varepsilon_{rp(l - 1, m - 1, n)} \right] = 0,
\]

\[
\left[ \frac{2 \lambda}{\mu^2} \tau_{rp} + \frac{v}{r} + 2 \gamma \Phi \left( \frac{\tau_{rp}}{\kappa} - 1 \right) \right]_{(l, m - 1, n - 1)} \Delta r
\]

\[
- \left[ v_{(l, m, n)} - v_{(l, m - 1, n - 1)} + \frac{\lambda}{\mu} \left[ \tau_{rp(l, m, n)} - \tau_{rp(l, m - 1, n - 1)} \right] = 0,
\]

\[
\left[ -\frac{2 \lambda}{\mu^2} \tau_{rp} + \frac{v}{r} + 2 \gamma \Phi \left( \frac{\tau_{rp}}{\kappa} - 1 \right) \right]_{(l - 1, m, n + 1)} \Delta r
\]

\[
- \left[ v_{(l, m, n)} - v_{(l - 1, m, n + 1)} + \frac{\lambda}{\mu} \left[ \tau_{rp(l, m, n)} - \tau_{rp(l - 1, m, n + 1)} \right] = 0.
\]

(3.97)
D. **Plane Waves in a Half Space.** Let \( \tilde{x}, \tilde{y}, \tilde{z} \) be Cartesian coordinates and consider an elastic/viscoplastic medium occupying the half space \( \tilde{x} \geq 0 \). Suppose that the plane \( \tilde{x} = 0 \) is exposed to the uniform pressure \( \tilde{p}(t) \).

The displacement field is characterized by

\[
(3.98) \quad u_x = u(\tilde{x}, t), \quad u_y = u_z = 0.
\]

The strain tensor has only one component that is not identically zero, namely

\[
(3.99) \quad \varepsilon_{xz} = \varepsilon_{xz}(\tilde{x}, t),
\]

and the normal stresses are independent of \( \tilde{y} \) and \( \tilde{z} \): \( \sigma_{xx}(\tilde{x}, t), \sigma_{yy}(\tilde{x}, t) = \sigma_{zz}(\tilde{x}, t) \), while the shearing stresses vanish identically.

The system (3.1) describes the plane problem in the inelastic regions, where

\[
(3.100) \quad U = \begin{bmatrix} \varepsilon \\ \sigma_{xz} \\ \varepsilon_{xz} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\frac{1}{\rho} & 0 \\ -\frac{3K + 4\mu}{3} & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},
\]

\[
B = \frac{4\mu \sqrt{3}}{3} \gamma \Phi \left[ \frac{3(\sigma_{xx} - K\varepsilon_{xx})}{2\sqrt{3\kappa}} - 1 \right].
\]

The characteristics (3.4) now take the form

\[
(3.101) \quad \tilde{x} = \text{const}, \quad \tilde{t} = \pm \lambda \tilde{t} + \text{const};
\]

along them the following conditions must be satisfied by (3.5):

\[
(3.102) \quad \frac{4\mu \sqrt{3}}{3} \gamma \Phi \left[ \frac{3(\sigma_{xx} - K\varepsilon_{xx})}{2\sqrt{3\kappa}} - 1 \right] d\tilde{t} + d\sigma_{xz} - \rho \lambda^2 d\varepsilon_{xz} = 0,
\]

\[
\pm \frac{4\mu \sqrt{3}}{3} \gamma \Phi \left[ \frac{3(\sigma_{xx} - K\varepsilon_{xx})}{2\sqrt{3\kappa}} - 1 \right] d\tilde{x} + d\sigma_{zz} \mp \rho \lambda d\nu = 0.
\]

Using the condition of kinematic continuity

\[
(3.103) \quad \varepsilon_{xz} + \frac{1}{\lambda} \nu = 0,
\]
the condition of dynamic continuity

\[(3.104) \quad \rho \lambda v + \sigma_{xx} = 0,\]

and the relation which is satisfied along the characteristic line \( \tilde{x} = \mu \), we obtain the following equation on the plane discontinuity:

\[(3.105) \quad \sigma_{xx} = \rho_0 - \int_{0}^{\tilde{x}} \Psi[\xi, \sigma_{xx}(\xi)] d\xi,\]

where

\[(3.106) \quad \Psi(\tilde{x}, \sigma_{xx}) = M + \gamma \Phi \left[ \frac{\sqrt{3}}{2k} \left( 1 - \frac{K}{\rho \lambda^2} \right) \sigma_{xx} - 1 \right],\]

and

\[(3.107) \quad M = \frac{2 \mu \sqrt{3}}{3 \lambda} .\]

Using iterations we may write the solution of Eq. (3.105) in the form

\[(3.108) \quad \sigma_{xx} = \lim_{n \to \infty} \sigma_{xx}^{(n)},\]

where

\[(3.109) \quad \sigma_{xx}^{(n)} = \rho_0 - \int_{0}^{\tilde{x}} \Psi[\xi, \sigma_{xx}^{(n-1)}(\xi)] d\xi,\]

and

\[(3.110) \quad \nu^{(n)} = -\frac{1}{\rho \lambda} \sigma_{xx}^{(n)}, \quad \varepsilon_{xx}^{(n)} = \frac{1}{\rho \lambda^2} \sigma_{xx}^{(n)}.\]

The condition for \( \tilde{x}^* \) (see Fig. 39) is

\[(3.111) \quad \sqrt{f_2(\tilde{x}^*)} = k.\]

The solution on the plane discontinuous wave for \( \tilde{x} > \tilde{x}^* \) is

\[(3.112) \quad \sigma_{xx} = \frac{2k}{\sqrt{3}} \left( 1 - \frac{K}{\rho \lambda^2} \right)^{-1}, \quad v = -\frac{2k}{\sqrt{3} \rho \lambda} \left( 1 - \frac{K}{\rho \lambda^2} \right)^{-1}, \quad \varepsilon_{xx} = \frac{2k}{\sqrt{3} \rho \lambda^2} \left( 1 - \frac{K}{\rho \lambda^2} \right)^{-1}.\]
Denoting the intersection points of the characteristic net as shown in Fig. 41, we can write the difference equations

\[ 4\mu \frac{\sqrt{3}}{3} \gamma \Phi \left[ \frac{\sqrt{3}(\sigma_{xx} - K\varepsilon_{xx})}{2\kappa} - 1 \right] \Delta t + \left[ \sigma_{xx}(i,m,n) - \sigma_{xx}(i-1,m,n) \right] = 0, \]

\[ - \rho\lambda^2 [\varepsilon_{xx}(i,m,n) - \varepsilon_{xx}(i-1,m,n)] = 0, \]

\[ 4\mu \frac{\sqrt{3}}{3\lambda} \gamma \Phi \left[ \frac{\sqrt{3}(\sigma_{xx} - K\varepsilon_{xx})}{2\kappa} - 1 \right] \Delta \bar{x} + \left[ \sigma_{xx}(i,m,n) - \sigma_{xx}(i,m-1,n) \right] = 0, \]

\[ - \rho\lambda [v(i,m,n) - v(i,m-1,n-1)] = 0, \]

\[ + \rho\lambda [v(i,m,n) - v(i-1,m,n+1)] = 0. \]

(3.113)

4. Application of the Method of Successive Approximations

Some problems of one-directional propagation of stress waves in an inelastic medium described by the system (3.1) may be reduced to second order partial differential equations. In these cases, the application of the method of successive approximations is easier, and solutions of the very general initial-boundary-value problems are available (cf. Section 1A and 1B). It will be proved that some problems previously discussed may be reduced to the solution of one of two general non-linear problems (see Section 1, Theorems 1 and 2).

The method of successive approximations permits a full discussion and examination of the convergence of the solution obtained.

As an example of application of the successive approximation method, we shall study the problem of propagation of shear waves in an infinite elastic/viscoplastic medium (see [139]).

The problem of propagation of shear waves in an infinite elastic/viscoplastic medium was first treated by V. V. Sokolovsky [163]. Certain generalizations were discussed in [130, 131, 135]. In all previous papers, the solution was obtained by means of finite differences taken on the characteristic net (compare with Section 3).

Assume now that shearing tractions \( \rho(t) \) or shear strain rates \( \omega(t) \) are uniformly distributed on the cylindrical surface with radius \( r_0 \) of the cavity in an infinite elastic/viscoplastic medium. In the cylindrical coordinates \( r, \varphi, z \), we have
The component of the shear stress is
\begin{equation}
\tau = \tau_{\theta \phi}(r,t),
\end{equation}
whereas all the remaining cylindrical components of stress vanish identically.

Case A. *Non-homogeneous material; nonlinear function $\Phi = \Phi[(\tau/k(r)) - 1]$.

It will be assumed that all functions describing the mechanical properties of the material vary with the radius $r$ only. We then obtain the semi-linear system of differential equations
\begin{equation}
U_t + AU_r + B = 0,
\end{equation}
where
\begin{equation}
U = \begin{bmatrix}
\tau \\
v
\end{bmatrix}, \quad
A = \begin{bmatrix}
0 & -\mu(r) \\
- \frac{1}{\rho(r)} & 0
\end{bmatrix},
\end{equation}
\begin{equation}
B = \begin{bmatrix}
\mu(r) \left[ \frac{v}{r} + 2\gamma(r)\Phi \left( \frac{\tau}{k(r)} - 1 \right) \right] \\
- \frac{2\tau}{r \rho(r)}
\end{bmatrix},
\end{equation}
and either one of the following two boundary conditions
\begin{enumerate}
\item \begin{equation}
[\tau(r,t)]_{r=r_0} = \dot{\rho}(t),
\end{equation}
\item \begin{equation}
\left[ \frac{\partial v}{\partial r} - \frac{v}{r} \right]_{r=r_0} = \omega(t).
\end{equation}
\end{enumerate}
The system (3.116) is hyperbolic, hence the eigenvalues of the matrix $A$
\begin{equation}
\lambda_{1,2}(r) = \pm [\mu(r)/\rho(r)]^{1/2}
\end{equation}
are real. If shearing tractions $\dot{\rho}(t)$ or shear strain rate $\omega(t)$ exceeding the plasticity limit, $\rho_0 > \rho_\gamma$ or $\omega_0 > \omega_\gamma$, are suddenly applied to the surface

* The problems which will be studied in the Case A are much more general than those discussed in Refs. [130, 131, 135].
of the cavity, then the characteristic line \( r - r_0 - \lambda(r)t = 0 \) will be a strong discontinuity (see Fig. 42).

By introducing new coordinates

\[
(3.120) \quad x = t + \int_{r_0}^{r} \lambda^{-1}(\xi) d\xi, \quad y = t - \int_{r_0}^{r} \lambda^{-1}(\xi) d\xi.
\]

the equations of motion (3.116) valid in the elastic/viscoplastic region \( \mathcal{D}^* \) (see Fig. 42) may be reduced to the second order partial differential hyperbolic equation:

\[
(3.121) \quad \tau_{xy} = f(x, y, \tau, \tau_x, \tau_y).
\]

\[\text{Fig. 42. The plastic region } \mathcal{D}^* \text{ in the } t, r \text{-plane.}\]

where

\[
f = \left\{ \frac{1}{4} \left[ \frac{\lambda(r)}{r} - \frac{\lambda'(r)}{\rho(r)} \frac{d\rho(r)}{dr} \right] (\tau_x - \tau_y) - \frac{\lambda^2(r)\gamma(r)\rho(r)}{2k(r)} \Phi \left[ \frac{\tau}{k(r)} - 1 \right] (\tau_x + \tau_y) \right. \\
- \lambda^2(r) \left[ \frac{1}{r^2} + \frac{1}{2\rho(r)} \frac{d\rho(r)}{dr} \right] \tau \left[ r = \lambda^{-1}(s - y)(\xi) \right].
\]

(3.122)

and

\[
(3.123) \quad f(r) = \int_{r_0}^{r} \lambda^{-1}(\xi) d\xi.
\]
The elastic/viscoplastic region $\mathcal{D}^*$ at Fig. 42 now takes the form of the region $\mathcal{D}$ at Fig. 43. The region $\mathcal{D}$ is bounded by the characteristic $y = 0$ and the line $x = y$ (not a characteristic).

The boundary conditions (3.118) now become

$$y = 0, \tau = \dot{\rho}(y) - \int_{\tau_0}^{\tau} \left\{ \frac{1}{2\eta} - \frac{(d/d\eta)(\lambda(\eta)\rho(\eta))}{2\lambda(\eta)\rho(\eta)} \right\} \tau + \rho(\eta)\lambda(\eta)\gamma(\eta)\Phi \left[ \frac{\tau}{\dot{k}(\eta)} - 1 \right] \right\} d\eta,$$

(3.124)

\[ x = y, \]

\[ \begin{cases} 1. \tau = \dot{\rho}(y), \\ 2. \tau_y = h(y,\tau,\tau_x) \end{cases} \]

**Fig. 43.** The plastic region $\mathcal{D}$ in the $x,y$-plane.

where

$$h := - \tau_x - 2\gamma(r_0)\mu(r_0)\Phi \left[ \frac{\tau}{\dot{k}(r_0)} - 1 \right] + \mu(r_0)\omega(y).$$

(3.125)

Thus, in either case the problem is reduced to the solution of a non-linear boundary-value problem for a semi-linear hyperbolic equation (cf. (3.29)).

Let us study in detail the second case of (3.124). Consider the linear space $C_{\mathcal{D}}^1$ of the functions $\tau(x,y)$. The norm in this space has the form

$$||\tau(x,y)|| = \sup_{\mathcal{D}} |\tau(x,y)| + \sup_{\mathcal{D}} |\tau_x(x,y)| + \sup_{\mathcal{D}} |\tau_y(x,y)|.$$

(3.126)

Assuming that the function $f$ given by (3.123), the integrand in (3.124), and the function $h$ given by (3.125), satisfy the Lipschitz inequalities with respect to $\tau$, $\tau_x$ and $\tau_y$, and the constants involved in these inequalities satisfy the restrictions (3.9), we can write the mapping $\mathcal{D}$ for that problem in the form
In practice, we shall have following iterative scheme:

\[
\tau_{(n+1)}(x,y) = \mathcal{A}[\tau_{(n)}(x,y)],
\]

\[
\tau_{x(n+1)}(x,y) = \mathcal{A}_x[\tau_{(n)}(x,y)],
\]

\[
\tau_{y(n+1)}(x,y) = \mathcal{A}_y[\tau_{(n)}(x,y)].
\]

The above restrictions imply that \(\{\tau_{(n)}(x,y)\}, \{\tau_{x(n)}(x,y)\}\) and \(\{\tau_{y(n)}(x,y)\}\) converge uniformly to the continuous limit functions \(\tau(x,y), \tau_x(x,y)\) and \(\tau_y(x,y)\), which constitute a solution of our problem.

To give an interpretation of these restrictions, and the limitations which they impose on the class of the materials, let us first study the Lipschitz conditions.

The functions \(f, h\), and the integrand in (3.124) satisfy the Lipschitz inequalities if they have at every point of the region \(\mathcal{D}\) bounded partial derivatives with respect to \(\tau, \tau_x, \) and \(\tau_y\) Thus we have to assume that the function \(\Phi[(\tau/k(r)) - 1]\) belongs to the class \(C^2\) with respect to \([\tau/k(r)] - 1\], and that the functions describing the physical properties of the material satisfy: \([\mu(r), \rho(r)] \in C^1, [\gamma(Y), \lambda(Y)] \in Co\), with respect to \(r\).

**Case B. Non-homogeneous material; linear function \(\Phi = (\tau/k(r)) - 1\).**

In this case, the equation of motion takes the linear form

\[
\tau_{xy} + a(x,y)\tau_x + b(x,y)\tau_y + c(x,y)\tau = 0,
\]

where

\[
a(x,y) = -\left[ \frac{1}{4} \left( \frac{\lambda(r)}{r} - \frac{\lambda(r)}{\rho(r)} \frac{d\rho(r)}{dr} - \frac{d\lambda(r)}{dr} - \frac{\lambda^2(r)\gamma(r)\rho(r)}{2k(r)} \right) \right]_{r = r - (x-y)/\beta},
\]
We seek the solution of (3.129), which satisfies the following boundary conditions

\[ y = 0, \quad \tau(x,0) = \sigma_1(x), \]

\[ x = y, \quad \begin{cases} 
1. & \tau(y,y) = \rho(y), \\
2. & \beta_0(y)\tau(y,y) + \beta_1(y)\tau_x(y,y) + \beta_2(y,y) = \sigma_0(y),
\end{cases} \]

where

\[ \sigma_1(x) = \left\{ \begin{array}{l}
\rho_0 - \int_{r_0}^{r} \left( \rho(\xi)\lambda(\xi)\gamma(\xi) \exp \left( \int_{r_0}^{\xi} \frac{1}{2\eta} - \frac{(d/d\eta)({\lambda(\eta)\rho(\eta)})}{2\lambda(\eta)\rho(\eta)} \right) \right. \\
\left. + \rho(\eta)\lambda(\eta)\frac{\gamma(\eta)}{k(\eta)} \left. \right|_{\eta} = 1 \right\} \exp \left( - \int_{r_0}^{x} \frac{1}{2z} - \frac{(d/dz)(\lambda(z)\rho(z))}{2\lambda(z)\rho(z)} \right) \right. \\
+ \rho(z)\lambda(z)\frac{\gamma(z)}{k(z)} \right. \\
\left. \int_{r_0}^{z} \right \} \\
\beta_0(y) = \frac{2\gamma_0\rho_0}{k_0}, \quad \beta_1(y) = 1, \quad \beta_2(y) = 1, \quad \sigma_0(y) = \mu_0[2\gamma_0 + \omega(y)].
\]

Thus the problem in the region \( \mathcal{D} \) (see Fig. 43) for the linear function \( \Phi \) is reduced to the solution of the special case of the linear problem (3.39).

**Case C.** Homogeneous material; linear function \( \Phi = \tau/k_0 - 1 \). This case was earlier discussed by V. V. Sokolovsky [163], who solved it by the finite difference method along characteristic lines.

For that case, the coefficients of (3.129) have the form

\[ a(x,y) = -\frac{\lambda_0}{2} \left\{ [2\gamma_0 + \lambda_0(x-y)]^{-1} - \lambda_0\gamma_0\rho_0/k_0 \right\}, \]

\[ b(x,y) = \frac{\lambda_0}{2} \left\{ [2\gamma_0 + \lambda_0(x-y)]^{-1} + \lambda_0\gamma_0\rho_0/k_0 \right\}, \]

\[ c(x,y) = 4\lambda_0^2 [2\gamma_0 + \lambda_0(x-y)]^{-3}. \]

The boundary conditions have a form similar to that of the previous case.
Solutions in the elastic region by successive approximations have been discussed in [130, 131, 135, 136]. In this section we shall follow the results presented in [135, 136].

By introducing new coordinates

$$x - 1 = \frac{\lambda}{2r_0} \left( t + \frac{r - r_0}{\lambda} \right), \quad y = \frac{\lambda}{2r_0} \left( t - \frac{r - r_0}{\lambda} \right),$$

for spherical waves, cylindrical radial waves, and cylindrical shear waves, and new coordinates

$$x - \frac{2x_0}{\lambda} = t + \frac{\bar{x}}{\lambda}, \quad y = t - \frac{\bar{y}}{\lambda},$$

(3.135)

where $x_0 = \text{const}$ for plane waves, the equation of motion in the elastic region may be written in the form

$$u_{xy} = \frac{\lambda}{(x - y)^2} u.$$  

The considered elastic regions in the $x,y$-plane are shown in Fig. 44. Equation (3.136) is valid in the regions $E_1$ and $E_2$.

Consider first the problem in the region $E_1$. This region is bounded by the characteristic $y = 0$ and the curve $x = \varphi(y)$. On $y = 0$, we have $u = 0$; on $x = \varphi(y)$ in the cases of spherical waves, shear waves, and plane waves the linear condition

$$c_1(u_x - u_y) + c_2u = c_3,$$

holds, and for cylindrical radial waves the nonlinear condition

$$u_y = \chi(y,u,u_x),$$

(3.137)
where
\[ \chi(y,u,u_y) = u_z + \frac{u}{y - \varphi(y)} \pm \left[ \frac{3k^2r_0^2}{\mu^2} - 3 \left( \frac{u}{y - \varphi(y)} \right)^3 \right]^{1/2}. \]

must be satisfied.

Hence for all four types of waves the problem in region $E_1$ reduces to the generalized Picard problem (see G. Majcher, [108]).

The solution of this generalized Picard problem may be expressed as follows

\begin{equation}
(3.139) \quad u(x,y) = \int_0^y V(x,y;x_0,\eta)\Omega(\eta)d\eta,
\end{equation}

where $V(x,y;\xi,\eta)$ denotes the Riemann function for (3.139) and $x_0 = \text{const}$. It is obvious that the solution (3.139) satisfies the condition $u = 0$ on $y = 0$. The function $\Omega(y)$ should be determined so that the solution satisfies the conditions (3.137) and (3.138) respectively. We obtain a Volterra equation of the second kind

\begin{equation}
(3.140) \quad \Omega(y) = \int_0^y \mathcal{N}(y,\eta)\Omega(\eta)d\eta + \mathcal{K}(y),
\end{equation}

where for spherical, cylindrical shear and plane waves

\begin{equation}
\mathcal{N}(y,\eta) = \frac{V_s(\varphi(y),y;x_0,\eta) - V_s(\varphi(y),y;x_0,\eta) - (c_1/c_2)V(\varphi(y),y;x_0,\eta)}{V(\varphi(y),y;x_0,\eta)},
\end{equation}

\begin{equation}
\mathcal{K}(y) = -\frac{c_0/c_1}{V(\varphi(y),y;x_0,\eta)},
\end{equation}

and for cylindrical radial waves

\begin{equation}
\mathcal{N}(y,\eta) = -V_s(\varphi(y),y;x_0,\eta)[V(\varphi(y),y;x_0,\eta)]^{-1},
\end{equation}

\begin{equation}
\mathcal{K}(y) = \frac{\chi[y, \int_0^y V(\varphi(y),y;x_0,\eta)\Omega(\eta)d\eta, \int_0^y V_s(\varphi(y),y;x_0,\eta)\Omega(\eta)d\eta]}{V(\varphi(y),y;x_0,\eta)}. \tag{3.142}
\end{equation}

The solution of the integral equation (3.140) may be written in the form

\begin{equation}
(3.143) \quad \Omega(y) = \int_0^y R(y,\eta)\mathcal{K}(\eta)d\eta + \mathcal{N}(y).
\end{equation}
where $R(y, \eta)$ is the resolving kernel of Eq. (3.140), that is,

$$R(y, \eta) = \mathcal{N}(y, \eta) + \sum_{n=1}^{\infty} \mathcal{N}_n(y, \eta),$$

with

$$\mathcal{N}_n(y, \eta) = \int_{\eta}^{y} \mathcal{N}(y, \xi) \mathcal{N}_{n-1}(\xi, \eta) d\xi, \quad (\mathcal{N}_0(y) = \mathcal{N}).$$

Thus in the case of the nonlinear condition (3.138), Eq. (3.140) is equivalent to the equation

$$\Omega(y) = \int_{0}^{y} \frac{R(y, \eta)}{V(\varphi(\eta), \eta; x_0, \eta)} \chi [\eta, \]

$$

$$+ \left[ V(\varphi(y), y; x_0, y) \right]^{-1} \chi [y,]

$$

$$\int_{0}^{\eta} V(\varphi(\eta), \eta; x_0, \xi) \Omega(\xi) d\xi \left[ \int_{0}^{\eta} V_x(\varphi(\eta), \eta; x_0, \xi) \Omega(\xi) d\xi \right] d\eta$$

$$= \int_{0}^{y} V(\varphi(y), y; x_0, \xi) \Omega(\xi) d\xi \left[ \int_{0}^{y} V_x(\varphi(y), y; x_0, \xi) \Omega(\xi) d\xi \right].$$

Equation (3.146) is a nonlinear integral equation with the unknown function $\Omega(y)$. The method of successive approximations will be used to define the functions

$$\Omega(0)(y), \Omega(1)(y), \ldots, \Omega(m)(y), \ldots$$

with the following recurrence formula for $\Omega_{(n+1)}(y)$:

$$\Omega_{(n+1)}(y) = \int_{0}^{y} \frac{R(y, \eta)}{V(\varphi(\eta), \eta; x_0, \eta)} \chi [\eta,]

$$

$$+ \left[ V(\varphi(y), y; x_0, y) \right]^{-1} \chi [y,]

$$

$$\int_{0}^{\eta} V(\varphi(\eta), \eta; x_0, \xi) \Omega_n(\xi) d\xi \left[ \int_{0}^{\eta} V_x(\varphi(\eta), \eta; x_0, \xi) \Omega_n(\xi) d\xi \right] d\eta$$

$$= \int_{0}^{y} V(\varphi(y), y; x_0, \xi) \Omega_n(\xi) d\xi \left[ \int_{0}^{y} V_x(\varphi(y), y; x_0, \xi) \Omega_n(\xi) d\xi \right].$$
<table>
<thead>
<tr>
<th>Type of wave</th>
<th>$A$</th>
<th>$\delta$</th>
<th>$\omega$</th>
<th>$\beta$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>1</td>
<td>$-2$</td>
<td>$-2$</td>
<td>1</td>
<td>$E(2(1 + \nu) r_0 \sqrt{3})^{-1}$</td>
<td>$E((1 + \nu) r_0 \sqrt{3} [\varphi(y) - y])^{-1}$</td>
<td>$\kappa$</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>$\mu/(2r_0)$</td>
<td>$\mu/[r_0(\varphi(y) - y)]$</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>Plane</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$2\mu/(a\sqrt{3})$</td>
<td>0</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$\frac{1}{4}$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$\frac{1}{4}$</td>
<td>$\chi(y, u, u_x) = u_x + \frac{u}{y - \varphi(y)} \pm \left[ \frac{3\kappa \nu r_0^2}{\mu^2} - 3 \left( \frac{u}{\varphi(y) - y} \right)^2 \right]^{1/2}$</td>
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\[ \chi(y, u, u_x) = u_x + \frac{u}{y - \varphi(y)} \pm \left[ \frac{3\kappa \nu r_0^2}{\mu^2} - 3 \left( \frac{u}{\varphi(y) - y} \right)^2 \right]^{1/2} \]
The Riemann function for Eq. (3.136) has the following closed form (see R. Courant [54])

$$V(x, y; \xi, \eta) = (x - y)^{\alpha}(\xi - y)^{\beta}(x - \eta)^{\beta}F(-\beta, -\beta; 1, \zeta),$$

where $F(-\beta, -\beta; 1, \zeta)$ is the hypergeometric function and is determined by the relation

$$F(-\beta, -\beta; 1, \zeta) = 1 + \beta^2 \zeta + \frac{\beta^2(1 - \beta)^2}{2 \cdot 2!} \zeta^2 + \ldots$$

(3.150)

$$= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(-\beta + n)\Gamma(-\beta + n)}{n! \cdot n!} \zeta^n,$$

and

$$\zeta = \frac{(x - \xi)(y - \eta)}{(x - \eta)(y - \xi)}.$$  

The values of the constants $\alpha, \beta, \omega$, and $\beta$ and the coefficients $c_1, c_2$, and $c_3$ are presented in Table 3. The case of a plane wave can be treated separately. In this case the Riemann function $V(x, y; \xi, \eta) = 1$ and the solution is trivial. In the region $E_2$ (Fig. 44) the problem for all four types of waves may be reduced to a similar generalized Picard problem. But in this region it is useful to apply the well-known Fourier transform method (see A. Kromm [98] and G. Hopkins [82]).

6. Numerical Examples

We shall discuss here the numerical examples presented in the paper [136].

As a first example, the propagation of plane waves in the half space is considered. In this case the geometric dispersion does not have such a great influence on the solution as in the other cases. In the spherical problem the convergence of the method of successive approximations, due to the geometric dispersion, is so quick that the difference between the first and the second iteration is practically insignificant, whereas in the case of plane waves the first and the second iteration and for certain small regions of $\xi$ even the next iterations can lead to different results (see curves $1 - 6$ of Fig. 45). Thus it seems that in the case of plane waves the influence of the work-hardening of the material may be studied more effectively.

For simplicity of the equations in practical applications, linear and power-law forms of the function $\Phi$ and linear work-hardening of the material have been assumed. Thus $\Phi$ has the form (cf. (2.88))

$$\Phi \left[ \frac{\sqrt{J_2}}{\kappa(W_\rho)} - 1 \right] = \left[ \frac{\sqrt{J_2}}{\kappa(W_\rho)} - 1 \right]^\delta;$$  

(3.152)
the coefficient of the work-hardening $\kappa(W_p)$ for linear work-hardening material may be determined by the relation (see [79], and [82])

$$\kappa(W_p) = (mW_p + 1)k,$$

where $m$ is a constant of the material depending on the tangent modulus and the yield stress in simple shear $k$. The tangent modulus and the coefficient of viscosity $\gamma$ have been determined from the experimental data of J. Harding, E. O. Wood, and J. D. Campbell [75], while $\delta = 1$ and $\delta = 3$ have been assumed. On the basis of the consideration of Section 3D, the solution (3.108) for perfectly plastic and for work-hardening material has the same closed form

(a) for linear $\Phi$:

$$\bar{x} = -\frac{1}{MC*\gamma_f} \log \frac{C*\sigma_{xx} - 1}{C*\rho_0 - 1}, \quad C* = \frac{\sqrt{3}}{2k} \left(1 - \frac{K}{\rho\lambda^2}\right),$$

and

(b) for a power-law form of $\Phi$

$$\bar{x} = \frac{1}{2MC*\gamma_3} \left[\frac{1}{(C*\sigma_{xx} - 1)^2} - \frac{1}{(C*\rho_0 - 1)^2}\right].$$

---

**Fig. 45.** The $n$th iterations of $\sqrt{\frac{F_3}{k}}$ versus $\bar{x}$ curve.
The data assumed for mild steel are collected in Table 4.

### Table 4

<table>
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<tr>
<th>Property</th>
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<td>Upper yield stress $\sigma_0$</td>
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<td>Shear yield stress $\kappa$</td>
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<td>760 sec$^{-1}$</td>
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<td>Bulk modulus $K$</td>
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<tr>
<td>Shear modulus $\mu$</td>
<td>$0.820 \times 10^8$ kG cm$^{-2}$</td>
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</tbody>
</table>

**Fig. 46.** Curves $\sqrt{f_0/\kappa}$ versus $\tilde{x}$. Comparison of the exact linear solution with the exact power solution.

In Fig. 46 the curves $\sqrt{f_0/\kappa}$ versus $\tilde{x}$ at the discontinuity are given. Full lines represent the results of the linear solution, broken lines correspond to the power-law solution. Material constants $\gamma_1$ and $\gamma_3$ have been assumed here the same as in the case of a perfectly plastic material.
The object of the second example is to find the curve (see Fig. 39) dividing the plastic and elastic region. The influence of such parameters as duration time of loading, the shape and sign of the loading curve, constant of material γ and work-hardening parameter k on the character of the curve \( F^* \) is also discussed. The comparison of the results for work-hardening and perfectly plastic material has been presented.

Computing the third iteration by (3.65) and (3.66), we obtain the following result at the discontinuity:

\[
\sigma_{rr}^{(3)} = \bar{M} \gamma_a (R^*\rho_0) \left[ \left( R^*\rho_0 \right)^2 \left( \frac{1}{4} D^3 + \frac{17}{6} D^2 + \frac{45}{4} D + 16 \right) - R^*\rho_0 \left( D^2 + \frac{15}{2} D + 15 \right) + \left( \frac{3}{2} D + 6 \right) \right] - \bar{M} \gamma_a \rho_0 (1 - D)
\]

\[
+ \left\{ \bar{M} \gamma_a \rho_0 \left[ D - (D + 1) \log r + \frac{1}{2} (\log r)^2 \right] \right\}
\]

\[
- \bar{M} \gamma_a R^*\rho_0 \left\{ (R^*\rho_0)^2 (D^3 + 6D^2 + 15D + 16) \right\}
\]

\[
- R^*\rho_0 (3D^2 + 12D + 15) + 3(D + 2) \right) \}
\]

\[
- \frac{3}{2} D \left[ (R^*\rho_0)^2 \left( D^3 + 3D + \frac{5}{2} \right) - R^*\rho_0 (2D + 3) + 1 \right] r^2
\]

\[
+ R^*\rho_0 D^2 \left[ R^*\rho_0 \left( D + \frac{3}{4} \right) - 1 \right] r^2 - \frac{1}{4} (R^*\rho_0)^2 D^3 r^4 \}
\]

\[
+ 3\bar{M} \gamma_a R^*\rho_0 \left\{ (R^*\rho_0)^2 (D^3 + 4D + 5) - 2R^*\rho_0 (D + 2) + 1 \right\} \log r
\]

\[
- R^*\rho_0 \left[ R^*\rho_0 (D + 2) - 1 \right] (\log r)^2 + \frac{1}{3} (R^*\rho_0)^3 (\log r)^3 \right\} \right)
\]

\[
- 3\bar{M} \gamma_a (R^*\rho_0)^2 D \left[ \left( R^*\rho_0 \left( D + \frac{3}{2} \right) - 1 \right) \log r
\]

\[
- \frac{1}{2} R^*\rho_0 (\log r)^2 \}
\]

\[
= \bar{M} \gamma_a (R^*\rho_0)^2 D^2 r^2 \log r
\]

(3.156)

where

(3.157)

\[
D = \frac{\bar{M} \gamma_a}{\rho_0} (R^*\rho_0 - 1)^3.
\]
Eqs. (3.67) are employed for the determination of the quantities \(\sigma, v,\) 
\(\epsilon_{rr}\) and \(\epsilon_{pp}\) at the discontinuity.

Condition (3.67), according to (3.152) and (3.163) together with (3.156),
may now be written as follows:

\[
R^*\sigma_{rr}(r^*) = 1.
\]

\(R^*\sigma_{rr}(r^*)\)

**FIG. 47.** Curves \(\sigma_{rr}\) versus \(r/r_0\) on the discontinuous wave.

**FIG. 48.** Curve \(r^*/r_0\) versus \(\sqrt{J_2(r_0)/k}\) for spherical wave.

In Fig. 47 we have two curves \(\sigma_{rr}\) versus \(r\) at the discontinuity for applied initial pressures \(p_0\) of 5800 Kg/cm\(^2\) and 3740 Kg/cm\(^2\), all material data being assumed as for the plane wave. The character of the curve \(r^*/r_0\) versus \(\sqrt{J_2(r_0)/k}\)
is presented in Fig. 48. In the elastic/plastic region, i.e. region $P^*$ in Fig. 39, the quantities $\sigma_{rr}$, $\sigma_{\varphi\varphi}$, $\nu$, $\varepsilon_{rr}$ and $\varepsilon_{\varphi\varphi}$ are determined from (3.71). The condition defining the curve $P^*$ in the plane is

$$\sigma_{rr} - \sigma_{\varphi\varphi} = \sqrt{3}\kappa(W_p),$$

where

$$\kappa(W_p) = \left[ \left( \int_0^t \sigma_{rr} \, d\varepsilon_{rr} + 2 \int_0^t \sigma_{\varphi\varphi} \, d\varepsilon_{\varphi\varphi} \right) m + 1 \right] k,$$

and

$$\varepsilon_{rr}^p = \varepsilon_{rr} - \frac{1}{E} (\sigma_{rr} - 2\nu\sigma_{\varphi\varphi}),$$

$$\varepsilon_{\varphi\varphi}^p = \varepsilon_{\varphi\varphi} - \frac{1}{E} [(1 - \nu)\sigma_{\varphi\varphi} + \nu\sigma_{rr}].$$

A more convenient formula for calculating the parameter $\kappa$ may be used, if we take into account that

$$W_p = \int_0^t \sigma_{ij} \varepsilon_{ij}^p \, dt.$$   

Using the constitutive equation valid in the plastic region

$$\varepsilon_{rr}^p - \varepsilon_{\varphi\varphi}^p = \sqrt{3}\gamma \Phi \left( \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\sqrt{3}\kappa} - 1 \right),$$

and the condition of plastic incompressibility

$$\varepsilon_{rr}^p + 2\varepsilon_{\varphi\varphi}^p = 0,$$

from (3.162), we obtain

$$W_p = \frac{2\sqrt{3}}{3} \gamma \int_0^t (\sigma_{rr} - \sigma_{\varphi\varphi}) \Phi \left[ \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\sqrt{3}\kappa(W_p)} - 1 \right] \, dt.$$  

With the notation $P(l,m,n)$ for the intersection point of the characteristic net as in Fig. 41, the following recurrence formula for determination of the parameter $\kappa$ for any fixed $n_0$ has been employed:
Fig. 49. Influence of the time duration on the character of boundary $I^*$ for linear form of pressure.

Fig. 50. Influence of the time duration on the character of boundary $I^*$ for power form of pressure.
This way of obtaining the $s$th iteration has been chosen in the numerical computations in order to satisfy the condition $|\kappa^{(s+1)} - \kappa^{(s)}| < 1$. The absolute value of an intensity $\sigma_{rr} - \sigma_{qp}$ has been taken because $W_p$ from the definition of the plastic work, should be positive. The following pressure functions have been assumed

$$
(3.167) \quad \rho(\tau) = \rho_0(2 - \tau), \quad \rho(\tau) = \rho_0(-2\tau^2 + 0.18\tau + 1.82),
$$

where $\tau = 2t/t_1$ is half of the loading period.

---

**Fig. 51.** Influence of the viscosity constant $\gamma$ on the shape of the curve $I^*$. 

In Figs. 49–51 and in Table 1 the results are presented. In Figs. 49 and 50 the curves $I^*$ for linear and quadratic form of $\rho(\tau)$ and for various loading periods have been plotted. It is seen that if the period increases, the character of $I^*$ changes. The influence of the material constant $\gamma$ on the shape of the curve $I^*$ may be observed in Fig. 51. The difference is seen only at the shock and in points lying close to it. From the results collected in Table 5 the small influence of the work-hardening effect is seen. However, it should be taken into consideration that the hardening parameter $\kappa$ depends first of all on the duration of the process of plastic deformation and on the amount
Table 5

\[ \Delta r = 0.04 \quad \Delta r = 0.01 \]

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\( t = 1.34576 \cdot 10^{-7} \)

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of pressure applied. In our example, the duration of pressure application is very short, and the process of plastic deformation is not yet fully developed.

The comparison of the results of the work-hardening and perfectly-plastic theories shows that in practical applications the influence of the work-hardening may be neglected at least for certain regions of strain rate and for initial plastic deformation.

IV. QUASI-STATIC SOLUTIONS

1. Spherical Problem

The quasi-static problem of a thick-walled spherical container with an elastic/viscoplastic material has been studied in Refs. [191, 194]. Two cases of boundary conditions have been treated. In the first a constant pressure $p$ is assumed and in the second a constant displacement $u_0$, both prescribed on the interior surface of the sphere.

We will consider in detail the first case. The full system of differential equations involving equation of equilibrium and constitutive equations, according to (3.1) and (3.49), has the form

$$\frac{\partial \sigma_{rr}}{\partial r} + 2 \frac{\sigma_{rr}}{r} - \sigma_{\theta \phi} = 0,$$

$$\frac{\partial \sigma_{rr}}{\partial r} - \frac{v}{r} = \frac{1}{2\mu} \frac{\partial}{\partial t} (\sigma_{rr} - \sigma_{\theta \phi}) + \sqrt{3} \gamma \Phi(F),$$

$$3K \left( \frac{\partial v}{\partial r} + 2 \frac{v}{r} \right) = \frac{\partial}{\partial t} (\sigma_{rr} + 2 \sigma_{\theta \phi}),$$

where the static yield function $F$ is defined by $F = \sqrt{\frac{3}{2}} k - 1$.

There exists a closed-form solution of the system (4.1), if we assume the linear function $\Phi(F) = F$ and a rectangular pressure pulse

$$\sigma_{rr}(a,t) = \rho H(t), \quad \sigma_{rr}(b,t) = 0,$$

where $a$ and $b$ denote the inner and outer radii respectively and $H(t)$ is the Heaviside function.

The solution of the system (4.1) by means of Laplace transform leads to the following formulae

$$\sqrt{J_3/k} = \frac{1}{3} \left\{ -\frac{3}{2} \left( \frac{b}{r} \right)^3 \left[ \left( \frac{b}{a} \right)^3 - 1 \right]^{-1} \left[ - \rho + 2k(1 - \exp(-m^* t)) \log \frac{b}{a} \right] + k \sqrt{3} [1 - \exp(-m^* t)] \right\},$$
where

\( u(r,t) = \left[ \frac{1}{3K} + \frac{1}{4\mu} \left( \frac{b}{r} \right)^3 \right] \left[ \left( \frac{b}{a} \right)^3 - 1 \right]^{-1} r \rho \)

\[ + \frac{1}{2} \left( \frac{b}{r} \right)^3 \left[ \left( \frac{b}{a} \right)^3 - 1 \right]^{-1} \gamma \left( \frac{b}{k} - 2 \sqrt{3} \log \frac{b}{a} \right) t \]

\[ + \left[ \left( \frac{b}{r} \right)^3 - \left( \frac{b}{a} \right)^3 \right] \left[ \left( \frac{b}{a} \right)^3 - 1 \right]^{-1} \frac{2\sqrt{3} k}{3K} \left[ 1 - \exp(-m^* t) \right] r \log \frac{b}{a}, \]

where

\[ m^* = 2\gamma \frac{\mu}{k} \frac{3K}{3K + 2\mu}. \]

![Graph](image)

**Fig. 62.** Curves \( J^2/k \) versus \( r/a \) for different time parameters (Ref. [194]).

The formulae for the components of the stress tensor \( \sigma_{rr}, \sigma_{\phi\phi} \) and \( \sigma_{\theta\theta} \) are given in [191], where the influence of viscosity and time on the stress distribution are also discussed. The interaction between elastic and inelastic
components of the stress tensor causes a certain levelling off in the distribution of $J_2$ along the radius $r$, Fig. 52.

2. Viscoplastic Flow of a Circular Plate

E. J. Appleby and W. Prager [4] have treated the viscoplastic flow of a circular plate that is simply supported along its edge and subjected to a uniformly distributed transverse load. The material of the plate is supposed to be incompressible and rigid/viscoplastic according to the linearized theory proposed by W. Prager [145] (see Chapter II, Section 12). Thus the solution has been based upon the Tresca yield condition. A similar problem based upon the Huber-Mises yield condition has been treated in Ref. [193]. We follow here the presentation given there.

The plate remains undisturbed if the applied pressure $p(t)$ does not reach the value of load carrying capacity $p = 6.51 \frac{M_0}{R^2}$, (see for instance [81]), where $R$ is the radius of the plate and $M_0 = \sigma_0 h^2$ the fully-plastic yield moment. The thickness of the plate is denoted by $2h$ and the yield stress in uniaxial tension by $\sigma_0$. The consequence of the assumed model of the material is that the load intensity $p' = p(R^2/M_0)$ can exceed the value $p' = 6.51$ and the displacement rate and displacement fields can be uniquely determined.

In the cylindrical coordinate system $r$, $\varphi$, $z$, ($z$ vertically downward) the only nonvanishing components of the stress tensor are the radial and circumferential stresses $\sigma_{rr}$ and $\sigma_{\varphi \varphi}$. The shearing stresses $\tau_{r\varphi}$ and $\tau_{\varphi r}$ vanish in view of rotational symmetry. Since the thickness of the plate is supposed to be small as compared with the radius $R$, $\sigma_{zz}$ and $\tau_{zz}$ will be small compared with $\sigma_{rr}$ and $\sigma_{\varphi \varphi}$. We assume that the shearing stress $\tau_{zz}$ does not enter into the constitutive equations.

The generalized stresses are the radial and circumferential bending moments $M_r$ and $M_\varphi$ and the corresponding generalized strain rates are the radial and circumferential rate of curvature $\dot{\kappa}_r$ and $\dot{\kappa}_\varphi$.

The whole analysis is carried out within the theory of thin plates. According to the Love-Kirchhoff hypotheses the rate of strain is related to the rate of curvature of the middle surface by

$$\dot{\varepsilon}_{rr} = \dot{\kappa}_r z, \quad \dot{\varepsilon}_{\varphi \varphi} = \dot{\kappa}_\varphi z. \tag{4.6}$$

Since the deflections of the plate are supposed to be small a linear relation between rate of curvature and rate of deflection is assumed:

$$\dot{\kappa}_r = - \frac{\partial^2 \bar{w}}{\partial r^2}, \quad \dot{\kappa}_\varphi = - \frac{1}{r} \frac{\partial \bar{w}}{\partial \varphi}. \tag{4.7}$$
If the inertia effects are neglected the equation of equilibrium takes the form

\[ \frac{\partial}{\partial r} (rM_r) - M_\phi = - \int_0^r \rho(t) r \, dr. \]  

The constitutive equations for incompressible strain-rate sensitive rigid-plastic material are used [cf. (2.45)]:

\[ \dot{\epsilon}_{ij} = \gamma^0 \Phi(F) \frac{\partial F}{\partial \sigma_{ij}} \]  
for \( F > 0, \) \( \dot{\epsilon}_{ii} = 0, \)

where \( F = \sqrt{J_2}/k - 1. \)

The principal idea of Eq. (4.9) is that the strain rate should be in general a non-linear but uni-valued function of the excessive stresses above the yield surface. Two particular types of the function \( \Phi(F) \) will be considered, namely, the linear function \( \Phi(F) = F \) and the power function \( \Phi(F) = F^\delta. \) In the case of the power function the constitutive equations (4.9) give two independent equations

\[ \dot{\epsilon}_{rr} = \frac{\gamma}{3} \left( \frac{\sqrt{J_2}}{k} - 1 \right)^\delta \frac{2\sigma_{rr} - \sigma_{\phi\phi}}{\sqrt{J_2}}, \]

\[ \dot{\epsilon}_{\phi\phi} = \frac{\gamma}{3} \left( \frac{\sqrt{J_2}}{k} - 1 \right)^\delta \frac{2\sigma_{\phi\phi} - \sigma_{rr}}{\sqrt{J_2}}, \]

where \( J_2 = \sigma_{rr}^2 - \sigma_{rr}\sigma_{\phi\phi} + \sigma_{\phi\phi}^2 \) and \( \gamma = \gamma^0/2k. \) All components of the stress and strain-rate tensors are functions of two coordinates \( r \) and \( z. \) It is therefore desirable to transform the constitutive equations (4.10) to the generalized stress and strain-rate space where all quantities are functions of a single space coordinate \( \gamma. \) The bending moments \( M_r \) and \( M_\phi \) will be expressed as

\[ M_r = \int_{-h}^h \sigma_{rr} zdz, \quad M_\phi = \int_{-h}^h \sigma_{\phi\phi} zdz. \]

The stresses \( \sigma_{rr} \) and \( \sigma_{\phi\phi} \) can be evaluated from (4.10) and substituted into (4.11). Taking account of the relations (4.6) the integration can be performed for any integer exponent \( \delta. \) This leads to the following formulae relating rate of curvature \( \kappa_r \) and \( \kappa_\phi \) with the radial and circumferential bending moments:
\[
\dot{\kappa}_r = B \{ [1 - M_0(M_r^3 - M_r M_\varphi + M_\varphi^3)^{-1/2}]^\delta \left[ \frac{[M_r^3 - M_r M_\varphi + M_\varphi^3]^{1/2}}{M_0} \right]^{\delta - 1} \\
 \cdot \left( 2M_r - M_\varphi \right)/M_0 \},
\]
\[
\dot{\kappa}_\varphi = B \{ [1 - M_0(M_r^3 - M_r M_\varphi + M_\varphi^3)^{-1/2}]^\delta \left[ \frac{[M_r^3 - M_r M_\varphi + M_\varphi^3]^{1/2}}{M_0} \right]^{\delta - 1} \\
 \cdot \left( 2M_\varphi - M_r \right)/M_0 \},
\]
\[(4.12)\]

where the constant \( B \) is
\[
B = \gamma / \sqrt{3} h [(2\delta + 1)/2\delta]^\delta.
\]
\[(4.13)\]

Note that no formal analogy between (4.10) and (4.12) exists. Equations (4.12) involve a new term which vanishes only for the linear function, i.e., for \( \delta = 1 \). This example shows that particular attention should be paid in constructing a stress strain-rate relation appropriate for the generalized quantities.

It is convenient to introduce dimensionless variables
\[
m_r = \frac{M_r}{M_0}, \quad m_\varphi = \frac{M_\varphi}{M_0}, \quad \bar{\rho} = \frac{\rho}{R}, \quad v = \frac{\dot{\omega}}{BR^3}.
\]
\[(4.14)\]

We have now five equations with five unknown functions \( m_r, m_\varphi, v, \dot{\kappa}_r \) and \( \dot{\kappa}_\varphi \). However only three of these are of interest to us. These functions are \( m_r, m_\varphi, \) and \( v \). After elimination of \( \dot{\kappa}_r \) and \( \dot{\kappa}_\varphi \) the following system of three ordinary quasi-linear differential equations is obtained
\[
\frac{dm_r}{d\bar{\rho}} = - \frac{\dot{\bar{\rho}}}{2} \frac{m_r - m_\varphi}{\bar{\rho}},
\]
\[(4.15)\]
\[
\frac{dm_\varphi}{d\bar{\rho}} = \frac{3(m_r - m_\varphi) - \{ \delta(2m_r - m_\varphi) g(m_r, m_\varphi) - 1 + (\delta - 1)(2m_r - m_\varphi) \} [(m_r^3 - m_r m_\varphi)
\]
\[+ m_\varphi^3]^{1/2} - 1 \} g(m_r, m_\varphi) \left[ \frac{\dot{\bar{\rho}}}{2} - (m_r - m_\varphi) \right],
\]
\[
\frac{dv}{d\bar{\rho}} = - \bar{\rho} [1 - (m_r^3 - m_r m_\varphi + m_\varphi^3)^{-1/2}]^\delta (m_r^3 - m_r m_\varphi + m_\varphi^3) \left( \delta - 1 \right) / 2 (2m_\varphi - m_r),
\]
where
\[
g(m_r, m_\varphi) = (2m_\varphi - m_r)(m_r^3 - m_r m_\varphi + m_\varphi^3)^{-1} \left[ (m_r^3 - m_r m_\varphi + m_\varphi^3)^{1/2} - 1 \right]^{-1}.
\]
\[(4.16)\]
At the centre of the plate $r = 0$, $m_r = m_\phi$ by rotational symmetry. At the simply supported edge $r = R$ the radial bending moment and rate of deflection vanish. Thus, the boundary condition can be written in the form

![Graph](image)

Fig. 53. Curves $m_r$ and $m_\phi$ versus $\bar{p}$ for $\delta = 1$ and several values of the loading parameter $p'$ (Ref. [193]).
The computations were carried out for two values of the exponent $\delta$ and for several values of the load intensity $\rho'$.

Figures 53–56 present the moment distribution and the corresponding velocity fields for all values of $\rho'$ and $\delta$ listed in Table 6. It was thought
desirable to compare the moment and velocity distribution for the case of non-linear and linear function $\Phi(F)$. This has been done in Figs. 57 and 58, where the solid line refers to the case $\delta = 3$ and the broken line corresponds to the case $\delta = 1$. Note that the moment distributions do not differ noticeably, but the difference in the rate of deflection is appreciable. This confirms the supposition that the rate of deflection is far more sensitive to the change in the function $\Phi(F)$ than the moment distribution.

Most numerical results in [193] were obtained for the linear function $\Phi(F) = F$. This has been done to get a detailed comparison of the solution based upon the Huber-Mises yield condition with the solution of a similar
Table 6

<table>
<thead>
<tr>
<th>$\varphi'$</th>
<th>7</th>
<th>8.5</th>
<th>10</th>
<th>11.5</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 1$</td>
<td>7</td>
<td></td>
<td>10</td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>$\delta = 3$</td>
<td>7</td>
<td></td>
<td>10</td>
<td></td>
<td>13</td>
</tr>
</tbody>
</table>

Fig. 57. Comparison between the distributions of the moments $m_r$ and $m_\varphi$ for non-linear and linear function $\Phi(F)$ (Ref. [193]).

problem based upon the Tresca yield condition, given by E. J. Appleby and W. Prager [4]. The linearization of the yield locus introduces much
simplification to the governing equations, and the corresponding solution is in closed form. In this case the plate is divided into three regions that are in different visco-plastic regimes, the radii $\bar{p} = \alpha_1$ and $\bar{p} = \alpha_2$ being the boundaries. A solid line in Figs. 59 and 60 represents the solution obtained in paper [193] for a chosen value of $\rho' = 10$, while the broken line plots the corresponding solution of E. J. Appleby and W. Prager. There is an excellent coincidence in both moment and rate of deflection distribution. The broken line in Fig. 60 does not exhibit a noticeable deviation from the solid line either in shape or even in the absolute value. This result is somewhat unexpected. It is known from the theory of perfectly plastic solids that the linearization of the yield criterion leads to a fairly good estimate of the stress fields, while the corresponding displacement fields are usually unrealistic. This is due to the piecewise linear yield condition since one component of the strain-rate tensor has a constant direction for each region while the others vanish. On the other hand in the majority of cases for visco-plastic material there are two or more non-vanishing components of the strain-rate tensor and the direction of the strain-rate tensor is no longer constant. It can be concluded that, at least for simply supported circular plates, the linearized theory of viscoplasticity due to W. Prager [145] may give a close qualitative and quantitative assessment of the deformation under a given condition of loading. Of course the above statement relates only to the linear function $\Phi(F)$ for which the comparison was carried out.

The above procedure involves the assumption that the deflection of the plate is small. It is straightforward to establish upper bounds on the values of pressure and duration of impulse so as to remain within the limits of the theory.
A certain peculiarity of the governing equations is the fact that the solution in dimensionless quantities depends neither on the dimension of the plate nor on the constants of the material $\sigma_0$ and $\gamma$. However, the influence of the type of function $\Phi(F)$ is fundamental.

![Graph showing comparisons between distributions of the moments $m_r$ and $m_\phi$ for the Huber-Mises and Tresca yield functions.](image_url)

Fig. 69. Comparison between the distributions of the moments $m_r$ and $m_\phi$ for the Huber-Mises and Tresca yield functions (Ref. [193]).

Since the applied pressure $p(t)$ was assumed to be constant the time is formally eliminated from the solution. Although there are no restrictions to extending of the solution over the range of variable pressure, a sufficiently rapid time variation of the load intensities would require consideration of inertia effects. The dynamics of a visco-plastic circular plate has been con-
sidered in Ref. [195]. The treatment makes the same assumptions as in the quasi-static problem, but the transverse inertia motion is taken into account. It is shown that the dynamic flow of a viscoplastic circular plate is described by an initial- and boundary-value problem for a quasi-linear parabolic system.

![Diagram](image)

**Fig. 60.** Comparison between the velocity distributions for the Huber-Mises and Tresca yield functions (Ref. [193]).

Other solutions for the quasi-static flow of a visco-plastic circular plate may be found in Refs. [22, 74].

V. OTHER DYNAMICAL PROBLEMS

1. *Wave Problems for Rods and Beams*

The problem of the propagation of the longitudinal plastic waves in rods of strain-rate dependent material, including effects of lateral inertia and shear, has been considered by H. J. Plass [141]. A similar problem for elastic/viscoplastic beams, i.e., the problem of propagation of bending and transverse waves has been treated by L. V. Nikitin [120].

The solution of the wave problem of infinite and semi-infinite elastic/viscoplastic beams by the finite-difference method has been discussed in Ref. [8]. In the equations of motion of a beam the effects of shear and rotatory inertia are included. The material of a beam is described by the constitutive equations (2.67).
Under the assumption that the propagation velocities of the moment and shear waves are the same, the wave problem for the elastic/viscoplastic beam may be solved by successive approximation (cf. Ref. [9]).

2. Impulsive Loading of a Spherical Container

Consider a sphere of rigid/viscoplastic material and denote inner and outer radii by $a$ and $b$, respectively. Let a time-variable pressure $p(t)$ be applied on the surface $r = a$. Assuming in (3.1) and (3.49) $\mu \rightarrow \infty$ and $K \rightarrow \infty$, we obtain the system of differential equations

$$\frac{\partial \sigma_{rr}}{\partial r} + 2 \frac{\sigma_{rr} - \sigma_{\phi \phi}}{r} = \rho \frac{\partial v}{\partial t},$$

(5.1)

$$\frac{\partial v}{\partial r} + \frac{v}{r} = \sqrt{3} y \Phi(F),$$

$$+ \frac{\partial v}{\partial r} + 2 \frac{v}{r} = 0.$$

We shall assume the power law $\Phi(F) = F^d$ [cf. (2.88)] and arbitrary pressure $p(t)$. The boundary and initial conditions have now the form

$$\sigma_{rr}(a,t) = -p(t), \quad \sigma_{rr}(b,t) = 0, \quad v(r,0) = 0.$$

(5.2)

In this case the general solution of (5.1) given in Ref. [192] has the following form

$$\sqrt{\frac{J_3}{k}} = 1 + \left(\frac{a}{r}\right)^{3/6} [y(t)]^{1/6},$$

(5.3)

$$v = (\sqrt{3/3})^d a \left(\frac{a}{r}\right)^{8/6} y(t),$$

(5.4)

where $y(t)$ is determined by the differential equation

$$\frac{dy}{dt} + A y^{1/6} = P(t),$$

(5.5)

with initial condition $y(0) = 0$ and

$$A = \frac{2k\delta}{\rho y a^2} \left[1 - \left(\frac{a}{b}\right)^{3/6}\right] \left(1 - \frac{a}{b}\right)^{-1}, \quad P(t) = \frac{p(t) + 2 \sqrt{3k} \log \frac{a}{b}}{(\sqrt{3/3})^d \rho y a^2 \left(1 - \frac{a}{b}\right)}.$$
Using successive approximations, the solution of (5.5) may be given in the form

\[ y(t) = \lim_{n \to \infty} y_{(n)}(t), \]  

where \( y_{(n)}(t) \) is determined by a recurrence formula

\[ y_{(n)}(t) = \int_0^t \{ P(\xi) - A [y_{(n-1)}(\xi)]^{1/\delta} \} d\xi. \]  

If the function \( \Phi(F) \) is linear, i.e., \( \delta = 1 \), (5.5) has a closed-form solution

\[ y(t) = \exp \left( -\frac{A}{\delta} \right) \int_0^t P(\xi) \exp \left( \frac{A}{\delta} \xi \right) d\xi. \]
The time dependence of \( J_d(t) \) for rectangular shape of the loading curve and the linear function \( \Phi(F) \) are plotted in Fig. 61.

In Ref. [192] the formulae for the components of stress \( \sigma_{rr}, \sigma_{\varphi \varphi}, \sigma_{\theta \theta} \), strain \( e_{rr}, e_{\varphi \varphi}, e_{\theta \theta} \) and strain-rate tensors are given and the solutions for other types of the function \( \Phi(F) \) are discussed.

In order to compare the dynamic and the quasi-static solutions (cf. Chapter IV, Section 1), the same material and identical boundary conditions in both cases should be assumed (see Ref. [194]). Consider the rigid/visco-plastic material, the rectangular impulse and the linear function \( \Phi(F) = F \).

The corresponding solution of the quasi-static problem can be obtained by putting into formulae (4.3)-(4.6) \( \mu \to \infty \) and \( K \to \infty \) with the result

\[
\sqrt{J_d/k} = 1 + \left( \frac{a}{r} \right)^3 \left[ 1 - \left( \frac{a}{b} \right)^3 \right]^{-1} \left( \frac{\sqrt{3} \dot{p}}{2k} + 3 \log \frac{a}{b} \right),
\]

\[
u(r,t) = \left( \frac{a}{r} \right)^3 \left[ 1 - \left( \frac{a}{b} \right)^3 \right]^{-1} \gamma \frac{r}{2} \left( \frac{\dot{p}}{k} + 2 \sqrt{3} \log \frac{a}{b} \right) t.
\]

The particular case of the dynamic solution given by (5.3), (5.4), (5.6), and (5.9) for \( \delta = 1 \) and \( \dot{p}(t) = \text{const} \), yields

\[
\sqrt{J_d/k} = 1 + \left( \frac{a}{r} \right)^3 \left[ 1 - \left( \frac{a}{b} \right)^3 \right]^{-1} \left( \frac{\sqrt{3} \dot{p}}{2k} + 3 \log \frac{a}{b} \right) [1 - \exp (- \bar{\lambda}t)],
\]

\[
u(r,t) = \left( \frac{a}{r} \right)^3 \left[ 1 - \left( \frac{a}{b} \right)^3 \right]^{-1} \gamma \frac{r}{2} \left( \frac{\dot{p}}{k} + 2 \sqrt{3} \log \frac{a}{b} \right)
\]

\[
\cdot \left\{ t + \frac{1}{\bar{\lambda}} [\exp (- \bar{\lambda}t) - 1] \right\}.
\]

In both cases the pressure \( \dot{p} \) has to exceed the minimum value

\[
\dot{p}_{\text{min}} = 2 \sqrt{3} k \log \frac{b}{a}.
\]

From the comparison it is evident that the quasi-static solution is an asymptotic solution for the dynamic problem. The final results of these solutions differ from each other only in terms involving a function of time. Thus the distribution of stresses and displacements along the radius of sphere is the same for the dynamic and quasi-static problems.

The above conclusions are illustrated in Figs. 62–63 for a mild-steel spherical container and \( a/b = 1.5 \). In these figures the dashed lines are plotted according to (4.3) and (4.4).
3. Strain-Rate Sensitive Beams under Impact

The elementary dynamic rigid-plastic theory is characterized by the neglect of elastic deformations, strain hardening, strain-rate sensitivity, shear deformation, and geometry changes associated with large deformations.

Rapid loading tests on mild-steel beams have been performed by E. W. Parkes [129] and by T. J. Mentel [114]. Their experimental deformation
values were consistently below theoretical predictions by factors ranging from about 0.3 to 0.8. They showed that approximate corrections for strain-hardening and dependence of the yield stress on strain rate could be made so as to bring the calculated deformations close to their experimental results.

G. R. Cowper and P. S. Symonds [55] have treated this problem more completely with strain hardening and strain-rate sensitivity included separately. In this treatment a strain-rate law (2.89) has been used, where the constants have been chosen to agree as closely as possible with the experimental results.

A more general discussion of the strain-rate effect on impulsive loading has been given by S. R. Bodner and P. S. Symonds [19, 20] and by T. C. T. Ting and P. S. Symonds [178].

The paper [20] presents the results of a test program on the dynamic loading of cantilever-beam specimens. These tests have been designed to evaluate the relative importance of the various factors that are neglected in the rigid-perfectly plastic theory of beams. The most important result of this work is the conclusion that the rigid-perfectly plastic theory can serve as a reasonable first-order theory as long as the energy ratio $\mathcal{E}$ (of the kinetic energy input to the maximum possible elastic energy) is not too small (at least greater than 3). Elastic vibrations do not have much effect on the results when the energy ratio $\mathcal{E}$ is greater than about 10. For sufficiently large $\mathcal{E}$, the trend of the test results indicates that the influence of strain
rate on the yield stress was primarily responsible for the deviations between theory and experiment. In theoretical treatment S. R. Bodner and P. S. Symonds [20] have assumed that the influence of plastic strain rate $\dot{\varepsilon}$ upon yield stress $\sigma$ obeys the relation (2.89). The numerical values for $\gamma$ and $\delta$ have been deduced from the experimental data of M. J. Manjoine [112] for mild-steel specimens, and from experimental values collected by E. W. Parkes [129] for various aluminium alloys (see Fig. 64). The predictions of the rate-dependent, rigid-plastic theory are generally in satisfactory agreement with test results for the final deformation (shape and magnitude), deformation time, and strain time-history. On the other hand the application of an overall strain-rate correction factor on the final deformation (cf. T. J. Mentel [114]) cannot be generally recommended since it may lead to serious errors.

The analysis in Ref. [20] depends on several special assumptions. In particular, it has been assumed that the plastic region is finite but small in length compared to the beam length and the deformations small enough so that geometry changes could be ignored. Comparisons both within the theory and with test results have shown that these assumptions are of doubtful validity in the range of interest. They are not made in the analysis given by T. C. T. Ting and P. S. Symonds [178] (see also T. C. T. Ting [189, 190]). This study shows that final plastic deformations are in good agreement with those measured in tests performed by S. R. Bodner and P. S. Symonds [20].

4. Longitudinal Impact on Viscoplastic Rods

The problem of plastic deformations in a cylinder striking a rigid target has been undertaken by several authors (see for instance G. I. Taylor [172], E. H. Lee and S. J. Tupper [202]). An exact analysis of this problem based on the theory of elastic and plastic wave propagation has been given by E. H. Lee and S. J. Tupper [202].

In several papers the attention has been confined to cases in which plastic strains are much larger than elastic strains, and the validity of rigid-plastic theory has been assumed. Such treatment is motivated primarily by the need to understand the essential features of the fields of stress, strain, and strain rate in a specimen undergoing plastic deformation as a result of high-speed impact. When a test of this type is performed in order to gain knowledge of material behavior at high strain rates, the results will be meaningful only if these essential features and the stress and strain history of typical elements of the specimen are reasonably well known (cf. A. C. Whiffin [189] and T. C. T. Ting and P. S. Symonds [181]).

An approximate analysis of a cylinder under impact load, disregarding the elastic strains, has been presented by G. I. Taylor [172] and by E. H. Lee
and H. Wolf [104]. These considerations have shown the very important influence of the strain-rate effect on the deformation.

A rigid-(linearly viscoplastic) formulation of the theory for longitudinal impact on a bar has been applied by V. V. Sokolovsky [184] to several problems of plane shear waves in a semi-infinite medium. Known solutions of the heat equations were used.

T. C. T. Ting and P. S. Symonds [181] in the analysis of the longitudinal impact on rigid/viscoplastic rods have used the linear stress-(strain rate) law (cf. (2.93))

\[ \dot{e} = \gamma^*(\sigma - \sigma_0). \]

(5.15)

Four cases have been solved: constant-velocity impact on a semi-infinite bar, constant-velocity impact on a bar of finite length, impact of a finite mass on a semi-infinite rod and impact of a finite mass on finite rod. In every case the problem has been reduced to the solution of the heat equation.

G. I. Barenblatt and A. Y. Ishlinsky [6] have extended Taylor's problem to the viscoplastic material.

In Ref. [7] successive approximations have been suggested as a method of analysis for the deformation of a strain-rate sensitive plastic cylinder and the theoretical results have been compared with Whiffin's experimental data. Discussions of similar problems may be found in Refs. [100, 101, 106, 150, 153, 190, 200].

References

(Titles of Russian publications are translated).

FUNDAMENTAL PROBLEMS IN VISCOPLASTICITY


FUNDAMENTAL PROBLEMS IN VISCOPLASTICITY

55. COWPER, G. R., and SYMONDS, P. S., Strain hardening and strain rate effects in
the impact loading of cantilever beams, Technical Report No. 28, Brown
University, September 1957.

56. CRISTESCU, N., Some problems of the mechanics of extensible strings, stress wave
in anelastic solids, in "I.U.T.A.M. Symposium, Brown University, April 1963"

57. CRISTESCU, N., Some dynamic problems of one-dimensional elastic/visco/plastic
bodies, Eleventh International Congress of Applied Mechanics, München,
August 30 to September 5, 1964, (forthcoming).

58. DEUTLER, H., Experimentelle Untersuchungen über die Abhängigkeit der Zug-

59. DIAZ, J. B., On an analogue of the Euler-Cauchy polygon method for the numerical

60. MAC DONALD, R. J., CARLSON, R. L., and LANKFORD, W. T., The effects of strain
rate and temperature on the stress-strain relations of deep-drawing steel, Proc.

61. DUWEZ, P. E., and CLARK, D. S., An experimental study of the propagation of
plastic deformation under conditions of longitudinal impact, Proc. Am.

York, 1982.


64. GURTIN, M., and STERNBERG, E., On the linear theory of visco-elasticity,

65. GIZATULINA, G. M., On the problem of deflection of a circular plate of visco-plastic

66. HARDING, J., WOOD, E. O., and CAMPBELL, J. D., Tensile testing of materials

67. HARTMAN, P., and WINTNER, A., On hyperbolic partial differential equations,
80. Hohenemser, K., and Prager, W., Über die Ansätze der Mechanik isotroper Kontinua, ZAMM 12, 216–226 (1932).


125. OLSZAK, W., and BYCHAWSKI, Z., Criterion of fracture for visco-elastic bodies (forthcoming).


141. Plass, H. J., and RIFPERGER, E. A., Current research on plastic wave propagation


186. **Vishman,** F. F., **Zlatin,** N. A., and **Modde,** V. S., Resistance to the deformation of metals at the rate of $10^{-4}$ to $10^{-3}$ mm./sec., Zhurn. Tekhn. Fiz. 19 (1949).
199. **Ziegler,** H., Thermodynamic considerations in continuum mechanics, The 1964 Minta Martin Lecture, Massachusetts Institute of Technology.