Optimal Swing Up and Stabilization Control for Inverted Pendulum via Stable Manifold Method

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Abstract—This brief addresses the problem of swing up and stabilization for inverted pendulum. It is shown that the stable manifold method, recently proposed for approximately solving Hamilton–Jacobi equation (HJE) in nonlinear optimal control problem, is capable of designing feedback control for this problem. The experimental results include two types of controllers (one-swing and two-swing), which indicates the nonuniqueness of solution for an HJE. This brief further provides a variational analysis method for investigating and enlarging a stable manifold and shows a detail structure of the stable manifold for a 2-D pendulum from which controllers from one-swing to five-swing can be derived.

Index Terms—Hamilton–Jacobi equation (HJE), inverted pendulum, nonlinear optimal control, stable manifold method.

I. INTRODUCTION

INVERTED pendulum system poses quite a few interesting and challenging problems for it is an underactuated mechanical system and exhibits strong nonlinearity. Especially, swing up and stabilization problem attracts the attention of nonlinear control researchers for decades [1]–[5]. Åström and Furuta [6] solved this problem by switching two different control laws, one that drives the pendulum to the upright region and another that stabilizes the pendulum around the upright position by linear control (see [7] for experimental results and the comparison of several control strategies). The problem is considered in [8]–[11] from the global stabilization viewpoint, which has an intrinsic difficulty from a topological reason. On the other hand, optimal controller design for the swing up and stabilization problem, despite its significance, is still a challenging problem, since it requires to solve a Hamilton–Jacobi equation (HJE).

There have been proposed several approaches for solving HJE, for example, Taylor series approach, policy iteration or successive Galerkin method, state-dependent Riccati equation method, and algebraic approach [12]–[24]. In this brief, we use an approach for numerically solving HJE, associated with the stable optimal regulator problem [25], called the stable manifold method [26], [27]. This method is an application of stable manifold theory and employs iterative computations for flows on stable manifold of a Hamiltonian system associated with an HJE. If one addresses the swing up and stabilization problem of inverted pendulum as an optimal control problem, the difficulty is in the computation of solution for the HJE in a large domain so that the pending position, which is the initial condition of the control, is included in the domain.

In this brief, we show, by experimental studies, that the stable manifold method is capable of designing single feedback controllers (without switching controllers) for the optimal swing up and stabilization for inverted pendulum. It will be even shown that the method produces two controllers working for the swing up experiment generated from one stable manifold. This nonuniqueness of solution is further investigated with a variational analysis method for a simpler problem of a 2-D pendulum system presenting from one-swing to five-swing controllers produced from one stable manifold. The analysis given here reveals the nature of this mechanical control system and 3-D figures of the stable manifold help understand how the controllers lie in the stable manifold (see [28], [29] as prior works for the analysis of a stable manifold in the optimal control for 2-D pendulum).

In the previous study [4], a horizontal inverted pendulum was used and optimal control experiment with one-swing was reported. This brief presents not only experimental results for rotational pendulum with one- and two-swings but also geometric analysis for the stable manifold from which the controllers are derived.

The organization of this brief is as follows. In Section II, the inverted pendulum system used in this brief is introduced with its mathematical model and experiment results of one-swing and two-swing stabilization are shown. The variational method for shifting flows on a stable manifold or enlarging the stable manifold is presented in Section III. The method is used in Section IV to further analyze the structure of stable manifold in the optimal control for 2-D pendulum. The 3-D figures made from the enlarged stable manifold are illustrated providing deeper understanding on the stable manifold and nonuniqueness of solution in an HJE. The stable manifold method for solving HJEs is summarized in the Appendix.

II. NONLINEAR OPTIMAL SWING-UP CONTROL OF THE ROTATIONAL INVERTED PENDULUM

In this section, we design swing up and stabilization controllers for a rotational inverted pendulum via the stable manifold approach. Two state feedback controllers will be designed, one of which effectively uses the resonance of the pendulum so that the control input is reduced. The experiments are successful for both controllers, but, they pose an issue of nonuniqueness of solution for HJEs.
TABLE I
PARAMETER OF THE ROTATIONAL INVERTED PENDULUM

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$ moment of inertia of rotational arm</td>
<td>$5.41 \times 10^{-2}$[kg m²]</td>
</tr>
<tr>
<td>$m_a$ mass of rotational arm</td>
<td>0.71[kg]</td>
</tr>
<tr>
<td>$m_p$ mass of pendulum</td>
<td>0.048[kg]</td>
</tr>
<tr>
<td>$l_a$ length of rotational arm</td>
<td>0.19[m]</td>
</tr>
<tr>
<td>$l_p$ length of pendulum</td>
<td>0.20[m]</td>
</tr>
<tr>
<td>$R_a$ armature resistance</td>
<td>3.1[Ω]</td>
</tr>
<tr>
<td>$K_E$ counter electromotive coefficient</td>
<td>0.053[V s/rad]</td>
</tr>
<tr>
<td>$K_T$ torque constant</td>
<td>0.053[N m/A]</td>
</tr>
<tr>
<td>$g$ gravity acceleration</td>
<td>9.8[m/s²]</td>
</tr>
<tr>
<td>$n$ Gear ratio</td>
<td>1:5</td>
</tr>
</tbody>
</table>

A. Rotational Inverted Pendulum System and Its Equations of Motion

The rotational inverted pendulum system and its schematic model built in our laboratory are shown in Fig. 1. This system consists of rotational arm, pendulum, encoders for both arm and pendulum, and dc motor. The pendulum can freely rotate and the control torque is applied to the arm by the dc motor through the gear. The values of the physical parameters are in Table I.

The Lagrangian is given as

$$L = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{6} m_p l_p^2 (\dot{\theta}^2 + \dot{\theta}^2 \sin^2 \theta)$$

$$+ \frac{1}{2} m_p l_a l_p \dot{\phi} \cos \theta + \frac{1}{2} m_p l_a \dot{\phi}^2 - \frac{1}{2} m_p l_p g \cos \theta$$

and the Lagrange equations are obtained as

$$M(\dot{\theta}) \ddot{\theta} = \begin{bmatrix} C_1(\phi, \dot{\phi}, \theta, \dot{\theta}) \n C_2(\phi, \dot{\phi}, \theta, \dot{\theta}) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

where

$$M(\dot{\theta}) = \begin{bmatrix} p_1 + p_2 \sin^2 \theta & p_3 \cos \theta \\ p_3 \cos \theta & p_2 \end{bmatrix}$$

$$C_1(\phi, \dot{\phi}, \theta, \dot{\theta}) = \begin{bmatrix} -2 p_2 \dot{\phi} \dot{\theta} \sin \theta \cos \theta + p_3 \dot{\phi}^2 \sin \theta \\ 2 p_2 \dot{\phi}^2 \sin \theta \cos \theta + p_4 \dot{\phi} \sin \theta \end{bmatrix}$$

$$p_1 = J + m_p l_p^2, \quad p_2 = \frac{1}{3} m_p l_p^2$$

$$p_3 = \frac{1}{2} m_p l_p l_a, \quad p_4 = \frac{1}{2} m_p l_p g$$

and $\tau$ is the torque on the arm. The dc motor generates $\tau$ with input voltage $u$, which is modeled by

$$\tau = -\mu_\phi \dot{\phi} + K_{dc} u$$

where

$$K_{dc} = \frac{n K_T}{R_a}, \quad \mu_\phi = \frac{n^2 K_T K_E}{R_a}$$

Denoting $x = [x_1, x_2, x_3, x_4]^T = [\phi, \theta, \dot{\phi}, \dot{\theta}]^T$ as state variables, the state space equation of the rotational inverted pendulum is written as follows:

$$\dot{x} = f(x) + g(x) u$$

where

$$f(x) = \begin{bmatrix} x_3 \\ \frac{x_4}{M^{-1}(x)} C_1(x) - \mu_\phi \dot{\phi} \\ \frac{x_4}{M^{-1}(x)} C_2(x) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ M^{-1}(x) K_{dc} \end{bmatrix}$$

B. Nonlinear Stable Optimal Regulator Design

For the stable optimal control problem to design a swing up control for (2), we define the cost function

$$J = \frac{1}{2} \int_0^\infty x^T Q x + u^2 dt, \quad Q = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In applying the stable manifold method in the Appendix, the weighting matrix $Q$ is chosen in such a way that the Linear Quadratic (LQ) control with (3) to keep the pendulum upright position does not have high gain and that the real parts of the closed loop eigenvalues are in the range of $[-10, -1]$ so that the convergence rate of the iterative computations (A.4) and (A.5) is not too different in each component of $x'$. The pair $(A, g(0))$, where $A = (\partial f/\partial x)(0)$, is stabilizable and $(\sqrt{Q}, A)$ is detectable and therefore, the stabilizing solution exists for the associated Riccati equation. This guarantees the existence of a stable manifold yielding a feedback controller that would swing up and stabilize the pendulum provided that the manifold can be extended to the pending position. The gain matrix of the LQ control is

$$[0.633 0.822 14.1 1.55]$$

and the closed loop eigenvalues are

$$-9.99, -9.23, -2.48 \pm 1.11i.$$
through the pending position. These trajectories are on the stable manifold with accuracy $|H| \leq 10^{-6}$, where $H$ is the left-hand side of the HJE, and some of them are shown in Fig. 2.

The next step is to approximate the stable manifold $p = p(x)$ by polynomial fitting, namely, construct a vector-valued polynomial function $p_{\text{pol}}(x)$ such that $p = p_{\text{pol}}(x)$ approximates the stable manifold in an appropriate sense. We assume that each element of $p_{\text{pol}}(x)$ [denote $p_{\text{pol}}^i(x)$] has the form

$$p_{\text{pol}}^i(x) = C^i Z(x), \quad i = 1, \ldots, 4$$

where

$$C^i = [c_1^i, c_2^i, \ldots, c_6^i] \in \mathbb{R}^{1 \times 67}, \quad i = 1, \ldots, 4$$

$$Z(x) = [x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_{67}]^T \in \mathbb{R}^{67}$$

and $\sum_{k=1}^{4} j_k = 5$. Here $Z: \mathbb{R}^n \to \mathbb{R}^{67}$ is a 67-D vector-valued function, whose elements are four-variable monomials. The coefficients $C^i$ of the polynomial are determined using least square technique based on the numerical data obtained by the iteration. The first-order coefficients, however, are given by the solution of Riccati equation, since the stable manifold of (A.2) is tangent at the origin to the plane defined by the solution of Ricatti equation, since the stable manifold iteration by changing the order of other swing up methods of inverted pendulum, such as energy-based control [6], [7], [10] or control using homoclinic fashion and takes several hours to get satisfactory results in simulation (Section II-C). Although the error can be decreased by taking larger polynomial order, considering actual implementation, it is better to have low-order polynomials. In the implementation, the polynomials are written in the Horner form to reduce computational time for evaluation at each sampling time.

The final form of the approximation of the optimal input is given as

$$u(x) = -\frac{1}{2} R^{-1} g(x)^T p_{\text{pol}}(x)$$

(5)

using (4).

C. Simulation and Experiment Results

The state feedback controller (5) is implemented in the actual experiment setup in Fig. 1. The control torque $\tau$ in (1) is generated by the dc motor (Tamagawa Seiki, TS1983) equipped with an amplifier, in which an inverting amplifier circuit is built using an operational amplifier (Texas Instrument, OPA541). The angles of the arm $\phi$ and the pendulum $\theta$ are measured by two encoders (Tamagawa Seiki, OSI38, 2,500(C/T)) and their derivatives are obtained using a filter $10,000 s/(s^2 + 140 s + 10000)$. The control programs are written in LabVIEW (National Instruments, NI) and they are implemented by real-time operation system, NI Compact RIO. The control period for the feedback controller is 1[ms]. Coulomb friction acting on the arm cannot be ignored and some compensation is necessary. The static friction significantly deteriorates the control performance and the friction compensation law is simply to add 0.2[V] to the input command for $0 \leq t \leq 0.2[\text{s}]$ and no compensation is made for $t > 0.2[\text{s}]$.

The simulation and experimental results are shown in Fig. 3. The angle of the pendulum $\theta$ starts from $\pi$ at $t = 0[\text{s}]$ and converges 0 around $t = 1[\text{s}]$ and the equilibrium $(\phi, \theta, \dot{\phi}, \dot{\theta}) = (0, 0, 0, 0)$ has been asymptotically stabilized. Compared with other swing up methods of inverted pendulum, such as energy-based control [6], [7], [10] or control using homoclinic
Fig. 4. Shooting line of two swing-up trajectories.

structure [30], our controller accomplishes the swing up with one swing using a single feedback controller. It is also seen in Fig. 3 that the closed loop behaviors of simulation and experiments well agree except that holding control at the upright position is still shaky, which is due to friction and backlash of gears.

We also verified the robustness of the controller. The same feedback controller did the swing up control for the pendulum 8 cm longer than the nominal one with a 5 g weight at the top. This shows a certain robustness of the stable manifold method. This robustness comes from the design method based on a bundle of trajectories with different initial points using the variational method described in Section III so that the states with parameter variations are within the controller domain.

D. Simulation and Experiment of Two-Swing-Up Controller

From the viewpoints of energy and resonance of the pendulum system, one expects that other swing up and stabilization schemes exist. In the initial computation data with a hundred of $\xi$’s, we found that trajectories similar to one-swing and two-swing coexist. From those similar to two-swing trajectories, we computed nearby trajectories in order to form a bundle of trajectories shown in Fig. 4, one of which passes exactly through the pending position. For this computation, we use the variational method, which is described in Section III in detail. In the top figure for $(\theta, \dot{\theta})$ in Fig. 4, the sign of $\dot{\theta}$ is positive when the trajectories start around $\theta = \pi$ and it gets negative until $(\theta, \dot{\theta}) \to (0, 0)$ while in Fig. 2, the sign of $\dot{\theta}$ is always negative. This shows that the arm to which the pendulum is attached moves one direction and then turns back to the opposite direction to efficiently get reaction so that the control torque is kept lower compared with the one-swing up control in Section II-C.

A feedback controller is constructed in the same way for (5) using polynomials with the first order for $\theta, \dot{\theta}$ and the fifth order for $\phi, \dot{\phi}$. In the simulation and experiment results shown in Fig. 5, one sees that the behaviors of $\theta, \dot{\theta}$ in the experiment are almost identical with those in the simulation. The difference of input responses is due to friction and gear backlash.

The values of cost function (3) for one-swing and two-swing controllers are

$$J_1 = 11.5050, \quad J_2 = 5.1034$$

respectively. This shows that the more swings the pendulum gets, the smaller the cost of the control becomes, which agrees to (10) for the analysis of stable manifold in the 2-D pendulum. The experimental verification of the control with more swings in the 4-D pendulum is a challenge for future work.

III. VARIATIONAL METHOD FOR STABLE MANIFOLD

The iterative computational theory for stable manifold presented in the Appendix needs more sophisticated tools from the following reasons. First, in order to obtain a feedback law that works in real applications, the domain of definition for the feedback function must be enlarged so that a set of initial points with adequate size containing the target initial point [pending position $(\phi, \theta, \dot{\phi}, \dot{\theta}) = (0, \pi, 0, 0)$ in our case] and the control system has robustness for parameter variations and uncertainties. Second, as Section II-D presents an example of nonunique solutions in an HJE for pendulum swing up, there needs to be a computational framework for better understanding of the structure of stable manifold that would elucidate the nature of optimal control problems for mechanical systems. More specifically, the controllers in Sections II-C and II-D work when initial condition is known to be in a neighborhood of $\theta = \pi$. When one seeks controllers that work in a larger domain of initial points, the geometry of the stable manifold tells that for certain direction it is hard to extend the domain due to the mechanism of the pendulum control system, which will be discussed in Section IV.

In this section, we develop a method, by variational approach, to compute nearby trajectories of a reference
should be zero. Tangent vectors on the stable manifold at $z_r(t) = [x_r(t), p_r(t)]$ on a stable manifold obtained by the stable manifold iteration in the Appendix. This method is utilized for the purposes mentioned above in finding $z^{'\prime}$ in (A.4) and (A.5) and the controllers for one- and two-swing up experiments in Sections II-C and II-D are obtained by this method. The idea of the method is similar in essence to the classical neighboring extremal control in [31], whereas we are concerned with the computation of stable manifold so that stability argument has to be made.

Let $z(t) = [\tilde{x}(t), \tilde{p}(t)]$ be errors from the reference trajectory. Then, $z(t)$ satisfies a linear time-varying equation, which is a variational equation of (A.2)

$$\dot{z} = A_z(t)z$$

where

$$A_z(t) = \begin{bmatrix} \frac{\partial}{\partial \tilde{x}} (\frac{\partial H}{\partial \tilde{p}})^T & \frac{\partial}{\partial \tilde{p}} (\frac{\partial H}{\partial \tilde{p}})^T \\ -\frac{\partial}{\partial \tilde{x}} (\frac{\partial H}{\partial \tilde{x}}) & -\frac{\partial}{\partial \tilde{p}} (\frac{\partial H}{\partial \tilde{x}}) \end{bmatrix} x = x_r(t), p = p_r(t)$$

Let $\Phi(t)$ be the $2n \times 2n$ matrix solution for

$$\dot{\Phi} = A_z(t)\Phi, \quad \Phi(0) = I_{2n}.$$ 

Since the linear part of the Hamiltonian system (A.2) has $n$ stable eigenvalues and $n$ unstable eigenvalues, the transition matrix $\Phi(t_f)$, where $t_f$ is the final time of the control, should also have $n$ stable eigenvalues with absolute value less than 1 and $n$ unstable eigenvalues with absolute value larger than 1. Let us denote a $2n \times n$ matrix $V_{sp \times p}^\dagger$, each column of which is a generalized eigenvector of the stable eigenvalue of $\Phi(t_f)$. Actually, if the trajectory certainly converges to the origin at $t = t_f$ and no numerical errors are present, the eigenvalues should be zero. Tangent vectors on the stable manifold at $[x_r(0), p_r(0)]$ are represented by $[V_{sp \times p}^\dagger]$ with $u \in \mathbb{R}^n$. If $V_{sp \times p}$ is nonsingular, taking $\Delta x = V_{sp}u$, $|u| \ll 1$, the tangent vectors are written as

$$\begin{bmatrix} \Delta x \\ V_{sp}V_{sp}^{-1}\Delta x \end{bmatrix}.$$  

Thus, approximations of nearby trajectories on the stable manifold of the reference solution $[x_r(t), p_r(t)]$ can be computed by solving (A.2) with initial conditions $[x_r(0) + \Delta x, p_r(0) + V_{sp}V_{sp}^{-1}\Delta x]$. Fig. 6 shows an example of nearby trajectories of a swing up solution in Section IV for a 2-D pendulum.

Blue dotted line represents a trajectory on stable manifold while red line represents a trajectory starting slightly off the manifold.

The expansion of the stable manifold is carried out repeatedly using the variational method described earlier. The trajectories $[x_r(t) + \tilde{x}(t), p_r(t) + \tilde{p}(t)]$ around a reference trajectory $[x_r(t), p_r(t)]$ computed by the variational method are not exact solutions of (A.2) and the iterative computation (A.4) must be employed. The initial conditions $(x_0, p_0)$ in (A.5) for the iteration are taken from the nearby trajectories (approximation) as $x_0 = x_r(t) + \tilde{x}(t), p_0 = p_r(t) + \tilde{p}(t)$, where $\tau$ are appropriately chosen so that the points are in the domain of convergence for the iteration.

### IV. Analysis of the Stable Manifold for Pendulum System

Sections II-C and II-D showed that multiple solutions exist in the HJE for the optimal control of inverted pendulum and they can be used for real experiments. While nonuniqueness of solutions in the first-order pdes is theoretically known [32], [33] and for a 2-D pendulum system nonunique solutions are numerically obtained in [29], in this section, we try to clarify the mechanism of nonuniqueness in the pendulum system by focusing on the neighborhood of the pending position. It is also useful to know the geometry of the stable manifold when one wants to extend the domain of a specific controller as will be shown in this section.

For the sake of simplicity, we will consider a 2-D inverted pendulum. This pendulum system is derived from a cart-pendulum system by focusing only on the pendulum dynamics or by taking the limit where the mass of the cart goes to 0 [8]. In this system, the angle of the pendulum is $\theta$ and $m, l, J,$ and $u$ are the mass, length, and inertia of the pendulum and force horizontally applied to the cart, respectively. An acceleration of the gravity is denoted as $g = 9.8 \text{[kg m/s}^2\text{]}$. The values of these physical parameters are given in Table II.

#### TABLE II

<table>
<thead>
<tr>
<th>Parameter of the 2-D Inverted Pendulum</th>
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</thead>
<tbody>
<tr>
<td>$J$ inertia of the pendulum</td>
</tr>
<tr>
<td>$m$ mass of the pendulum</td>
</tr>
<tr>
<td>$l$ length of the pendulum</td>
</tr>
</tbody>
</table>

Taking state variables as $x = [x_1, x_2]^T = [\theta, \dot{\theta}]$, the state equation is given as follows:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ m l g \sin x_1 - m l^2 x_1^2 \sin x_1 \cos x_1 - l \cos x_1 \dot{u} \end{bmatrix}.$$  

For this system, consider the nonlinear optimal control problem with the following cost function:

$$J = \int_0^\infty x^T Q x + Ru^2 dt, \quad Q = \begin{bmatrix} 10^{-5} & 0 \\ 0 & 10^{-5} \end{bmatrix}, \quad R = 2.$$  

Fig. 6. Stable and unstable eigenvector for 2-D pendulum system.
of the pending position (obtained trajectories, we choose those passing a neighborhood trajectories with one-, two-, and three-swings. In Section III. For the sake of better visibility, we focus on optimal control problem (8) and (9) via the variational method in Section IV-B. Structure of the Stable Manifold With Variational Analysis

A. Nonuniqueness of the Solution

In this section, we will show that it is possible to obtain swing up trajectories in the Hamiltonian system associated with the optimal control problem (8) and (9) with one swing to five swings. First, we assign several hundreds of initial $\xi$'s in (A.5) on a 1-D circle in $x_1^\prime-x_2^\prime$ plane with radius 0.1 and apply the stable manifold iteration with $k = 15$. From the obtained trajectories, we choose those passing a neighborhood of the pending position $(x_1, x_2) = (\pi, 0)$ and then, apply the variational method in Section III to obtain trajectories exactly passing through the pending position. Fig. 7 shows them and one sees that the trajectories intersect, which indicates the nonuniqueness of solutions for the HJE equation for the optimal control problem (8) and (9). For now, it is not possible to obtain trajectories with more than five swings and further investigation or improvement is required for it. We have computed the values of the cost function for each trajectory, which are shown in the following:

\[
J_1 = 0.0073, \quad J_2 = 0.0033 \\
J_3 = 0.0026, \quad J_4 = 0.0023 \\
J_5 = 0.0022 \\
\]

where $J_i$ is the value for the trajectory with $i$ swings. It can be seen that the more swings, the less the value of the cost function takes and the difference of the values gets smaller as the number of swings increases. Although the cost function (9) is bounded below, it is an open question whether certain control actually achieves the minimum of $J$ especially when $Q = 0$. These solutions all satisfy the same HJE and it should be understood that all of them are local minimum for the optimal control problem. These computations prompt to better analyze the structure of the stable manifold, which will be shown in Section IV-B.

B. Structure of the Stable Manifold With Variational Analysis

In this section, we further investigate the structure of the stable manifold in the Hamiltonian system associated with the optimal control problem (8) and (9) via the variational method in Section III. For the sake of better visibility, we focus on trajectories with one-, two-, and three-swings.

The closed loop eigenvalues with the Riccati solution obtained by linearization are

\[
\lambda = -8.3557, \quad -10.6659.
\]

Using the computational data in Section IV-A, the stable manifold around one-, two-, and three-swing trajectories is constructed and its 3-D figures are shown in Figs. 8 and 9. Fig. 8 shows the manifold in $x_1$-$x_2$-$p_1$ space and Fig. 9 shows it in $x_1$-$x_2$-$p_2$ space. In both figures, red, blue, and green trajectories correspond to one-, two-, and three-swing trajectories, respectively, starting from vertical poles at $(x_1, x_2) = (\pi, 0)$. It can be seen from these figures that three leafs of the manifold overlap around $(x_1, x_2) = (\pi, 0)$ indicating that one cannot define a single solution for the HJE for this optimal control problem in a classical sense (see [33] for the notion of viscosity solution in HJEs). In Fig. 9 in which the vertical axis $p_2$ is associated with the input value, one can see that as the number of swings increases, the maximum input decreases, which is naturally deduced from the physical consideration using the resonance of pendulum.

Fig. 10 shows an enhanced figure of the stable manifold for $0 \leq x_1 \leq \pi/2$. It is seen that one-swing trajectories with initial points approaching $\pi/2$ sharply change in the $p_1$ values (red line). This shows that the horizontal position $x_1 = \pi/2$ is a vertical asymptote of the one-swing leaf and its $p_1$ value goes infinite as initial value $x_1$ approaches the asymptote, indicating
that no control can swing up the pendulum with one swing from the horizontal initial position. The green line is the three-swing trajectory starting at \( x_1 = \pi \) and there is a leaf under the three-swing trajectory, which is an enlarged leaf from the three-swing trajectories.

The trajectories composing the manifold in Figs. 8–10 are computed by the variational method in Section III shifting one trajectory to another. The transition matrix \( \Phi(t) \) includes valuable information of the geometry of the manifold. As is shown in (7), the \( x \)-component of the tangent vector of the stable manifold is given by \( \Delta x = V_{sx}u, u \in \mathbb{R}^n \), where \( V_{sx} \) consists of \( x \)-component of the generalized eigenvector for stable eigenvalues. This shows that when \( V_{sx} \) is nonsingular, the manifold can be enlarged for any direction in \( x \)-space while if not, there is a direction for which the manifold does not exist anymore. Let us define the angle of two \( x \)-component vectors in the stable eigenspace of \( \Phi(t) \) by

\[
\theta_x = \arccos \left( \frac{V_{s1}(1 : 2) \cdot V_{s2}(1 : 2)}{|V_{s1}(1 : 2)||V_{s2}(1 : 2)|} \right)
\]

where \( V_{s1} \) and \( V_{s2} \in \mathbb{R}^4 \) are the generalized eigenvectors corresponding to stable eigenvalues. If \( \theta_x \) is sufficiently larger than 0, the stable manifold can be extended to any direction and if \( \theta_x \) is close to 0, two stable eigenvectors are almost identical, meaning that one cannot extend the manifold except for the direction of the trajectory itself.

Two \( \theta_x \)'s are computed for the points on one-swing leaf and three-swing leaf at \( x_1 = 1.614(\sim \pi/2) \) and we found that \( \theta_x = 0.0211 \) [rad] on the one-swing leaf while \( \theta_x = 1.22 \) [rad] on the three-swing leaf. It can be concluded that \( x_1 = \pi/2 \) is not an asymptote for the manifold enlarged from three-wing up trajectories and it is possible to enlarge this part of the manifold over the horizontal initial position. This is naturally understood because three-swing up control effectively uses the falling motion of the pendulum from horizontal position and the larger the initial potential energy of the pendulum is, the easier it is to swing up the pendulum (blue line in Fig. 10).

V. Conclusion

In this brief, we have shown, by laboratory experiment, that the optimal swing up and stabilization problem for inverted pendulum is solvable by the stable manifold method, which approximately solves HJE. The experimental results raise an issue of nonuniqueness for solution of an HJE and the variational analysis, which is also introduced in this brief, helps elucidate the structure of the stable manifold from which solutions with one or more swings can be derived.

Appendix

Stable Manifold Method

The stable optimal regulator problem is to find a state feedback controller \( u = u(x) \) for a nonlinear control systems \( \dot{x} = f(x) + g(x)u \) that locally asymptotically stabilize the system around its equilibrium \( x = 0 \) and minimizes the cost function \( J = \int_0^\infty x^TQx + u^TRudt \), where \( Q \) is positive semidefinite and \( R \) is positive definite constant matrices with appropriate dimensions (see [25] for the linear stable regulator problem and the analysis of the associated Riccati equation). An HJE associated with this problem has the form

\[
(HJ)H(x, p) = p^Tf(x) - \frac{1}{4}p^T\tilde{R}(x)p + x^TQx = 0 \quad (A.1)
\]

where \( \tilde{R}(x) = g(x)R^{-1}g(x)^T \) and \( p_1 = \partial V/\partial x_1, \ldots, p_n = \partial V/\partial x_n \). We assume that \( [A, g(0)] \) is stabilizable, where \( A = (\partial f/\partial x)(0) \), and \( (f, Q, A) \) is detectable. Then, the linearizing Riccati equation for (A.1) has the stabilizing solution and it is known that (A.1) has a local solution \( V(x) \) with which \( \dot{x} = f(x) - (1/2)\tilde{R}(x)(\partial V/\partial x)^T \) is asymptotically stable around \( x = 0 \).

The associated Hamiltonian system for (A.1) is

\[
\dot{x} = \frac{\partial H}{\partial \dot{p}}, \quad \dot{p} = -\frac{\partial H}{\partial x} \quad (A.2)
\]

and (A.2) can be block diagonalized as

\[
\begin{bmatrix}
\dot{x}' \\
\dot{p}'
\end{bmatrix} =
\begin{bmatrix}
F & 0 \\
0 & -F^T
\end{bmatrix}
\begin{bmatrix}
x' \\
p'
\end{bmatrix} + \text{higher order terms}
\quad (A.3)
\]

where \( F \in \mathbb{R}^{n \times n} \) is asymptotically stable and \( T \begin{bmatrix}
\dot{x}' \\
\dot{p}'
\end{bmatrix} = \begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix} \) is a linear coordinate transformation using the stabilizing solution of the Riccati equation that approximates (A.1). The stable manifold iteration is

\[
\begin{cases}
\begin{aligned}
x'_{k+1}(t) &= e^{Ft}x + \int_0^t e^{F(t-s)}h_1(x'(s), y'(s)) \, ds \\
y'_{k+1}(t) &= -\int_t^\infty e^{-F(t-s)}h_2(x'(s), y'(s)) \, ds
\end{aligned}
\end{cases}
\quad (A.4)
\]

with \( k = 0, 1, 2, \ldots \) and arbitrary \( \xi \in \mathbb{R}^n \). It is proved that for sufficiently small \( |\xi| \) (domain of convergence), \( x_k(t, \xi) \), \( p_k(t, \xi) \) defined by

\[
\begin{bmatrix}
x_k(t, \xi) \\
p_k(t, \xi)
\end{bmatrix} \approx T
\begin{bmatrix}
x'_k(t, \xi) \\
p'_k(t, \xi)
\end{bmatrix}
\]

converge to a solution of (A.2) and that \( x_k(t, \xi), p_k(t, \xi) \to \infty \) as \( t \to \infty \) for all \( k \in \mathbb{N} \).
The stable manifold method for optimal control design is summarized as follows.

1) Apply iteration (A.4) and (A.5) $k$-times for (A.3) to get an approximation of stable manifold of (A.2)

$$\Omega_k = \{ (x_k(t, \xi), p_k(t, \xi)) \mid t < t_0, |\xi| < 1 \}$$

where $t_0$ is a negative value so that $\Omega_k$ covers the domain of interest for the control problem.

2) Obtain a function $\pi_k(x)$ such that $p = \pi_k(x)$ approximates the set $\Omega_k$.

3) $k$th approximation of the optimal control is given by

$$u = -\hat{R}(x)\pi_k(x).$$

The detail of the stable manifold method can be found in [26] and [27] and its theoretical background material is in [34]–[36].

REFERENCES


