Analysis Tools for Mass Movement Assessment

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ABSTRACT
Slope stability assessment is a fundamental step in evaluating landslide hazard and for the safe design of structures and infrastructures. We analyze the physical principles that underlie both the most frequently used computational methods and some less common but accurate methods. The most important limit equilibrium methods, and the most widely adopted by practitioners, are presented in detail in a unified conceptual framework, and then limit analysis approaches are introduced and discussed. The latter have the advantages of performing parametric analyses of slope stability under slightly simplified conditions to obtain dimensionless stability charts. The basic principles of finite element, finite difference, and distinct element methods are described, and their advantages and disadvantages are listed.

13.1 INTRODUCTION
In this chapter, we outline the physical principles underpinning the onset of landslides. Put into simple terms, the information of interest is how big a potential landslide will be at a chosen site, under what conditions it can occur, and how close the slope is to releasing material through landsliding. To answer the first question, geotechnical engineers and geologists need to identify the location of the underground failure surface, which determines the shape of the failing soil/rock mass and therefore the landslide volume. This task is simple in the ideal case of homogeneous slope materials, but it may become very challenging in the cases of complex stratigraphy and/or the presence of discontinuities since there may be several potential failure surfaces within the investigated slope on which there may be very different mechanical resistances. In this case, a small error in the evaluation of the parameters governing the ground resistance (e.g., the cohesion $c$ and internal friction angle $\phi$) along one of the candidate failure surfaces may have a profound effect on the assessment...
of the landslide volume. For instance, a shallower failure surface may be identified as the most critical one, whereas the most critical surface is in fact a much deeper one whose resistance has been slightly overestimated (this may also occur due to the fact that the level of uncertainty on the determination of the properties of mechanical resistance for the various candidate failure surfaces can be significantly different). Typically, when more than one potential failure surface (or more than one failure mechanism) is considered in a geomechanical analysis, they are compared in terms of a dimensionless number, in most cases the factor of safety (less often, the so-called stability number is used), which according to engineering practice is defined as the ratio between the resistance offered by the ground within the slope (in this case, the resistance developed along the failure surface) and the factors causing instability, the main one being due to the self-weight of the volume involved in the landslide, together with the weight of any installation on the slope, seepage, and/or seismic forces. To answer the second question, “Under what conditions will a landslide occur?”, a knowledge of the most critical failure surface and its associated factor of safety is not enough. In fact, it requires also a reasonable estimate of potential future variations of the factors that affect the slope (e.g., hydrogeological conditions and anthropic activities) and of chemophysical factors that may decrease the ground resistance over time (e.g., creep behavior in cohesive soils and weathering in rocks).

The acquisition of reliable information on the geology of the slope is paramount for any geomechanical analysis of the safety of the slope, and for subsequent geotechnical work undertaken to increase the stability of the slope, since any type of analytical or computational calculation and design is based on data from geological maps (especially concerning the potential stratigraphy) and in situ investigations (e.g., borehole stratigraphic and lithologic logs, ground properties, structural features, hydrogeological modeling, and geophysical surveys).

The aim of this chapter is to outline the methods available for analyzing the critical failure mechanisms (hence the potential landslide volume) and the factor of safety of slopes. This is useful to geologists, engineering geologists, and geotechnical engineers and to various other professionals engaged in acquiring data and in slope stability analyses or hazard and risk assessment. For instance, information on the location of layers of different mechanical properties and the determination of these properties through standard laboratory and in situ tests provide more comprehensive information than can be derived simply from the mineralogical composition of the materials.

13.2 THE COMPUTATIONAL TOOLS AVAILABLE

Nowadays, there is a variety of tools available for slope stability analyses and modeling. In this chapter, the ones most used in practice are emphasized. The first method (and nowadays still the one most used in practice for the
assessment of the stability of slopes) is the so-called limit equilibrium method (LEM: Fellenius, 1927). Limit analysis, especially the upper bound form, is very important for slope stability since several stability charts generally employed by practitioners (e.g., Michalowski, 2002; Duncan and Mah, 2004; Utili, 2013) have been produced from this method. Unfortunately, the underlying analytical procedure is less well known among practitioners since they can directly use the derived standard charts without necessarily having to know the theory behind it. However, we believe it is important to include limit analysis in any textbook dedicated to the stability of slopes for its importance in deriving stability charts largely used by practitioners. In addition, finite element analyses are becoming increasingly popular (especially for cohesive frictional soils) to run stability analyses of slopes with complex stratigraphy. In slopes where several discontinuities are present (typically rock slopes), an equivalent continuum approach is not feasible because the scale of the discontinuities is too large in comparison with the scale of the slope, so discontinuous analyses are required. To this end, block theory was developed by Goodman and Shi (1985), and the Distinct Element Method (DEM; Cundall, 1971) is becoming increasingly popular. In this chapter, we will outline the basic tenets of LEM, limit analysis, finite element method (FEM), and some elements of discontinuity analysis, for example, Distinct Element Method and block theory.

The methods of analyses illustrated are very different in terms of both the conceptual framework of reference (e.g., continuum versus discontinuum soil mechanics) and the underlying physical assumptions (e.g., the validity of constitutive equations at the level of a representative element volume versus equations governing the mechanical interaction of blocks, and imposition of global equilibrium at the level of a finite wedge of soil/rock versus imposition of local equilibrium at the level of an infinitesimal element of soil/rock). Thus, it is important to start with an overview of the different conceptual frameworks of reference for the various methods illustrated.

Limit equilibrium methods are based on subdividing the mass of potentially unstable ground to be analyzed into (often) vertical slices of a finite size (if the slices are not vertical, they are sometimes called wedges), imposing the equations of (global) equilibrium on each slice, and assuming reaction forces along the boundaries of the slices according to some physical assumptions concerning both the interslice forces (forces exchanged between slices) and the forces at the base of each slide, which stem from the reaction offered by the ground underneath the failing mass and the water pressure. Typically, the water pressure acting at the base of each slice is either assumed on the basis of a hypothetical phreatic line or taken from a seepage analysis performed on the basis of some known hydraulic boundaries (for instance, the position of a horizontal phreatic line at a considerable distance from the slope). In some methods, some of the equations of equilibrium may not be satisfied, and the methods are called non-rigorous, whereas if all the equations are satisfied, the
methods are called rigorous. These methods provide no information on the stress state inside the failing mass, nor the deformations, nor the displacements.

In limit analysis, there are two methodologies that have been employed for slope stability: the so-called kinematic (or upper bound) and static (or lower bound) methods. The upper bound method is the more popular among practitioners, and for this reason, we shall present the upper bound method only. In this method, a failure mechanism has to be assumed as in LEM, but with the additional constraint of being kinematically compatible. This means that the failure mechanism has to satisfy equations imposing the constraint that the body can deform but remains a continuum at all times, that is, if we consider two adjacent points located at an infinitesimal distance from each other, neither detachment nor penetration between them is allowed. Then, the energy balance between the rate of external work done by the load applied on the failing mass and the rate of internal energy dissipation, that is, the energy dissipated by the deforming soil, is imposed for all the potential failure mechanisms considered. The critical failure mechanism is identified as the mechanism giving rise to the minimum (lowest) stability number. The energy balance equation translates the well-known principle of virtual work. Both methods assume that the materials constituting the slope behave as an elasto-perfectly plastic body, that is, they assume the validity of the normality rule according to which plastic deformations occur proportionally to the incremental stresses applied according to the so-called associate constitutive law. Considering a linear failure criterion, such as the Mohr–Coulomb criterion, means that the so-called dilation angle is assumed to be equal to the angle of internal friction. However, real frictional-cohesive soils and rocks do not obey the normality rule. In fact, soft rocks, overconsolidated clays, and cemented sands are usually characterized by a dilation angle that is smaller than the friction angle. Unfortunately, the limit theorems are only applicable to materials obeying a nonassociated flow rule in the case of translational failure (Drescher and Detournay, 1993), which is in general far less critical than rotational mechanisms. According to the limit analysis upper bound theorem (Chen, 1975), the collapse load for a material with a nonassociated flow rule is smaller than those obtained for the same material when an associated flow rule is assumed. Manzari and Nour (2000) were the first to examine the effect of soil dilatancy on homogeneous slopes, performing nonlinear finite element analyses of slopes by the strength reduction technique. They showed that the stability numbers obtained from limit analysis are not conservative (i.e., higher than the real value) for soils exhibiting dilation angles smaller than friction angles. Recently, Crosta et al. (2013) ran FEM analyses on straight homogeneous $c$, $\phi$ slopes with both the associative flow rule as assumed in limit analysis ($\Psi = \phi$) and with a dilation angle $\Psi = 1/4\phi$, typical for materials with little dilatancy, for a range of slope inclinations of 20–30°, with $\phi$ values ranging from 8 to 28°. It emerged that the influence of the dilation angle on the volume of the sliding mass is negligible. This is due to the fact that the
soil/rock is little constrained from a kinematic point of view in a slope (or in other words, the degree of confinement on the material is small), whereas dilatancy may have a very important effect in the case of high confinement (e.g., tunneling).

As in LEM, the presence of water pressure acting on the sliding mass is either assumed on the basis of a hypothetical phreatic line or taken from a seepage analysis performed on the basis of some known hydraulic boundaries (e.g., elevation of a horizontal phreatic line at a considerable distance from the slope). In the lower bound method, a stress field satisfying the local equations of equilibrium (i.e., the differential equations of static equilibrium for a deformable body) is imposed together with the already mentioned associated flow rule. This method is dual (i.e., where duality is in the sense of an optimization problem for which both the primal and the dual solutions are possible, and are associated, one with the lower bound and the other with the upper bound of the optimal sought value; Boyd and Vandenberghe, 2004. Their difference is called the duality gap) to the upper bound since the maximum (highest) stability number is sought. The upper bound limit analysis method will be expanded in section 13.4, since it is the most important for slope stability calculations.

The FEM is a very large subject to which several books have been devoted. Some cornerstone textbooks are Zienkiewicz and Taylor (2005), and, specifically for geotechnical engineering, Potts and Zdravkovic (1999, 2001). Here, we provide a brief overview of the main aspects of finite element analyses performed for slope stability. Unlike LEM and limit analysis, any constitutive law can be considered, so there are no restrictions on the type of mechanical behavior that can be considered for the soil/rock of the slope analyzed. This is a continuum mechanics approach since the materials constituting the slope are assumed to be one continuum or several continua separated by known boundaries (e.g., between different strata) along which a mechanical law ruling the interaction has to be specified. The differential equations of classical solid mechanics are applied, that is, equations imposing equilibrium (two in two-dimensional (2D) and three in 3D) on the stress field, equations imposing kinematic compatibility (three in 2D and six in 3D) on the deformation or strain field, and constitutive equations imposing the law of material behavior (three in 2D and six in 3D) linking stresses to strains. The equations of fluid mechanics ruling the behavior of the water (assumed to be in laminar regime, i.e., Darcy’s law) and its interaction with the solid phase (seepage forces exchanged between the solid and the fluid phases) are also considered. These differential equations are coupled (in 3D, they amount to a set of 16 independent differential equations), and therefore, no analytical solutions are available. However, they can be solved by discretization via the FEM so that they are imposed at the level of each small element used to discretize the continuum considered (at times, different discretizations may be used for the solid and the fluid phases).
In order to find the potential failure surface, usually the so-called strength reduction technique is employed. First, a solution is found for the whole slope in its current stable state, then the strength parameters of the slope (e.g., \( c \) and \( \phi \) or the Hoek–Brown parameters) are decreased by steps, with a new solution being sought after each step of strength decrease has been applied. After each step, the slope suffers extra deformations that typically tend to localize in a narrow band called a shear band that identifies the failure surface in the slope. However, loss of convergence (i.e., a solution satisfying all the differential equations ruling the problem cannot be found) often occurs before a shear band is clearly identifiable within the slope, so that some considerable approximation may be involved in the definition of the failure surface and in the assessment of the stability factor. In the latter case, for instance, if the loss of convergence is due to numerical reasons (i.e., the strength reduction stopped due to a numerical issue rather than because the strength at which failure occurs has been found), the calculated safety factor may be significantly lower than the real one. More recently, analyses where the failure surface is sought by gravity increase (Li et al., 2009; Nishimura et al., 2010) have been presented where the ground strength assigned is kept fixed and the self-weight of the slope is increased until a failure surface is detected. Whether the two methods give rise to similar results is still to be seen, but some studies support this hypothesis (Scholtes and Donzé, 2012). The advantage of using the FEM lies in the accuracy of the constitutive model that can be used. However, a drawback is that very often the failure surface found is poorly defined, and there is no certainty whether the factor of safety found reflects a loss of convergence of the static solution for physical or numerical reasons.

The DEM was introduced by Cundall (1971) for the analysis of rock mechanics problems and then applied to soils by Cundall and Strack (1979). A thorough description of the method is given in the two-part paper of Cundall (1988) and Hart et al. (1988). A distinct element code simulates the mechanical behavior of a system that comprises a collection of arbitrarily-shaped bodies that displace independently from one another and interact only at contacts or interfaces between the particles. In the Distinct Element Method, also called the soft contact approach of the more general discrete element method, the blocks are assumed to be rigid, and the behavior of the contacts is characterized using relative displacement–force laws. The mechanical behavior of such a system is described in terms of the movement of each particle and the interparticle forces acting at each contact point. Finite rotations and displacements of any body of the assembly, including complete detachment between bodies, are allowed. The interaction of the particles is treated as a dynamic process. The calculations alternate between the application of Newton’s second law to the particles and a force–displacement law at the contacts. Newton’s second law is used to determine the motion of each particle arising from the contact and body forces acting upon it, while the force–displacement law is used to update the contact forces arising from the relative motion at each contact.
The DEM is used in rock slope problems where the presence of single joints and discontinuities in the rock mass plays a dominant role with respect to the overall behavior. For instance, Allison and Kimber (1998) used the UDEC code (Itasca, 2006) to simulate the possible failure mechanisms of a rocky cliff featured by two sets of joints. More recently, Boon et al. (2014) simulated the occurrence of the famous Vaiont slide employing a 3D analysis. The DEM has great potential since unlike the other methods described so far (LEM, limit analysis, and to a certain extent finite element), it has the potential to simulate the whole landslide from its onset throughout the collapse and runout phases until the material comes to a full stop. However, a simulation (especially in 3D) of a full landslide is still computationally very onerous so that very important simplifications have to be undertaken. The most important of these is the upscaling of the block size employed so that potentially unrealistic failure surface and runout distances will be obtained. Although this tool is still mainly confined to academic research, we believe that its description is justified in light of the fact that the increasing computational power available to practitioners makes it likely that this may soon become a mainstream tool for the assessment of slope stability.

### 13.3 LIMIT EQUILIBRIUM METHODS

#### 13.3.1 Introduction

Limit equilibrium methods have been used in geotechnical engineering for decades to assess the stability of slopes. The idea of discretizing a potential sliding mass in vertical slices was introduced early in the twentieth century. In 1916, Petterson (1955) presented the stability analysis of the Stigberg Quay in Gothenberg, Sweden, where the slip surface was taken to be circular. But the first method of slices is associated with Fellenius (1927, 1936). His method (also known as the Ordinary method, the Swedish circle method, the conventional method, and the US Bureau of Reclamation method) assumes no interslice forces, and the factor of safety is achieved by the overall moment equilibrium around the center of a circular slip surface.

In the mid-1950s, Janbu (1954) and Bishop (1955) made advances in the method. Janbu developed his method for generic slip surfaces, whereas Fellenius and Bishop developed their methods for circular surfaces only (later Bishop extended his method to generic surfaces). In the 1960 and 1970s, most other methods were invented, some making the LEM a more powerful and refined tool of slope stability analysis (Spencer, Morgenstern & Price, Sarma methods) and others making it more suitable for hand calculations (force equilibrium methods). Many articles were published in these years on this topic: some of them made a real contribution to the improvement of the method, whereas in others, only slight modifications or different formulations of earlier methods were given.
In the late 1950s, these methods began to be implemented in computer codes. The advent of powerful desktop personal computers in the 1980s made it economically viable to develop commercial software based on LEMs. Nowadays, these methods are routinely used for stability analyses in geotechnical engineering practice, and many programs are available (examples will be supplied further on).

Methods of slices can be classified according to different criteria:

1. suitable only for circular failure surfaces or applicable to any shape of surface.
2. Rigorous or simplified: the former satisfy all the equilibrium equations, whereas the latter satisfy only some of the equilibrium equations. Some authors developed two versions (simplified and rigorous) of the same method: Bishop, Sarma, Janbu, etc. Within simplified methods, a large group is covered by the methods of forces (Lowe & Karafiath, 1960; Corps of Engineers method, Seed and Sultan, 1967).
3. Depending on assumptions made to render the problem statically determinate: three groups of methods can be recognized on the basis of the hypotheses introduced about the interslice forces (Espinoza et al., 1992, 1994).
4. Based on the parameter used to determine the critical surface: the traditional factor of safety $F_s$ or other parameters such as the critical horizontal uniform acceleration (Sarma, 1973, 1979; Spencer, 1978).

All methods approximate the bottom boundary of slices with linear bases. Formulations are based either on differential equations (e.g., Janbu, 1954; Bishop, 1955; Spencer, 1967) or algebraic equations, making it difficult for the inexperienced reader to compare different methods. As the factor of safety is calculated by algebraic equations, and LEMs are based on a slope divided into a discrete number of slices, here the latter formulation is preferred.

13.3.2 Assumptions That Make the Problem Determinate

13.3.2.1 First Group

In the first group, assumptions address the inclination of the resultants of interslice forces with respect to the horizontal direction:

$$T(x) = \lambda_1 f_1(x) E(x),$$  \hspace{1cm} (13.1)

where $\lambda_1$ is a dimensionless scaling parameter to be evaluated with the factor of safety, and $f_1(x)$ is a chosen scalar function of the abscissa ($x$) representing the distribution of the inclination of the interslice forces. Morgenstern and Price (1965) were the first to propose this type of assumption. To solve their method, they used the Newton–Raphson numerical technique.
Subsequently, Fredlund (1974) at Saskatchewan University implemented a different numerical procedure (Slope code) based on the so-called “method of best fit regression” (Fredlund and Krahn, 1977). This technique is illustrated since it is common to most methods of slices and it gives the reader a better understanding of the use of equilibrium equations in determining the factor of safety than does the Newton—Raphson technique. Moreover, the structure of the algorithm is almost the same as that implemented, later on, in other computer codes (Slope/w, 2002).

According to Fredlund and Krahn, the sensitivity of $F_{ff}$ and $F_{mm}$ to the distribution of the inclination of the interslice forces $f_i(x)$ is very different (Fredlund and Krahn, 1977). In fact, $F_{ff}$ shows a strong dependence on $f_i(x)$, whereas $F_{mm}$ shows no significant variations with $f_i(x)$ (Figure 13.1). However, in the case of uniform slope, the global factor of safety $F$ shows very little dependence on $f_i(x)$.

Spencer (1967) proposed a simpler expression than the Morgenstern & Price assumption: $T(x) = \tan \theta E(x)$. This assumption corresponds to $f_i(x) = 1$.

**FIGURE 13.1** Effect of different assumptions relative to the distribution of the inclination of the interslice forces (constant, sine, and clipped-sine) on the factors of safety. The analyzed slip surface is circular and crosses a uniform slope (after Fredlund and Krahn, 1977).
and $\lambda_1 = \tan \theta$, where $\theta$ is the angle between the interslice resultants and the horizontal direction. Therefore, it is a particular case of the Morgenstern & Price method. In the case of $\lambda_1 = 0$, the factor of safety coincides with $F_{\text{Bishop}}$.

According to Spencer’s static assumption, all interslice forces are parallel. However, a variation of the inclination of the interslice resultants along slices must be expected since physics suggests that the soil mass above the slip line is characterized by different stress states: it could be roughly divided into an active region, a transition region, and a passive region. Therefore, the assumed interslice force distribution is not realistic.

Lowe and Karafiath (1960) assumed the direction of the resultants of the interslice forces $\tan \theta$ equaled the average of the slope surface and the slip surface. The U.S. Corps of Engineers method (1970) takes $\tan \theta$ to be equal to either the changing slope of the ground surface or the average slope of the slip surface between the two end slices. Neither method introduces an extra unknown. In fact, they compute only $F_{ff}$, calculated with the prescribed distribution of $\tan \theta$, assuming the factor of safety of forces, as the final factor of safety. Of course, this factor does not satisfy all the equilibrium equations, and it can be very different from the factor of safety calculated by rigorous methods. In fact, $F_{ff}$ is very sensitive to the assumptions made (Figure 13.2).

Chen and Morgenstern (1983) were the first to focus on the physical admissibility of solutions; however, the main interest here concerns the new relationship among interslice forces, which they proposed. They took into consideration the slices located at the edges of slopes (end slices: 1, $n$). Assuming the slices to be infinitesimal and homogeneous, equilibrium considerations together with the Mohr–Coulomb failure criterion led them to infer that the direction of the interslice resultants of the end slices must be equal to the slope of the ground surface above the slices. They concluded that this condition is necessary to achieve a physically admissible solution.

But, this is not true. In fact, their demonstration is based on the implicit hypothesis of a uniform state of stress at failure throughout the end slices. In the case of finite slices, this hypothesis is not acceptable anymore. Therefore, taking the interslice resultants of the end slices to be equal to the slope of the ground surface cannot be judged a condition of physical admissibility. Nevertheless, Chen and Morgenstern’s requirement on end interslice resultants is reasonable since the stress field relative to these slices is, of course, well approximated by a uniform active and passive stress field.

Sarma (1979) suggested assuming a local factor of safety constant and equal to the factor of safety along the slip surface. In this case, the same degree of mobilization of the strength parameters along all the vertical slices is assumed. Sarma’s method assumes that the whole shear strength along the slip surface is mobilized under the action of a uniform horizontal acceleration $K$, defined as a fraction of the acceleration of gravity $K = a/g$. Therefore, a critical
acceleration factor $K_c = a_c/g$ is sought instead of the factor of safety. At the end of an iterative procedure, $K_c$ and the extra unknown $\lambda$ are determined.

To obtain the static factor of safety, the value of the factor of safety that gives zero critical acceleration needs to be computed. To this end, the strength parameters of the material along the slip surface have to be reduced by a trial factor of safety $F_i$, and the critical acceleration $K_{Ci}$ has to be computed. If $K_{Ci}$ is positive, the new trial factor must be $F_{i+1} > F_i$; otherwise, $F_{i+1} < F_i$. A few trials suffice to produce a relationship between $K_c$ and $F$ from which the static factor of safety can be inferred. Note that the surface with the lowest $K$ may not have the lowest $F$ and the surface with the lowest $F$ may not have the lowest $K_c$. This has been recognized by Sarma as well (Sarma and Bhave, 1974).

13.3.2.2 Second Group

In the second group of methods, assumptions concern the shape of internal shear force distribution $T(x)$:

$$T(x) = \lambda_2 f_2(x), \quad (13.2)$$
where $\lambda_2$ is a scaling factor with the dimension of a force and $f_2(x)$ a chosen scalar function. The expression proposed by Sarma belongs to this class of hypotheses:

$$T(x) = \lambda_2 f_2(x) \left[ c_{avg} H(x) + k' \frac{\gamma H^2}{2} \tan \phi_{avg} \right],$$

where $k'$ depends on the soil strength parameters and the geometric characteristics of the analyzed slice (Sarma, 1973). An assumption of this type was also proposed by Correia (1988).

Note that methods belonging to the first two groups do not use moment equilibrium equations for each slice in determining the factor of safety. The line of thrust can be determined at the end of the iterative process, once $F_s$ and $\lambda$ have been found, in order to assess the reasonableness of a solution. To do so, working from the first slice to the last, the points of action of the interslice forces $h_i$ are found by taking the equilibrium of moments for each slice in turn. The independent moment equilibrium equations available are $n - 1$, since the overall moment equilibrium has been used to determine $F_{mm}$, and they coincide with the remaining unknowns: $h_2, \ldots, h_n$.

13.3.2.3 Third Group

Unlike the methods previously discussed, the third group of methods of slices makes use of moment equilibrium equations for each slice in determining the factor of safety.

13.3.3 Limit Equilibrium Solutions: Physical Admissibility and Optimal Solutions

Chen and Morgenstern (1983) stated two conditions of physical admissibility. The first condition applies to interslice shear forces, which must not exceed the shear strength that can be mobilized along the slip surface, that is,

$$-1 \leq \rho \leq 1,$$

where $\rho$ is defined as the ratio between the local factor of safety averaged along the vertical face of each slice:

$$F(x) = \frac{[c_{avg} H(x) + E(x) \tan \phi_{avg}]}{T(x)}$$

and the factor of safety along the slip surface. The second condition concerns the line of thrust that must not lie outside the vertical surface of slices:

$$0 \leq h(x) \leq H(x).$$

According to Zhu et al. (2003), two more conditions can be stated: the effective normal forces $P_i$ and the effective interslice forces $E_i$ must be
nonnegative. These conditions apply to granular and cohesive soils, but do not apply to rock. In fact, LEMs are mainly used to analyze cohesive and non-cohesive slopes rather than rock slopes. For rock slopes, it is reasonable to accept negative interslice forces limited by the rock tensile strength. If the condition of Eqn (3) is violated, the factor of safety determined refers to a surface that is no longer the slip surface. In fact, slip should occur along the vertical interslice face where Eqn (3) is violated and along the remaining part of the analyzed slip surface.

Eqn (4) is required by elementary principles of mechanics, since the line of action of a force must intersect the body that the force is acting on. Zhu et al. (2003) showed that the condition of Eqn (3) may not be satisfied in simplified as well as rigorous methods (Figure 13.3).

Physical admissibility is a necessary requirement to achieve a reasonable solution from LEMs, but it is not yet sufficient. In fact, a solution may be physical admissible, but not reasonable if the achieved interslice force distribution does not “agree” with the state of stress within the slope mass which, in simple cases, can be guessed by engineering experience or otherwise needs to be determined by a finite element analysis. To perform such an analysis is a simple task as it is required to analyze the slope in its current stable state (far from collapse). No problems due to localization of deformations intervene, and even simple constitutive relations may be used. In fact, the purpose of such an analysis is to determine horizontal and shear stresses \((\sigma_{xx}, \tau_{xy})\) which, integrated along vertical lines from the slip to the ground surface, give rise to a distribution of “mobilized interslice forces” suitable to be compared from a qualitative point of view with the distribution resulting from the limit equilibrium analysis. If more than one failure surface has been analyzed by LEMs, only the distribution of forces relative to the most critical surface has to be judged.

![FIGURE 13.3](image)

(a) Composite slip surface. (b) Rho distribution obtained by the Morgenstern & Price method. (c) Rho distribution obtained by Janbu’s rigorous method (after Zhu et al., 2003).
Chen and Morgenstern (1983) aimed to find bounds to $f(x)$ from physical admissibility conditions. They analyzed two example cases, one of which is shown in Figure 13.4(a), by the Morgenstern & Price method. They used the numerical procedure based on the Newton–Raphson technique developed at the University of Alberta in order to find the factor of safety (Morgenstern and Price, 1967). They selected $f_0(x) = \sin x$ as the starting function expressing the variation of the inclination of the interslice resultants and determined the associated $F_s$. The starting function $f_0(x)$ was then modified by repeatedly adding $\Delta f(x) = K(\eta_1(x) + \eta_2(x))$ where $\eta_1(x)$ and $\eta_2(x)$ were arbitrarily chosen functions (parabolic and elliptic) and $K$ is a constant value that makes $\Delta f(x)$ sufficiently small in comparison to the corresponding value of $f(x)$ in order to not lose convergence. They then determined a new $F_{s_{i+1}}$ for each iteration.

**FIGURE 13.4** (a) The slope and the circular failure surface. (b) and (d) The distribution of the inclination of the interslice force resultants, initially sinusoidal, varied toward two opposite directions. (c) and (e) The variation of the local factor of safety relative to (b) and (d) distributions, respectively. The limiting condition is reached at $F_s = 1$ (after (Chen and Morgenstern, 1983)).

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\( f_{i+1}(x) = f_i(x) + \Delta f_i(x) \). According to the authors’ intentions, the iterative process should have ended when the function \( f_{\min}(x) \), giving rise to the minimum value of \( F_{\min} \), was found. In all cases examined, the iterative procedure was stopped on reaching the limiting condition in one or more slices (Figure 13.4(c,e)). Therefore, the \( F_{\min} \) determined, in terms of mathematical analysis, is a constrained minimum and not a free minimum.

In general, adopting different \( \eta_i \) functions, any type of distribution \( f(x) \) could be tested. But it is not necessary to use complicated functions, since the cubic functions adopted (\( \eta_1, \eta_2 \)) were enough to get a reasonable distribution (Figure 13.4(d)). From this work stems the idea of determining the best interslice force distribution function \( f(x) \), defined as the function among all the possible ones giving rise to the minimum factor of safety, as an optimization problem with constraints (physical admissibility conditions). Considering that the determination of the most critical slip surface is an optimization problem too, determining the most critical surface having assumed the best interslice force distribution becomes a double optimization problem.

Therefore, implementing an efficient and time-inexpensive algorithm capable of seeking the optimum interslice force distribution \( f(x) \) for each slip surface analyzed appears to be a difficult task, since the variation in the factor of safety for the assumed functions was small and relative minima of Fs were not found, but only constrained minima. Anyway, to assess the influence of interslice force distribution on Fs, an extensive numerical campaign is required, and this task appears prohibitive since Fs depends on many factors such as stratigraphy, soil strength, number of slices, and slip surface assumed.

To obtain physically admissible solutions, simple modifications to the existing algorithms of limit equilibrium computer codes could be introduced. Of course, further constraints may lead to nonconvergence since, in some situations, there are no physically admissible solutions for any interslice force distribution assumed. In these cases, the only way to achieve a safety factor consistent with a physically sound interslice force distribution is by repeating the analysis with another LEM. In general, methods belonging to the third group are less likely to converge to physically admissible solutions.

### 13.3.4 Some Critical Considerations

In simple cases, that is, when the slip surfaces analyzed are circular, Bishop’s simplified method is a good method since \( F_{\text{mm}} \) is only slightly sensitive to the extra unknown \( \lambda_1 \) (Figure 13.1), and it is lower than the factor of safety found by rigorous methods; therefore, it is on the safe side. This has been found to be an accurate method of analysis for circular slip surfaces in any case (Duncan and Wright, 1980). The Ordinary method, by contrast, gives bad results in the case of flat slopes and deep failure surfaces.

However, the stratigraphy of many slopes features the presence of thin weak layers (e.g., lenses of sand) or bedrock at shallow depths. Hence, the slip
surface becomes composite (partly planar and partly circular). In this case, $F_{mn}$ may be strongly sensitive to $\lambda_1$, and above all, it is no longer lower than $F$ (Figure 13.5). Therefore, in all cases of noncircular slip surfaces (composite, logspiral, etc.), rigorous methods have to be preferred since it is not guaranteed that $F_{\text{Bishop}}$ is an approximation on the safe side of the factor of safety found by rigorous methods.

Force equilibrium methods determine a safety factor $F_{ff}$ that already shows strong sensitivity to $\lambda_1$ for simple cases. In the past, when computers were not readily available, these methods were advantageous because they made it possible to calculate the factor of safety only by drawing force equilibrium polygons, and they had some success. But, nowadays, they should be avoided as they are unnecessary in any case.

Among rigorous methods, there is no physical evidence of the superiority of one method over another. Moreover, all methods can be reformulated in a unified framework (Espinoza et al., 1992, 1994), and there are computer codes where the main methods are implemented so that comparative analyses may be easily run. In any case, methods that make use of the moment equilibrium equation for each slice (third group) require the analyst to guess the line of thrust. This is a nonintuitive task especially for complex slopes. In these cases, methods belonging to the first group should be preferred as an engineer is usually better able to make a mental picture of the inclination of the interslice resultants than of their location. In other words, it is easier to guess forces than points of action as dealing with equilibrium of forces is usually simpler than equilibrium of moments.

Sarma’s method (based on the search for the critical acceleration factor) may be useful for slope stability analysis when the main cause of collapse is earthquake shaking. Of course, an analysis based on LEMs that considers the seismic effect only by means of a uniform static force appears very poor in comparison with other methods of analyses (FEM, spectral element method, boundary element method, coupled finite and spectral element method, etc.)
and cannot take into account fundamental phenomena such as soil liquefac-
tion, which intervene during the seismic event. But if a first rough estimation
by a pseudostatic analysis is sought, Sarma’s method appears to be the best
among all the LEMs. In fact, it makes it possible to determine the acceleration
level causing the slope to collapse and the failure surface as result of the
analysis. Then, a safety factor may be defined as the ratio between the critical
acceleration \( g \cdot K_c \) and the design seismic acceleration PGA (peak ground
acceleration) of the chosen return period. This safety factor has the advantage
of being based on the main cause of collapse. By contrast, all the other
LEM\s determine the factor of safety as the reduction of soil strength causing
slope collapse. The seismic effect is represented by a uniform inertial
force assumed equal to the PGA acceleration multiplied by the soil mass. The
factor of safety found is not based on the cause of collapse and the slip surface
found as critical is likely to not coincide with that determined by Sarma’s
method.

The concept of physically admissible solutions, first applied to LEM\s by
Chen and Morgenstern, is useful but not sufficient to obtain physically
meaningful solutions in terms of interslice forces. This is due to a limitation
intrinsic to all the LEM methods: they are based on satisfying the equations of
global static equilibrium written for finite soil slices, but they do not provide
any information on the actual state of stress within the soil mass. To obtain
such information, finite element analyses of the soil wedge (or more recently
DEM analyses) are needed.

A final observation concerns the very large literature available on LEM\s. Unfortu-
nately, as already mentioned, there is no uniform way of presenting the
methods, for example, variational formulation versus discrete formulation.
Moreover, a great deal of difficulty is due to the fact that, in many papers,
people who publish results referring to algorithms relative to methods of slices,
do not clearly and exhaustively report which equations they implemented in
their codes, but only quote the method that they intended to implement or to
generalize. Unfortunately, most times, their implementation does not corre-
spond to the original method and therefore it is very difficult to understand to
what extent the method used is different from the original one and to use the
results published to compare the performances of the methods they refer to.

13.4 LIMIT ANALYSIS

13.4.1 The Limit Analysis Upper Bound Theorem

Considering a 3D solid, a virtual rate of displacement that satisfies the
following relations:

\[
\dot{W}_{\text{ext}} = \int_V F_j \cdot \dot{u}_j dV + \int_{S_F} f_j \cdot u_j dS > 0 \text{ and } \dot{u}_i = 0 \text{ on } S_u, \quad (13.5)
\]
\begin{align*}
\dot{\epsilon}_{ij} &= \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}), \quad (13.6) \\
\dot{\epsilon}_{ij} &= \frac{\partial \mathbf{g}}{\partial \sigma_{ij}} \dot{\lambda} \text{ and } \dot{\lambda} \geq 0 \quad (13.7)
\end{align*}

with \( \mathbf{g} \) vector of plastic modes making a convex domain in the stress space, gives rise to a kinematically admissible act of motion. Assuming such an act of motion, the upper bound limit analysis theorem states that “the loads determined by equating the rate at which the external forces do work:

\begin{equation}
\dot{W}_{\text{ext}} = \int_{V} F_{j} \cdot \dot{u}_{j} dV + \int_{S_{f}} f_{j} \cdot \dot{u}_{j} dS \quad (13.8)
\end{equation}

to the rate of internal dissipation:

\begin{equation}
\dot{W}_{d} = \int_{V} \sigma_{ij} \dot{\epsilon}_{ij} dV \quad (13.9)
\end{equation}

will be either higher than or equal to the actual limit load.” Therefore, it can be inferred that the lowest load among all the loads relative to admissible failure mechanisms, determined by equating the rate of external work to the rate of energy dissipation, is the best approximation to the limit load. This load is an upper bound on the limit load.

In this case, the only force present is the weight force (a body force) and no tractions are present on the solid boundary. Eqn (13.7) are satisfied since a \( c - \varphi \) soil type has been assumed. Further, the problem is 2D.

A further assumption about kinematics has been made: rigid body motions are considered. This means that strains only develop along a narrow separation layer (discontinuity surface) between a sliding rigid body and a fixed one (Figure 13.6(a)) where all energy dissipation occurs. According to the assumptions made, the rate of energy dissipation is given by

\begin{equation}
\dot{W}_{d} = \int_{\Gamma} (\sigma \dot{\epsilon} + \tau \dot{\gamma}) d\Gamma. \quad (13.10)
\end{equation}

Strains develop according to an associated flow rule (Figure 13.6(b)).

The slope self-weight is given by \( F = mg = \rho g A \). Since the area \( A \) is proportional to the slope height \( H \), the load is proportional to \( H \) as well. Finally, the energy balance equation,

\begin{equation}
\dot{W}_{\text{ext}} = \dot{W}_{d} \quad (13.11)
\end{equation}

makes it possible to determine a function \( c = c \) (considered mechanisms) by which the most critical mechanism can be determined. The maximum of this function gives an upper bound on the limit value.
For most of the problems of practical interest in engineering geology and geotechnical engineering, the integration of the governing system of partial differential equations is possible only by means of approximate numerical methods. As mentioned above, among the various numerical techniques adopted to solve geomechanical problems, two methods are commonly used in practice for geomaterials characterized by nonlinear and irreversible behavior. These are the finite difference (FDM) and the finite element (FEM) methods that are based on the discretization of the governing partial differential equations. The derivative operator becomes the difference quotient in the FDM and the solution improves with reduction in the discretization step. Unfortunately, for complex material behaviors, the solution to the problem does not always exist, nor is it always unique, as is the case for linear governing equations. In the case of nonlinear equations (e.g., for coupled seepage and static problems), the efficiency and accuracy of FDM reduce and FEM is preferred, unless very fine discretization steps (in both space and time) are adopted.

The FEM approach is based on a local formulation of the governing partial differential equations and the transformation from a differential to an algebraic problem. FEM is based again on a spatial discretization, but the unknown functions are approximated through some functions with a constrained number
of unknown scalar coefficients that represent the values of the unknown functions at specific points. A detailed presentation of the FEM method for general problems in solid mechanics is given in Zienkiewicz and Taylor (2005). A specific textbook dedicated to the application of finite elements into geotechnical engineering with considerable emphasis on slope stability is that of Potts and Zdravkovic (1999).

The main advantage of FEM is the possibility of including complex constitutive laws (e.g., creep via viscosity) and inclusion of progressive failure (e.g., gradual development of the failure line) implying the redistribution of stresses in the continuum and along the failure surface. The disadvantage of FEM is the onerous computational time when parametric analyses or back analyses are sought, and approximations in the computation of the landslide volume because of the mesh size. The latter can be solved by the use of more advanced meshing techniques (e.g., adaptive remeshing). Recently, FEM analyses based on an arbitrary lagrangian eulerian framework have been adopted to simulate complex landsliding inclusive of the initial instability, the collapse, and runout and the associated erosion of basal material (Crosta et al., 2009a,b, 2013; Roddeman, 2008).

Finally, during 1994–2004, the combined finite-discrete element method has been developed and applied to the solution of complex solid mechanics problems including failure, fracture, fragmentation, collapse, and extensive material damage mechanisms (Munjiza, 2004).

13.6 DISTINCT ELEMENT METHOD

For jointed rock masses intersected by discontinuities, the design philosophy is different between sparsely and moderately jointed rock masses (Hoek et al., 1995). For sparsely jointed rock masses with large joint spacing, stability is governed by key blocks whose shapes permit free kinematic movement. Failure involves either sliding or falling of key blocks. The study of key blocks has led to established analytical design procedures using stereographic projection techniques and block theory (Londe, 1970; Goodman and Shi, 1985; Goodman, 1995). However, slope stability assessments based on key blocks are not suitable for slopes with moderately to heavily jointed rock masses. Sliding of large wedge structures is not normally encountered in such materials, as maintained by Hoek et al. (1995). In contrast, failure usually involves raveling or loosening of rock mass material (Utili and Crosta, 2011a,b). The failed material usually consists of numerous rock blocks. The complexity of the problem has led rock engineers to develop, for the purpose of routine design, rock mass classifications based on past field data, such as the Rock Mass Rating (Bieniawski, 1983) and the Q-system (Barton et al., 1974).

Jointed rock masses are typically made up from numerous polyhedral rock blocks, whose faces are determined by discontinuities in the rock field. The spatial distribution, size, and orientation of these discontinuities are rarely
regular and are usually assumed to follow probabilistic distributions. As a result, the sizes and shapes of each block in the jointed rock mass are different. For the purpose of distinct element modeling (DEM) or discontinuous deformation analysis (DDA), one has to invest significant effort to identify polyhedral blocks from the discontinuities, whose orientations are typically defined using their dip directions and dip angles (Figure 13.7).

Broadly, there exist two approaches in block generation algorithms, a summary of which is given in Boon (2013). The first approach is based on subdivision, in which discontinuities are introduced sequentially (Warburton, 1985; Heliot, 1988; Yu et al., 2009; Zhang and Lei, 2013). Each discontinuity is introduced one at a time. If the discontinuity intersects a block, the parent block is subdivided into a pair of so-called child blocks. This process is repeated until all the discontinuities have been introduced. The number of blocks increases as more “slices” are introduced, and a data structure of every block is maintained throughout the slicing process. The blocks generated through sequential subdivision are convex because a discontinuity has to terminate at the face of a neighboring block. Concave blocks can, nonetheless, be generated through the use of clustering, which can be automated (Yu et al., 2009) or guided by specifying fictitious construction joints (Warburton, 1985). Blocks subdivided by a construction joint are clustered together by imposing a kinematic constraint that prevents any relative movement between the two sides of the joint. Likewise, nonpersistent joints, that is, joints of finite sizes (Einstein et al., 1983; Zhang and Einstein, 2010), can be modeled through clustering or specifying fictitious construction joints. This is discussed in more detail in the next paragraph.

On the other hand, in the second approach (“face-tracing” based on homology theory), discontinuities are introduced all at once. All the vertices and edges in the domain are first calculated from the intersections between the discontinuities. From these vertices and edges, there are ways by which the faces and polyhedra in the rock mass can be identified (Lin et al., 1987; Ikegawa and Hudson, 1992; Jing, 2000; Lu, 2002). The necessary algorithms are, however, rather complex. The advantage of this approach is that convex and concave blocks are identified in the same manner. Nonpersistent joints and dangling joints, that is, joints that terminate inside intact rock without

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**FIGURE 13.7** Definition of strike, dip, and dip direction according to Hoek et al. (1995).
contributing to block formation (Jing, 2000), are also treated in the same manner as persistent joints, that is, joints of infinite size. Depending on the type of mechanical analysis that is to be performed on the generated rock mass, these dangling joints may have to be removed; for instance, they have to be removed if either the DEM (Cundall, 1988; Itasca, 2006, 2007) or DDA is used later on for analysis, but they do not need to be removed if fracturing has to be modeled, for instance, by employing the discrete FEM (Pine et al., 2006).

13.7 CONCLUSIONS

Assessing the degree of stability of natural or artificial slopes requires a series of steps: first, geological and geomorphological characterization; second, assessment of mechanical properties; third, choice of the boundary conditions; and finally, choice of a calculation method for assessing slope stability. This chapter addresses the last point, outlining limit equilibrium and limit analysis methods, and the basic principles and features of finite element, finite difference, and discrete element methods. Limit equilibrium methods are still the most widely used for practical applications, and for obtaining a rapid assessment of slope stability under complex geological conditions as well. Recently, 3D LEMs and codes have become increasingly available, enlarging the possible applications, and details can be found in the specialized literature. In this chapter, we have not discussed this subject, but the main principles of 2D methods are straightforwardly extendible to them. Limit analysis is a more robust method that has the great advantage of supporting parametric slope stability studies and for which the initial solutions have been extended to more complex conditions (e.g., slope geometry, tension cracks, and reinforcements). Numerical methods, for both continuum and discontinuum materials, are extremely powerful and versatile. The former are employed for frictional-cohesive slopes; the latter are used for rock slopes where the stability of the slope is ruled by the presence of discontinuities. Nevertheless, they require specific expertise in preparing data, characterizing the material behavior, and setting up the numerical model (e.g., discretization and meshing, boundary conditions), and their presentation requires a dedicated chapter or even a volume. Finally, the choice of the best method to be used depends on the quality of the input data and the target level of accuracy. The use of more than one method is recommended for providing a more complete understanding of the significant variables and parameters controlling the stability of the considered slope.

REFERENCES


Chapter | 13  Analysis Tools for Mass Movement Assessment


