High order finite difference WENO schemes for fractional differential equations

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A B S T R A C T

This letter develops high order finite difference weighted essentially non-oscillatory (WENO) schemes for fractional differential equations. First, the αth, $1 < \alpha \leq 2$, Caputo fractional derivative is split into a classical second derivative and a weakly singular integral. Then the sixth-order finite difference WENO scheme is used to discretize the classical second derivative and the Gauss–Jacobi quadrature is applied to solve the weakly singular integral. The constructed scheme of approximation for the fractional derivative has high order accuracy in smooth regions and maintains a sharp discontinuity transition. Finally, numerical experiments are performed to demonstrate the effectiveness of the proposed schemes.

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1. Introduction

Weighted essentially non-oscillatory (WENO) schemes are designed for numerically solving problems with piecewise smooth solutions containing discontinuities. The first WENO scheme was introduced in the form of a third-order finite volume scheme by Liu et al. [1]. Jiang and Shu provided a more efficient fifth-order WENO scheme (WENO5) in [2] with a general framework for the design of smoothness indicators and nonlinear weights. These schemes focus on discretizing the first space derivative. Recently a sixth-order finite difference WENO (WENO6) discretization was proposed for the second-derivative term [3].

Numerical methods for solving fractional differential equations have been developing rapidly over the last few decades. For example, the method of lines and the finite difference method are used to solve space fractional differential equations [4,5]; the finite element methods for space/time–space fractional differential equations are developed in [6,7]; and the spectral method for solving time–space fractional diffusion equations is presented in [8]. However, it seems that there are few works on numerically solving fractional differential equations with discontinuous solutions.

The aim of this letter is to extend the WENO scheme to a fractional differential equation, whose solution may be discontinuous. The basic idea is to split the αth, $1 < \alpha \leq 2$, Caputo derivative into a classical second derivative and a weakly singular integral, and then use WENO6 to discretize the second derivative and Gauss–Jacobi quadrature to solve the weakly singular integral. The numerical results confirm sixth-order convergence of the scheme and show the maintenance of sharp discontinuous transitions.

2. The numerical scheme

We are interested in designing WENO schemes for solving the following fractional differential equation:

$$\frac{\partial u(x, t)}{\partial t} = c_1 D_\alpha^x u(x, t) + c_2 D_\beta^u u(x, t) + f(x, t),$$  \hspace{1cm} (1)
where \(1 < \alpha < 2\), \(x \in [a, b]\), \(c_1\) and \(c_2\) are nonnegative constants and \(c_1c_2 \neq 0\), and the left and right \(\alpha\)th Caputo fractional derivatives are as defined by Podlubny [9]:
\[
\frac{\partial}{\partial t}^\alpha u_t(x) = \frac{1}{\Gamma(2 - \alpha)} \int_a^x \frac{u_t''(\xi)}{(x - \xi)^{\alpha - 1}} d\xi,
\]
(2)
\[
\frac{\partial}{\partial x}^\alpha u(x) = \frac{1}{\Gamma(2 - \alpha)} \int_x^b \frac{u''(\xi)}{(\xi - x)^{\alpha - 1}} d\xi.
\]
(3)

First, the fractional derivatives are split into a second derivative and an integral part:
\[
\mathcal{L} = \frac{\partial^2}{\partial x^2} u(x) \quad \rightarrow \quad \mathcal{L} = \frac{1}{\Gamma(2 - \alpha)} \int_a^x \frac{v(\xi)}{(x - \xi)^{\alpha - 1}} d\xi,
\]
(4)
\[
v(x) = \frac{\partial^2}{\partial x^2} u(x).
\]
(5)

Then (1) can be rewritten as
\[
\frac{\partial}{\partial t} [u(x, t)] = \frac{1}{\Gamma(2 - \alpha)} \left\{ c_1 \int_a^x \frac{v(\xi, t)}{(x - \xi)^{\alpha - 1}} d\xi + c_2 \int_x^b \frac{v(\xi, t)}{(\xi - x)^{\alpha - 1}} d\xi \right\} + f(x, t),
\]
(6)
\[
v(x, t) = \frac{\partial^2}{\partial x^2} u(x, t).
\]
(7)

For the second derivative in (7), we use WENO6 to do the discretization [3], i.e., a direct WENO discretization for the second derivative in conservation form. For (6), by simple linear transformations, there exist
\[
\int_a^x \frac{v(\xi, t)}{(x - \xi)^{\alpha - 1}} d\xi = \left( \frac{x - a}{2} \right)^{2 - \alpha} \int_{-1}^1 \frac{v \left( \frac{\alpha + x}{2} + \frac{\alpha - \eta}{2} \right)}{(1 - \eta)^{\alpha - 1}} d\eta,
\]
(8)
and
\[
\int_x^b \frac{v(\xi, t)}{(\xi - x)^{\alpha - 1}} d\xi = \left( \frac{b - x}{2} \right)^{2 - \alpha} \int_{-1}^1 \frac{v \left( \frac{b + \xi}{2} + \frac{b - \eta}{2} \right)}{(1 + \eta)^{\alpha - 1}} d\eta,
\]
(9)
respectively. On the basis of (8) and (9), we use the Gauss-Jacobi quadrature with weight functions \((1 - \eta)^{1 - \alpha}\) and \((1 + \eta)^{1 - \alpha}\) to solve the two weakly singular integrals in (6) respectively.

Let \(\tau\) be the time step size, \(\tau = T/M\), and \(h\) the spatial step size, \(h = (b - a)/N\), where \(M\) and \(N\) are, respectively, the numbers of subintervals divided in the time and spatial directions. Denote as \(u(x_i, t_n)\) and \(u^n_t\) the exact solution and the numerical solution at mesh point \((x_i, t_n)\), respectively. When performing the Gauss-Jacobi quadrature, usually it cannot be guaranteed that all the Gauss nodes are subsets of the points \(x_i\). So we need to do the interpolation to get the values of the intermediate functions \(v(x, t)\) at the Gauss nodes. And we suitably choose the degree of the interpolant polynomial such that the error caused in this approximation is of the same order as the error of the corresponding WENO scheme.

After doing the spatial discretizations, we get a classical nonlinear ODE system with its variables \(u_i\). The third-order TVD Runge–Kutta method [10] is used to solve the ODE system obtained.

3. Numerical results

By means of two examples, we demonstrate the effectiveness of the algorithms provided, including the convergence order and the maintenance of sharp discontinuity transitions. We compute the fractional differential equation with periodic boundary condition to verify the convergence order in the first example. The second example focuses on numerically solving the fractional differential equations with discontinuous initial condition and piecewise smooth solution.

Example 1. Consider the following fractional differential equation:
\[
\frac{\partial}{\partial t}^\alpha u_t(x) = \frac{\partial}{\partial x}^\alpha u(x) + f(x, t), \quad x \in [0, 1],
\]
with periodic boundary conditions and the initial condition
\[
u(x, 0) = \sin(4\pi x) - 2\sin(2\pi x),
\]
Table 1
The $L_\infty$ error and convergence order for the numerical solution of Example 1 with the step sizes $h = 1/N$, $\tau = 0.4h^2$ at $T = 0.1$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L_\infty$ error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.96e-004</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>2.28e-006</td>
<td>6.44</td>
</tr>
<tr>
<td>80</td>
<td>3.10e-008</td>
<td>6.20</td>
</tr>
<tr>
<td>160</td>
<td>4.73e-010</td>
<td>6.04</td>
</tr>
</tbody>
</table>

Table 2
The $L_\infty$ error and convergence order for the numerical solution of Example 1 with the step sizes $h = 1/N$, $\tau = 0.3h^{1.6}$ at $T = 0.1$.

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<td>6.20</td>
</tr>
<tr>
<td>160</td>
<td>4.72e-010</td>
<td>6.04</td>
</tr>
</tbody>
</table>

Table 3
The $L_\infty$ error and convergence order for the numerical solution of Example 1 with the step sizes $h = 1/N$, $\tau = 0.3h^{1.4}$ at $T = 0.1$.

<table>
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<th>$L_\infty$ error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.06e-004</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>2.28e-006</td>
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</tr>
<tr>
<td>160</td>
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<td>6.14</td>
</tr>
</tbody>
</table>

and where the exact solution is taken as $u(x, t) = e^{-t}[\sin(4\pi x) - 2\sin(2\pi x)]$. And $f(x, t)$ is numerically obtained. Table 1 shows the $L_\infty$ error and convergence order. The sixth-order convergence of the proposed scheme is confirmed.

From Tables 2 and 3, it can be seen that the CFL condition of the numerical scheme is not worse than $\tau \sim h^\alpha$, where $\alpha$ is the order of the fractional derivative. From the observed good convergence orders, we know that the numerical errors are dominated by space discretization.

**Example 2.** Consider the following fractional differential equation:

$$\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial x} = D \nabla^\alpha u, \quad x \in [-L, L],$$

with the initial condition

$$u(x, 0) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

and the boundary conditions

$$u(-L, t) = 1, \quad u(L, t) = 0,$$

where $2\nabla^\alpha u = \xi D^\alpha_x u + \xi^\alpha D^\alpha_x u$, $V = 0.5$, $D = 0.2$, and $L = 1$ or 10.

In the simulations of Figs. 1 and 2, we take $L = 1, N = 100, h = 2/N, \tau = 0.4h^2$. Fig. 1 shows numerical solutions for different orders of the fractional derivative, $\alpha = 1.2, 1.4, 1.6, 1.8$, at $T = 0.01$. It can be noted that the numerical simulations retain sharp transitions for different $\alpha$.

In Fig. 2, we plot the numerical solutions computed by using the scheme in which the classical second spatial derivative is approximated by using the second central difference scheme. Comparing with the results in Fig. 1, it is obvious that the numerical solutions in Fig. 2 exhibit oscillatory behavior.

Fig. 3 shows the numerical solutions for $D = 0.2$, $\alpha = 1.8$ at $T = 1, T = 2$, and $T = 3$. In the simulations, the parameters are chosen as: $L = 10, N = 200, h = 20/N$, and $\tau = 0.4h^2$.

Fig. 4 shows the numerical solutions for $D = 0.02$, $\alpha = 1.8$ at $T = 1, T = 2$, and $T = 3$. In the simulations, the parameters are chosen as: $L = 10, N = 200, h = 20/N$, and $\tau = 0.3h^{1.8}$. 
Fig. 1. The numerical solutions of Example 2 obtained using the WENO scheme with $h = 2/100$, $\tau = 0.4h^2$ for different orders of the fractional derivative at $T = 0.01$.

Fig. 2. The numerical solutions of Example 2 obtained using the second central difference scheme for the classical second spatial derivative with $h = 2/100$, $\tau = 0.4h^2$ for different orders of the fractional derivative at $T = 0.01$.

Fig. 3. The numerical solutions of Example 2 obtained with $h = 20/200$, $\tau = 0.4h^2$ for $\alpha = 1.8$ at $T = 1, T = 2, T = 3$. 
Fig. 4. The numerical solutions of Example 2 obtained with $h = 20/200$, $\tau = 0.3h^{1.8}$ for $\alpha = 1.8$ at $T = 1$, $T = 2$, and $T = 3$.

4. Conclusions

This paper discusses numerical schemes for efficiently solving a fractional differential equation with piecewise smooth solutions. We first split the fractional derivative into a classical second derivative and a weakly singular integral; then the second derivative is approximated by using the WENO6 scheme and the weakly singular integral is computed by Gauss–Jacobi quadrature. Then we obtain the classical ODE system, which is computed by the TVD Runge–Kutta method. The numerical scheme is verified to have sixth-order convergence for the smooth solution, and to retain sharp transitions for discontinuous solutions. Upcoming work should aim to extend the ideas of this letter to multidimensional cases and nonlinear fractional systems.

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