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The tensile armour behaviour of unbonded flexible pipes close to end fittings under axial tension

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An analytical model is given to investigate the tensile armour behaviour of unbonded flexible pipes close to end fittings under axial tension with no friction. The deviation from the initial lay angle is taken to describe the path of the single armour wire and is determined by minimisation of the strain energy functional using the Euler equation. The obtained simultaneous differential equations are transformed into a boundary value problem which is solved numerically. An analytical solution is found by neglecting the twist of the wire cross-section with respect to the wire centre-line. The developed model is verified with a finite element simulation. Good agreement of armour wire path and maximum lateral bending moment is observed between the model predictions and the finite element results. The discrepancies in the bending and twisting moment distributions are attributed to the twisting constraint assumed in the analytical solutions. The verified model is then applied to typical flexible pipe designs to find the level of the greatest increases in stress. Inclusion of the cross-section twist decreases the lateral bending stress. The potential effect of friction on the results is also discussed.

Keywords: unbonded flexible pipe; tensile armour; end fitting; transverse slip; stress increase

1. Introduction

Flexible pipes (Figure 1) have been largely used as the offshore industry advances into deeper waters and harsher environments where a high level of bending deformation is expected. As shown in Figure 2, all the strength members in the pipe’s construction are terminated at an end fitting, so that axial loads and bending moments can be transmitted into the end connector without adversely affecting the fluid-containing layers (API 2008). In deep-water applications, such as a flexible riser, significant dynamic tensile loads may occur due to the tangential drag forces along the riser. In addition, large bending effects take place near the end fitting owing to the joining of rigid and flexible system elements. Therefore, stress concentrations and higher fatigue will result. A bend stiffener is usually designed to protect the riser from overbending and from the accumulation of fatigue damage in the vicinity of the rigid connection (Dong et al. 2013a).

Although flexible pipe distant from any restraint has been extensively analysed (Feret and Bournazel 1987; Claydon et al. 1992; Witz and Tan 1992a; Witz and Tan 1992b; McIver 1995; Witz 1996; Kraincanic and Kebadze 2001; Leroy and Estrier 2001; Custodio and Vaz 2002; Ramos and Pesce 2004; Bahtui 2008; Dong et al. 2013; Dong et al. 2013b; Dong et al. 2014), there has been little work concentrating on the mechanical behaviour of flexible pipes close to end fittings. Most of the models, theoretical or numerical, are based on the hypothesis that the length of the pipe is assumed sufficiently long for the clamping conditions to be negligible. Together with the greater likelihood of failure in this region, it calls for a detailed study of this topic. The following is a brief review of the main works regarding the analyses of structures incorporating helical layers with end restraints.

Using Lutchansky’s shear interaction model (Lutchansky 1969) which is capable of examining the end restraint effect, Raoof and Hobbs (1984) addressed the bending behaviour of spiral strands and armoured cables both near and remote from the terminations. It was concluded that slippage between wires close to the fixed termination was the factor controlling the cable’s fatigue life. In a series of later publications, Raoof and co-workers put substantial effort into fatigue life estimation of spiral strands and cables at terminations (Raoof 1992; Raoof 1993a; Raoof 1993b; Raoof and Davies 2005; Raoof and Davies 2014). Interwire/interlayer fretting was assumed in their theory to be the main mechanism responsible for wire fractures under cyclic loading, and a contact stress–slip parameter was suggested for estimating the free-bending fatigue life of spiral strands. Ramsey (1991) applied his thin rod theory (Ramsey 1988) to analyse the effect of constraint due to clamp or socket end connections on changes in the lay angles of the constituent wires in multilayered cables. The case of axisymmetric loading in the absence of friction was examined,

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2. Analytical model

Many papers address the phenomenon of tensile armour slip which governs a flexible pipe’s response to bending. Slip is considered by some to be adequately measured by the consideration of axial slip along the strained helix, while others assume that a movement to a new geodesic curve is the controlling factor. Both curves are considered as the limit curves for wire slip, and the actual path is probably somewhere in between.

As the pipe is loaded, the wire in the end fitting region moves away from the strained helix to a new position which cannot be a geodesic due to the end restraint, despite the absence of friction. The new configuration of the wire

and a boundary layer effect was observed in wire stresses. Starting with Love’s equations of equilibrium (Love 1944), Rawlins (2005) studied the flexure behaviour of a simple stranded cable in the vicinity of a clamp. Martindale (2006) performed a study of axial and transverse slip in helically wound tensile armour wires on a flexible pipe under the influence of an end fitting. The developed model uses small changes in helical angle along the wire to describe tensile armour configuration as the pipe is stretched and bent. The path adopted by the wire is found by minimising total strain energy to find stress concentrations and slip characteristics. The solution to the case in which the pipe is under axial tension was presented independently by Thorsen (2011).

The development of finite element (FE) analysis methods has allowed numerical analyses of flexible pipe internal structure. Sævik (1992, 1993, 2011) developed a curved beam element for FE modelling of a flexible pipe tensile armour strip. Curvature distribution along the pipe and end restraint effect on the single armour can be taken into account. The use of sophisticated FE packages is another numerical modelling option. Jiang and Henshall (1999) developed an FE model of a seven-wire strand for the analysis of termination effects on the contact forces and the resulting relative movements between wires along the contact lines. Shen et al. (2008) and de Sousa et al. (2013) addressed the stress characteristics induced in flexible pipe tensile armour wires inside an end fitting.

The aforementioned models are, as noted, mainly developed for the analyses of spiral strands or cables, partly because these structures have a longer history than flexible pipes. This work concentrates on the analysis of flexible pipe tensile armour slip and stresses in the region of an end fitting under axial tension. The existing models (Martindale 2006; Thorsen 2011) are extended to account for the axial strain effect and the twist of the wire cross-section with respect to the wire centre-line.
centre-line is what is required and can be described by the deviation through a small angle from the initial lay angle. The distribution of this angle can be determined by the minimisation of strip strain energy due to axial strain, bending and twist, with an appropriate constraint.

2.1. Assumptions and definitions
The main assumptions used throughout the analysis are as follows.

1. The armour strip has a uniform rectangular cross-section. In any cross-section, the principal axis drawn through the centre and perpendicular to the longer side is coincident with the normal to the supporting cylindrical surface. This assumption is shown to be fulfilled when the strip is exposed to reasonably high stresses, as during normal load conditions of a flexible pipe. The fact that one layer is restrained by the next layer should, however, make this assumption reasonable even if compression occurs.

2. The strip is treated as a thin rod with the initial shape of a helix in the unloaded state. The radius of the helix is assumed to remain unchanged due to the presence of interlocking pressure armour layer and the total radial constraint at the end fitting. The change in the strip thickness, i.e. the Poisson effect, is also ignored since the strip stiffness is relatively high.

3. The armour wires are fully fixed at the end fitting. This assumption tends to give conservative results but its accuracy depends on the end fitting construction.

4. All deformations and displacements are sufficiently small so that non-linear terms can be neglected and the material remains in the elastic range.

5. Friction is not included. Although it can simplify the analysis, this assumption does have some influence on the results, which will be discussed later.

Now consider a helix of length $L$ and lay angle $\alpha_0$ wound on a cylindrical element of length $L_p$ subjected to axial tension defined by global strain $\varepsilon_p$. The element is developed so that the original and strained helices are taken as continuous lines, as shown in Figure 3. As the pipe is exposed to axial tension, the lay angle of the armour wire must change slightly. If no end restraints are present, this change in lay angle would be the same at all points along the wire and can be determined as

$$\beta = \alpha_0 - \alpha_1 = \alpha_0 - \tan^{-1} \frac{\tan \alpha_0}{1 + \varepsilon_p}.$$  (1)

where $\alpha_1$ is the lay angle of the deformed wire. From geometry shown in Figure 3, the linearised axial strain in the deformed wire is found to be

$$\varepsilon = \varepsilon_p \cos^2 \alpha_0.$$  (2)

The displacement pattern ($\alpha_1$) and strain ($\varepsilon$) are only possible for cases with no end restraints. The wire is actually fixed at the end fitting where the change in lay angle is not allowed. This implies that the lay angle must remain at $\alpha_0$ at the termination and approach $\alpha_1$ gradually as one moves away. The actual change in lay angle denoted as $\alpha$ is what we require and can be determined by the minimisation of the strain energy in the deformed wire.

2.2. Wire strain energy
The internal strain energy of the armour strip, taking axial strain, bending and twist into account, is

$$\Pi = \int_0^L \frac{1}{2} \left( E \varepsilon_a^2 + EI_1 \kappa_1^2 + EI_2 \kappa_2^2 + G I_3 \kappa_3^2 \right) dS,$$  (3)

where $S$ is the arc length along the strip centre-line in the undeformed state; $E$ and $G$ are Young’s modulus and the shear stiffness of the material, respectively; $A$ is the cross-sectional area of the strip; $\varepsilon_a$ is the axial strain of the strip following the actual path; $I_i$ ($i = 1 \sim 3$) is the moment of inertia around the principal axes of the strip cross-section; and $\Delta \kappa_i$ ($i = 1 \sim 3$) is the change in curvature or torsion. The estimation of $\varepsilon_a$ and $\Delta \kappa_i$ addressed in the following sections.

2.2.1. Wire axial strain
An alternative solution to this task is to determine the tensile load in the wire using Love’s equations (LeClair and Costello 1988; Martindale 2006). This procedure is preferable in bending analysis where there are some difficulties in estimating the axial strain directly. The loading case considered in this study, however, allows the axial strain to be.
obtained from geometric considerations:
\[
\epsilon_a = \frac{ds - dS}{dS},
\]

(4)
where \( s \) is the arc length along the strip centre-line in the
deformed state. The angle between the actual path and the
strained helix is \( \beta - \alpha \), as shown in Figure 3, so we have
\[
\cos (\beta - \alpha) = \frac{dS (1 + \epsilon)}{ds}.
\]

(5)
Substituting Equation (5) into Equation (4) and linearising
the expression results in
\[
\epsilon_a = \epsilon + \frac{1}{2} (\beta - \alpha)^2.
\]

(6)

2.2.2. Wire bending and twist

Love’s thin rod theory (Love 1944) has been used exten-
sively in the analytical analysis of structures incorporating
helical layers. Witz and Tan (1992a) were among the first
to apply Love’s equations of equilibrium for the analysis
of flexible pipes. However, as Ramsey (1988) pointed out,
most authors mistakenly identified Love’s change in torsion
with the change in the twist of the rod centre-line. Love’s
variable \( \tau \), which measures the torsion of the rod, depends
not only on the twist of the rod centre-line, but also on
an angle \( f \) which describes the orientation of the principal
axes of the rod cross-section with respect to the principal
normal of the rod centre-line. In the twisting of a straight
rod, Love’s angle \( f \) measures directly the rotation of the
rod cross-section. Moreover, the effect of axial strain on the
bending and torsion of the rod was ignored in Love’s theory.
Ramsey (1988) took this effect into account but the devel-
oped method is not applicable in Love’s coordinate system.
The modified expressions of curvature and torsion incre-
ments considering the axial strain effect and the rotation of
the rod cross-section are derived as follows.

A local coordinate system \( \mathbf{U} \) is defined, as shown in
Figure 4, to describe the centre-line of the strip, with a
set of unit vectors \( \{ \mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z \} \) for the coordinate axes. To
consider the rotation of the cross-section relative to the
centre-line, another local coordinate system is introduced
with base vectors \( \mathbf{a}_i \). The orientation is such that \( \mathbf{a}_1 \) is always
directed along the inwards surface normal and fixed to the
cross-section principal axes (see the first assumption) while
\( \mathbf{a}_3 \) is parallel to \( \mathbf{u}_z \), directed along the tangent of the strip
centre-line. The last vector \( \mathbf{a}_2 \) is determined by the right-
hand rule. Considering the relative angle \( \varphi \) between the two
systems, as shown in Figure 4, we have
\[
\begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix}
=
\begin{bmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_x \\
\mathbf{u}_y
\end{bmatrix},
\]

(7)

The following equation holds for the \( \mathbf{a}_i \) system:
\[
\frac{d}{dS}
\begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix}
=
\begin{bmatrix}
0 & \kappa_3 & -\kappa_2 \\
-\kappa_3 & 0 & \kappa_1 \\
\kappa_2 & -\kappa_1 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix},
\]

(8)
where
\[
\kappa_i = \kappa_i^0 + \Delta \kappa_i, \ (i = 1 \sim 3).
\]

(9)
Here, \( \kappa_i^0 (i = 1 \sim 3) \) is the initial curvature or torsion of
the strip in the undeformed state. Similarly, for the system
\( \mathbf{U} \) we have
\[
\frac{d}{ds}
\begin{bmatrix}
\mathbf{u}_x \\
\mathbf{u}_y \\
\mathbf{u}_z
\end{bmatrix}
=
\begin{bmatrix}
0 & \kappa_z & -\kappa_y \\
-\kappa_z & 0 & \kappa_x \\
\kappa_y & -\kappa_x & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_x \\
\mathbf{u}_y \\
\mathbf{u}_z
\end{bmatrix},
\]

(10)
where \( \kappa_x \) and \( \kappa_y \) are the curvature components of the strip
centre-line; \( \kappa_z \) is the torsion.

Writing the third equality of Equation (10) results in
\[
\frac{du_z}{ds} = \kappa_y \mathbf{u}_x - \kappa_x \mathbf{u}_y.
\]

(11)
Considering the local coordinate systems defined above and the axial strain in the strip, the left side of Equation (11) can be rewritten as

\[
\frac{du_z}{ds} = \frac{da_3}{ds} \simeq (1 - \varepsilon_a) \frac{da_3}{dS}.
\]  
(12)

Substituting the third equality of Equations (8) and (9) into Equation (12) and neglecting the non-linear terms gives

\[
\frac{du_z}{ds} = (\kappa_2^0 - \kappa_2 \varepsilon_a + \Delta \kappa_2) a_1 - (\kappa_1^0 - \kappa_1 \varepsilon_a + \Delta \kappa_1) a_2.
\]  
(13)

Combining Equations (11) and (13) and considering Equation (7) yields

\[
\begin{align*}
\kappa_x &= (\kappa_1^0 - \kappa_1 \varepsilon_a + \Delta \kappa_1) \cos \varphi - (\kappa_2^0 - \kappa_2 \varepsilon_a + \Delta \kappa_2) \sin \varphi, \\
\kappa_y &= (\kappa_1^0 - \kappa_1 \varepsilon_a + \Delta \kappa_1) \sin \varphi + (\kappa_2^0 - \kappa_2 \varepsilon_a + \Delta \kappa_2) \cos \varphi.
\end{align*}
\]  
(14a, 14b)

The following equation can be obtained using the similar procedure:

\[
\kappa_z = \frac{d\varphi}{dS} + (\kappa_2^0 - \kappa_2 \varepsilon_a + \Delta \kappa_3).
\]  
(14c)

Noticing the fact that

\[
\kappa_1^0 = \kappa_1^0, \kappa_2^0 = \kappa_2^0, \kappa_3^0 = \kappa_3^0
\]  
(15)

in the initial state and solving \(\Delta \kappa_j\) from Equation (14) leads to

\[
\begin{align*}
\Delta \kappa_1 &= \kappa_x \cos \varphi + \kappa_y \sin \varphi - \kappa_x^0 (1 - \varepsilon_a) \\
\Delta \kappa_2 &= -\kappa_x \sin \varphi + \kappa_y \cos \varphi - \kappa_y^0 (1 - \varepsilon_a) \\
\Delta \kappa_3 &= \frac{d\varphi}{dS} + \kappa_z - \kappa_z^0 (1 - \varepsilon_a).
\end{align*}
\]  
(16a, 16b, 16c)

The derivation of Equation (16) is independent of loading cases, and therefore it holds under both axisymmetric loads and bending. If the axial strain effect and the rotation of the strip cross-section are ignored, Equation (16) is reduced to

\[
\begin{align*}
\Delta \kappa_1 &= \kappa_x - \kappa_x^0 \\
\Delta \kappa_2 &= \kappa_y - \kappa_y^0 \\
\Delta \kappa_3 &= \kappa_z - \kappa_z^0.
\end{align*}
\]  
(17a, 17b, 17c)

Equation (17) is used extensively in the constitutive relations for bending and twisting moments in a helical strip. However, from the derivation shown above, it is slightly inappropriate to use it directly, despite the fact that the usage will not cause significant accuracy loss in the analysis.

Now consider the curvature components and the twist of the strip centre-line, involved in Equation (16), before and after deformations. The initial curvatures and twist are given by the well-known formulae:

\[
\begin{align*}
\kappa_x^0 &= 0, \kappa_y^0 &= \frac{\sin^2 \alpha_0}{R}, \kappa_z^0 &= \frac{\sin \alpha_0 \cos \alpha_0}{R}
\end{align*}
\]  
(18)

where \(R\) is the radius of the centre-line. In the deformed state, the lay angle of the strip centre-line changes to \(\alpha_0 - \alpha\) and Equation (18) becomes

\[
\begin{align*}
\kappa_x &= 0, \kappa_y &= \frac{\sin^2 (\alpha_0 - \alpha)}{R}, \kappa_z &= \frac{\sin (\alpha_0 - \alpha) \cos (\alpha_0 - \alpha)}{R}.
\end{align*}
\]  
(19)

Substituting Equations (18)–(19) into Equation (16), using double angle formulae and retaining only terms to first order in \(\alpha\) and \(\varphi\) results in

\[
\begin{align*}
\Delta \kappa_1 &= \frac{\sin^2 \alpha_0}{R} \varphi \\
\Delta \kappa_2 &= -\frac{\sin 2 \alpha_0}{R} + \frac{\sin^2 \alpha_0}{R} \varepsilon_a \\
\Delta \kappa_3 &= \frac{d\varphi}{dS} - \frac{\cos 2 \alpha_0}{R} \alpha + \frac{\sin \alpha_0 \cos \alpha_0}{R} \varepsilon_a.
\end{align*}
\]  
(20a, 20b, 20c)

A helix is a specific geodesic on a cylindrical surface, and therefore, at any point on the helix, its normal is parallel with the surface normal and the helix has no transverse curvature. However, it will experience transverse curvature if the wire follows a line other than the geodesic path. Therefore, an additional term should be included in Equation (20a), using a similar way as for straight beams:

\[
\Delta \kappa_1 = -\frac{d\alpha}{dS} + \frac{\sin^2 \alpha_0}{R} \varphi.
\]  
(21)

2.3. Minimisation of the strain energy functional

The generalised strains involved in Equation (3) have been discussed in the preceding section. Since the strains are all functions of \(\alpha\) and \(\varphi\) which in turn are functions of the arc length \(S\), it is possible to establish the total strain energy in the wire as a functional of these angles. The problem is now reduced to find the extremal functions which minimise the strain energy functional.
For a functional

$$\Pi = \int_{y_1}^{y_2} F(x; y_1, y_2, \ldots, y_N; y_1', y_2', \ldots, y_N', y_1^{(n)}, \ldots, y_N^{(n)}) \, dx$$

(22)

involving $n$th order differentials of the functions $y_i(x)$ ($i = 1, 2, \ldots, N$), the founded extremal functions must satisfy the Euler equation (Chien 1980):

$$\sum_{m=0}^{n} (-1)^m \frac{d^m}{dx^m} \left( \frac{\partial F}{\partial y_i^{(m)}} \right), \quad (i = 1, 2, \ldots, N).$$

(23)

Applying Equation (23) to the problem on hand leads to

$$\begin{align*}
\frac{\partial F}{\partial \alpha} - \frac{d}{dS} \left( \frac{\partial F}{\partial \alpha'} \right) &= 0 \\
\frac{\partial F}{\partial \varphi} - \frac{d}{dS} \left( \frac{\partial F}{\partial \varphi'} \right) &= 0
\end{align*}$$

(24)

Substituting Equations (6), (20b), (20c) and (21) into Equation (24) and denoting by $F$ the integrand in the expression of the strain energy yields

$$F = \frac{1}{2} \left( EA \left( \varepsilon + \varepsilon_0 \frac{1}{2} \beta - \alpha \right)^2 + EI_1 \left( -\frac{\partial \alpha}{\partial \alpha} + \frac{\sin^2 \alpha_0 \alpha}{R} \varphi \right)^2 \right. $$

$$+ EI_2 \left( -\frac{\sin 2\alpha_0}{R} - \frac{\sin^2 \alpha_0}{R} \left( \varepsilon + \frac{1}{2} \beta - \alpha \right)^2 \right) $$

$$+ GI_3 \left( -\frac{\cos 2\alpha_0}{R} - \frac{\sin \alpha_0 \cos \alpha_0}{R} \right) $$

$$\left. \times \left( \varepsilon + \frac{1}{2} \beta - \alpha \right)^2 \right)$$

(25)

Substituting Equation (25) into Equation (24) and, once again, neglecting the non-linear terms, we can obtain the following simultaneous differential equations:

$$\begin{align*}
EI_1 \alpha'' - T^* \alpha - M^* \varphi' + Q &= 0 \\
GI_3 \varphi'' - N^* \varphi + M^* \alpha' &= 0
\end{align*}$$

(26)

where

$$T^* = EA \varepsilon + EI_1 \left( \frac{\sin^2 \alpha_0}{R^2} + \frac{4 \sin^3 \alpha_0 \cos \alpha_0}{R^2} \frac{\beta}{R^2} \right) + GI_3 \left( \frac{\cos^2 \alpha_0}{R^2} + \frac{\sin 4\alpha_0}{R^2} \frac{\beta}{R^2} + \frac{1}{4 R^2} \right)$$

$$M^* = EI_1 \frac{\sin^2 \alpha_0}{R} \beta + GI_3 \left( \frac{\cos 2\alpha_0}{R} \frac{\beta}{R} + \frac{\sin \alpha_0 \cos \alpha_0}{R} \beta \right)$$

(27)

which can be simplified to

$$T^* = EA \varphi + EI_2 \left( \frac{\sin^2 \alpha_0}{R^2} - \frac{4 \sin^3 \alpha_0 \cos \alpha_0}{R^2} \varphi \right) + GI_3 \frac{\cos 2\alpha_0}{R^2}$$

$$M^* = EI_1 \frac{\sin^2 \alpha_0}{R} \varphi + GI_3 \frac{\cos 2\alpha_0}{R^2}$$

$$Q = \left( \frac{\sin 4\alpha_0}{R^2} + \frac{\sin^2 \alpha_0}{R^2} \varphi \right)$$

(28)

if the effect of axial strain on the bending and torsion of the strip is ignored in Equations (20b) and (20c).

### 2.4. Solutions to the differential equations

#### 2.4.1. Numerical solution ($\varphi \neq 0$)

It is difficult to obtain closed-form solutions to Equation (26). Numerical techniques are thus employed here. Introducing the following notation

$$\alpha = y_1, \alpha' = y_2, \varphi = y_3 \text{ and } \varphi' = y_4$$

(29)

and substituting them into Equation (26), then combining Equations (26) and (29) we can express the transformed governing equations as

$$\begin{align*}
y_1' &= y_2 \\
y_2' &= \frac{1}{EI_1} (T^* y_1 + M^* y_4 - Q) \\
y_3' &= y_4 \\
y_4' &= \frac{1}{GI_3} (-M^* y_2 + N^* y_3)
\end{align*}$$

(30)

A set of boundary conditions must be specified to solve the problem:

$$y_1 (0) = 0, \quad y_1 (L) = \beta, \quad y_3 (0) = 0 \text{ and } y_3 (L) = 0,$$

(31)

where $L$ should be large enough to make sure that the boundary conditions at the far end are valid. Equations (30)–(31) now constitute a boundary value problem (BVP) which can be solved numerically.

Although we assume that the helix radius remains unchanged (Assumption 2), the effects of varying radius along the pipe length, a possibility if the pipe is restricted at the end fitting but otherwise more free to expand, could also be assessed by introducing a new small variable $\Delta R$. Using
a further Euler equation in $\Delta R$, another differential equation of $\alpha$, $\phi$, $\Delta R$ and their derivatives can be obtained. The two more necessary boundary conditions are taken as $\Delta R(0) = 0$ and $\Delta R(L) = \Delta R_L$, where $\Delta R_L$ is the radius variation at the far end. This results in a new BVP and can be solved using a numerical method.

### 2.4.2. Analytical solution ($\phi = 0$)

Neglecting the rotation of the strip cross-section makes it possible to find an analytical solution to the problem. In this case, instead of using Equation (31), a different approach is taken to apply the boundary condition.

The distribution of $\alpha$ represents a deviation of the wire from a known path. Due to decline of the end restraint, the actual wire position tends to coincide with the strained helix, described by $\beta$, as one approaches the far end. From Figure 3, this lateral constraint can be expressed as

$$\int_0^L (1 + \varepsilon_{\alpha}) \sin (\beta - \alpha) \, dS = 0. \quad (32)$$

Assuming small deformations and displacements, Equation (32) can be simplified to

$$\int_0^L (\alpha - \beta) \, dS = 0. \quad (33)$$

Under the constraints

$$\int_{x_1}^{x_2} \phi_j (x; y_1, y_2, \ldots, y_N) \, dx = 0, \quad (j = 1, 2, \ldots, K)$$

with $\beta_j (j = 1, 2, \ldots, K)$ being constant, the Euler equation becomes (Chien 1980)

$$\sum_{m=0}^n (-1)^m \frac{d^m}{dx^m} \left( \frac{\partial F^*}{\partial y_i^{(m)}} \right), \quad (i = 1, 2, \ldots, N), \quad (35)$$

where

$$F^* = F + \sum_{j=1}^K \lambda_j \phi_j. \quad (36)$$

Here, $\lambda_j (j = 1, 2, \ldots, K)$ are known as Lagrange multipliers.

Equation (24) is now becomes

$$\frac{\partial F^*}{\partial \alpha} - \frac{d}{dS} \left( \frac{\partial F^*}{\partial \alpha'} \right) = 0. \quad (37)$$

Denoting by $\phi$ the integrand in the left side of Equation (33), substituting $F$ and $\phi$ into Equation (36) and then into Equation (37), the governing differential equation is obtained as

$$EI \alpha'' - T^* \alpha = \lambda - Q. \quad (38)$$

Equation (38) is a second-order linear differential equation with constant coefficients and has the solution:

$$\alpha = C_1 e^{ks} + C_2 e^{-ks} + \frac{Q - \lambda}{T^*}. \quad (39)$$

where $k = \sqrt{\frac{T^*}{EI}}$. Intuitively, $\alpha$ cannot be allowed to grow without limits as $S$ increases, which implies that $C_1 = 0$. The constant $C_2$ is found from the boundary condition $\alpha(0) = 0$. To complete the solution, the multiplier $\lambda$ needs to be evaluated from the overall constraints (Equation (33)). The solution for $\alpha$ is ultimately obtained as

$$\alpha = \bar{\beta} (1 - e^{-ks}), \quad (40)$$

where $\bar{\beta} = \frac{\beta}{1 - e^{-ks}}$. In deep-water applications, $L$ is sufficiently large so that $\bar{\beta} \simeq \beta$, which simplifies Equation (40) to

$$\alpha = \beta (1 - e^{-ks}). \quad (41)$$

The obtained expression of $\alpha$ would be the same with that given by Martindale (2006) if $T^*$ in Equation (28) rather than in Equation (27) is used in the expression of the decay rate $k$, which confirms that the effect of axial strain on the bending and torsion of the strip is ignored in his model. This assumption is surely reasonable as the axial strain in the wire is much smaller than unity (Equation (16)). Also ignoring the axial strain effect, Thorsen (2011) took the restrained warping of the strip into account. This introduces a warping stiffness term in the denominator of the radicand in the expression of $k$. However, it was found that the warping stiffness contribution is totally negligible.

### 2.5. Stress increase at the end fitting

Stresses caused by the end restraint increase the maximum longitudinal stresses experienced, and this is of interest from a fatigue point of view and also regarding maximum loading. The axial stress in the armour wire due to tension is taken to be uniformly distributed over the cross-section, while longitudinal stresses due to wire local bending are found using the simple beam theory. Neglecting the rotation of the strip cross-section allows the explicit expressions of the stress components at the end fitting ($S = 0$) to be
obtained as
\[ \sigma_a = E \left( \varepsilon + \frac{1}{2} \beta^2 \right) \simeq E \varepsilon \] (42)
\[ \sigma_{b1,\max} = \frac{1}{2} Ek \beta b \] (43a)
\[ \sigma_{b2,\max} = \frac{Et \sin^2 \alpha_0}{2R} \left( \varepsilon + \frac{1}{2} \beta^2 \right) \simeq \frac{Et \sin^2 \alpha_0}{2R} \varepsilon. \] (43b)

The significance of the increase in maximum longitudinal stress due to local bending can be evaluated by examining the proportional increase over the direct axial stress resulting from pipe tension:
\[ \frac{\sigma_{b,\max}}{\sigma_a} = \frac{\sigma_{b1,\max} + \sigma_{b2,\max}}{\sigma_a} = \frac{k \beta b}{2 \varepsilon} + \frac{t \sin^2 \alpha_0}{2R}. \] (44)

3. FE verification of the model
The FE approach is taken to verify the analytical model developed in the preceding sections. Line elements such as beams and links included in FE packages seem well suited to the modelling of thin rods. The characteristics of the armour wires used in Thorsen’s case study (Thorsen 2011), as shown in Table 1, are selected in this verification.

3.1. Description of the FE model
The FE model is constructed using the ANSYS program. The BEAM188 element is selected to model the armour wire. A series of 101 nodes is arranged along a helix, equally spaced and covering one pitch length. As a result, the helical wire is modelled by an arrangement of 100 elements. To apply loading, the nodes at the ends in the helical arrangement are fixed against all rotations. Both nodes are moved a prescribed equal but opposite distance in the pipe axial direction while circumferential displacements are kept to zero. The radial constraint is provided by restraining the radial displacement of the nodes in the polar coordinate system.

3.2. Solutions and comparisons
Comparison of the change in lay angle for the 9 mm × 3 mm cross-section is shown in Figure 5, for an applied pipe strain of 0.40%. It should be noted that the length of the strip is required to be sufficiently large to ensure the approximation \( \bar{\beta} \simeq \beta \) and the validity of the boundary conditions at the far end (Equation (31)). For this reason, the length of the ANSYS model should have been set to a much larger value. This is, however, prone to unexpected errors in ANSYS, and therefore only one pitch length is specified. Actually the “effective length” of the model is half a pitch only due to the symmetry of the loading, which does not satisfy the above condition. Therefore, Equation (40)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values and units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross section ((b \times t))</td>
<td>9 mm × 3 mm and 15 mm × 6 mm</td>
</tr>
<tr>
<td>Radius ((R))</td>
<td>0.10 m</td>
</tr>
<tr>
<td>Lay angle ((\alpha_0))</td>
<td>35 deg</td>
</tr>
<tr>
<td>Young’s modulus ((E))</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio ((\nu))</td>
<td>0.30</td>
</tr>
</tbody>
</table>

![Figure 5. Change in lay angle (9 mm × 3 mm). (This figure is available in colour online.)](image-url)
should be used to the determination of $\alpha$ for the analytical solution. The replacement of $\beta$ by $\overline{\beta}$ in Equation (31) is, however, not enough to produce a reasonable distribution of $\alpha$ for the numerical solution. Although not presented here, the result is found to be even larger than $\overline{\beta}$ in some region. In other words, $\alpha$ does not approach $\overline{\beta}$ asymptotically as expected. To overcome this issue, both the boundary conditions (Equation (31)) and the lateral constraint (Equation (33)) are applied. This treatment introduces the unknown $\lambda$, and therefore one more boundary condition is required. $\alpha'(L) = 0, \gamma_2(L) = 0$ following the notation of Equation (29), is hence specified to control the variation of $\alpha$.

Using the nodal rotations directly and combining the global output to find rotations about the surface normal gives the ANSYS equivalent of the variable $\alpha$ which defines the path of the wire on the surface. The results given in Figure 5 show a good correlation regarding the wire position. The numerical solution is found to be closer to the ANSYS result than the analytical solution.

The difference between the current model and the existing models (Martindale 2006; Thorsen 2011) lies mainly in the inclusion of the cross-section rotation. The angle of rotation ($\phi$) for the 9 mm × 3 mm cross-section varying with the arc length is shown in Figure 6. The distribution of $\phi$ has the expected overall shape, starting with zero at the end fitting, reaching the maximum and approaching zero as one moves to the far end. The counterpart of $\phi$ in ANSYS is not accessible directly for all nodes in one single result coordinate system due to the helical configuration. However, the twisting moment is available by creating an element table and filling it using the sequence number method. Comparison of the twisting moment is shown in Figure 7 in which the results for both the numerical and the analytical solution are calculated using the constitutive relation $M_3 = GJ_1\Delta \kappa_3$. The numerical solution agrees well with the ANSYS result, while a marked discrepancy is observed between the analytical solution and the other two sets of results. This is not surprising as the cross-section rotation is ignored in the analytical solution, and therefore the derivative term of $\phi$ is not included in Equation (20c).

Special attention is once again paid to the change in lay angle shown in Figure 5, in which $\alpha$ is still found to be slightly larger than $\overline{\beta}$ in some region. This is accounted for by the strip length selected in the verification, as explained earlier. The variation of $\phi$ close to the far end has the same implication regarding the lack of strip length, as shown in Figure 6. However, this lack is not serious as $\phi$ does approach zero in an asymptotic manner.

It is believed that the lateral bending about the strong axis is the major contributor to the increased stress at the end fitting. Using a similar way, comparison of the lateral bending moment can be performed among the three approaches, as shown in Figure 8. Here the maximum values are in good agreement but the numerical solution suggests a slightly faster decay rate than does the analytical solution, which is confirmed by the ANSYS result. It is easy to understand as the rotation of the strip cross-section relieves the lateral bending requirement caused by the end restraint (Equation (21)).

The maximum lateral bending moment response as the pipe strain is increased from zero through to 0.40% is shown in Figure 9. The results obtained from different approaches are in close agreement with each other, showing the same non-linearity of the response.

Comparisons of the results for the 15 mm × 6 mm cross-section, as shown in Figures 10–14, come to similar
conclusions. Here the wire is found to approach the final position, described by $\beta$, in a slower rate than does for the 9 mm × 3 mm cross-section. For instance, at the arc length corresponding to one-tenth of the pitch length, 90% more or less of the maximum has been reached for the 9 mm × 3 mm cross-section, while only approximately 75% is achieved for the 15 mm × 6 mm cross-section. A simple explanation can be made by inspecting the decay rate $k$. The axial strain term in $T^*$ and the bending stiffness about the strong axis $EI_1$ are regarded as the dominant parameters which defines the behaviour of the wire close to the end restraint. The decay rate $k$ is then found to be inversely proportional to the width of the strip. For the 15 mm × 6 mm cross-section, a larger width of the strip and hence a smaller decay rate results in a slower increase of $\alpha$. Associated with this is a more serious lack of model length, as shown in Figure 11.

4. Stress increase at the end fitting: case study and discussion

To get a picture of the relative magnitude of the local bending stress at the end fitting, Thorsen (2011) performed comparisons between the analytical solutions and the FE results obtaining from the Aflex (Sævik 1992) and Bflex (Sævik
2010) analyses. His technique is reapplied here. The ratios of local bending stress to direct axial stress against the axial stress are shown in Figures 15–17. The following is a discussion on the results.

(1) Cross-section dimensions and shape of curves

It is observed in Figure 15 that the results for the two cross-sections are close to each other despite the relatively large differences in strip dimensions. Therefore, it is of interest to see whether or not cross-section dimensions have effect on the bending stresses at the end fitting. Using an approximate expression of $k$ which contains only the dominating parameters ($EA\varepsilon$ and $EI_1$), Equation (43a) becomes

$$\sigma_{b1,max} = \sqrt{3}E\beta\varepsilon^{0.5}. \quad (45)$$

This verifies the fact that the maximum lateral bending stress at a given pipe strain is virtually independent of the cross-section dimensions. However, the cross-section size does affect the stress due to local bending about the weak axis which is found to be proportional to the strip thickness (Equation (43b)).
Substituting the approximate expression of $k$ into Equation (44) results in

$$\frac{\sigma_{b,\text{max}}}{\sigma_a} = \sqrt{3}\beta \varepsilon^{0.5} + \frac{t\sin^2\alpha_0}{2R}. \quad (46)$$

Considering Equation (2), Equation (1) can be linearised as

$$\beta = \tan\alpha_0 \varepsilon. \quad (47)$$

Substituting Equation (47) into Equation (46) yields

$$\frac{\sigma_{b,\text{max}}}{\sigma_a} = \sqrt{3}\tan\alpha_0 \varepsilon^{0.5} + \frac{t\sin^2\alpha_0}{2R}. \quad (48)$$

This indicates that the proportional increase of local bending stress is approximately proportional to $\sigma_a^{0.5}$ which is in agreement with the shape of curves shown in Figures 15–17.

(2) Cross-section rotation and axial strain effect

Figure 11. Rotation of strip cross-section (15 mm × 6 mm). (This figure is available in colour online.)

Figure 12. Twisting moment over boundary region (15 mm × 6 mm). (This figure is available in colour online.)
The rotation of the strip cross-section relieves the lateral bending requirement caused by the end restraint (Equation (21)), which decreases the lateral bending stress and hence the total bending stress, as shown in Figures 16 and 17. In this case study, the proportional decrease is not more than 0.48% and 0.89% for the 9 mm × 3 mm and 15 mm × 6 mm cross-sections, respectively. The significance of the decrease becomes less important as the applied pipe strain increases.

The effect of axial strain on the bending and torsion of the strip is ignored in the existing models (Martindale 2006; Thorsen 2011). The second term on the right-hand side of Equation (20b) vanishes under this assumption. Accordingly local bending about the weak axis makes no contribution to the bending stress at the end fitting (Equation (43b)). The proportional increase of stress due to local bending about the weak axis corresponds to the second term on the right-hand side of Equation (44), which is a constant.
for a given cross-section. Inclusion of this local deformation increases the results by 0.5% and 1.0% for the two cross-sections, respectively.

As discussed above, cross-section rotation and axial strain in the wire affect the bending stress in the opposite direction. The numerical solution taking the effect of the two factors into account produces predictions very close to the FE results. The analytical solution ignoring the cross-section rotation overestimates the maximum bending stress slightly.

(3) Potential influence of friction

A major shortcoming of the developed model is the lack of friction. The presence of friction will increase the local bending stress, as the armour wire in the region away from the end restraint is incompatible with any lateral slip, which would require some lateral curvature. This has been confirmed by the FE modelling (Thorsen 2011). Besides, the location of the boundary varies as the pipe strain increases, leading to a transitional region of slip.
5. Conclusions

This paper presents a dedicated mode to evaluate the end restraint effect on the tensile armour behaviour of unbonded flexible pipes under axial tension. The work sets out to find the position adopted by the armour wire and to evaluate its significance in terms of stress increase in the vicinity of the end fitting. General expressions of curvature and torsion increments are derived to extend the existing models to account for the axial strain effect and the twist of the wire cross-section with respect to the wire centre-line.

Under axial tension, the effect of the end restraint is to keep the armour wire oriented in its original direction thus causing a transition or boundary layer over which there is a lateral bending of the wire. Beyond this region which is typically less than half a pitch length, the wire assumes a new helical form. Stress increase as a result of the lateral bending depends on the level of tension and on the initial lay angle, but not on the cross-section dimensions. Cross-section rotation relieves the lateral bending requirement caused by the end restraint thus decreases the lateral bending stress. Inclusion of the axial strain effect introduces another bending stress component and the proportional increase is found to be constant for a given cross-section as the pipe strain increases.

This model avoids the shortfalls of adapting previous models designed either for similar but different structures or for application away from any end fitting restraint. The inclusion of friction in the analytical model, which needs more attention, is being pursued currently.

Disclosure statement

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References


Martindale HGA. 2006. The behaviour of flexible riser tensile armour in the region of an end fitting [PhD thesis]. [London]: University College London.


Rawlins CB. 2005. Flexure of a single-layer tensioned cable at a rigid support. Proceedings of the 6th International Symposium on Cable Dynamics; 2005 Sep; Charleston, USA. Li`ege: AIM (Association of Engineers from the Montefiore Electrical Institute).


