Dynamic programming in stochastic control of systems with delay

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We consider optimal control problems for systems described by stochastic differential equations with delay (SDDE). We prove a version of Bellman’s principle of optimality (the dynamic programming principle) for a general class of such problems. That the class in general means that both the dynamics and the cost depends on the past in a general way. As an application, we study systems where the value function depends on the past only through some weighted average. For such systems we obtain a Hamilton–Jacobi–Bellman partial differential equation that the value function must solve if it is smooth enough.

The weak uniqueness of the SDDEs we consider is our main tool in proving the result. Notions of strong and weak uniqueness for SDDEs are introduced, and we prove that strong uniqueness implies weak uniqueness, just as for ordinary stochastic differential equations.

**Keywords:** Stochastic delay equations; Optimal stochastic control; Dynamic programming; Hamilton–Jacobi–Bellman equations

**2000 Mathematics Subject Classification:** Primary: 34K50; Secondary: 93E20, 49L20

**INTRODUCTION**

Consider a controlled stochastic differential delay system of the form

\[
\begin{align*}
    dX(t) &= b(t, X_t, u(t)) \, dt + \sigma(t, X_t, u(t)) \, dW(t), \quad t \in [s, T], \\
    X(t) &= \varphi(t - s), \quad t \in [s - \delta, s],
\end{align*}
\]  

(1.1)
where $\delta \equiv 0$ is the fixed delay, $X_t$ is the segment of the path from $t - \delta$ to $t$, $\varphi \in C[-\delta, 0]^n$ is the initial path, and $u$ is the control to be chosen in some prescribed class $\mathcal{U}$ of strategies. We measure the performance of the control with the cost functional

$$J(s, \varphi, u) := E^x_{\varphi, u} \left[ \int_s^T f(t, X_t, u(t)) \, dt + h(X_T) \right].$$

The problem is to find the value function $V(s, \varphi) := \inf_{u \in \mathcal{U}} J(s, \varphi, u)$ and an optimal control $u = u^*$ such that $V(s, \varphi) = J(s, \varphi, u^*)$. Because the space $C[-\delta, 0]^n$ of initial conditions is infinite dimensional for delay systems, the problem is conceptually much more difficult than in the no-delay case. There seems to be no general solution methods available. However, if one considers systems where the maps $b$, $\sigma$, $f$, and $h$ depend on the past only through $X(t - \delta)$ and some sliding average of previous values, several authors have been able to reduce the problem to finite dimensions and results have been obtained using variational methods. See e.g. the monograph [10] by Kolmanovskii and Shaikhet and the references therein, or [4] by Elsanousi and Larssen who used a verification theorem and obtained classical solutions to the associated HJB-equation for the value function for $f$ and $g$ of HARA utility type.

The contribution of this paper is to the general theory of stochastic control problems for systems with delay. We show that a dynamic programming principle (DPP) holds:

$$V(s, \varphi) = \inf_{u \in \mathcal{U}} E^x_{\varphi, u} \left[ \int_s^{\hat{s}} f(t, X_t, u(t)) \, dt + V(\hat{s}, X_{\hat{s}}^u) \right]$$

where $\hat{s} \in [s, T]$ is a stopping time with respect to the filtration generated by the driving Brownian motion $W$ in Eq. (1.1). This has the following interpretation: To achieve optimal cost, let the system evolve from $s$ to $\hat{s}$ with some control $u$ and pay the cost on this interval; then continue optimally from where the system is at time $\hat{s}$ and take the infimum over all controls $u$.

This result for discrete time systems without delay originates in the works of Bellman and his group in the 1950s, see e.g. Ref. [1]. More recent accounts for continuous time systems without delay has been given by e.g. Krylov [11], Fleming and Soner [5], and Yong and Zhou [19].

When $\delta = 0$ the system (1.1) degenerates to an ordinary stochastic differential equation. In this case, the solution has the strong Markov property. A natural approach in proving a DPP is to use this property. For delay systems ($\delta > 0$) no such property holds. What we do instead is to adapt a method described by Yong and Zhou in Chapter 4 of Ref. [19]. In this method one needs
the weak uniqueness of solutions to stochastic differential equations. For this purpose, we introduce notions of weak and strong solutions to stochastic differential delay equations (SDDE) along with some other notation and definitions in the second section.

Conditions for strong, or pathwise, uniqueness of solutions to delay systems are given by Mohammed’s existence-uniqueness results, see Refs. [15] or [16]. In the third section, we use a method of Yamada and Watanabe [18] to show that pathwise uniqueness implies weak uniqueness also for SDDEs. This section may be skipped if one is willing to accept the result.

In the fourth section, we introduce the strong and weak formulations of the control problem, in the spirit of Yong and Zhou [19], and prove the main result (Theorem 4.12). We will see that, as in the no-delay case, the weak uniqueness is crucial. Theorem 4.12 is then a generalization of Theorem 3.3 in Ref. [19].

In the fifth section, as an application of our result, we prove a converse of the verification theorem used in Ref. [4]; that is, we show that if the value function is sufficiently smooth, it solves an associated second-order Hamilton–Jacobi–Bellman partial differential equation.

We also expect the DPP will be useful in developing a theory of viscosity solutions to HJB-equations coming from control problems of the class considered in Ref. [4].

Another interesting application of the DPP would be to investigate whether there is a connection between dynamic programming and the maximum principle for delay systems as in the no-delay case. Recently, Øksendal and Sulem (see Ref. [17]) proved a maximum principle for certain delay systems.

NOTATION AND DEFINITIONS

Fix real numbers $0 \leq T < \infty$ and $0 \leq \delta < \infty$, and integers $n, r \geq 1$. Let $C[-\delta, 0]^n$ denote the Banach space of continuous paths $\gamma : [-\delta, 0] \to \mathbb{R}^n$ with the supremum norm $\|\gamma\| = \|\gamma\|_{[-\delta, 0]^n} := \sup_{-\delta \leq s \leq 0} |\gamma(s)|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. As a rule, we use double bars, $\|\cdot\|$, for norms in function spaces and single bars, $|\cdot|$, in Euclidean spaces. As a metric space we associate with $C[-\delta, 0]^n$ its Borel $\sigma$-algebra $\mathcal{B}(C[-\delta, 0]^n)$. Define the spaces $C[0, T]^n$ and $C[0, T]'$, accordingly.

For $\eta : [-\delta, T] \to \mathbb{R}^n$ and $0 \leq t \leq T$ we denote by $\eta_t$ the function defined by

$$\eta_t(s) = \eta(t + s), \quad -\delta \leq s \leq 0,$$

i.e. $\eta_t$ is the segment of the path of $\eta$ from $t - \delta$ to $t$. 

Suppose we are given two Borel-measurable mappings $b = (b_1, \ldots, b_n)^T$ and $\sigma = (\sigma_{ij})_{i,j=1,1}^n$ such that

\[
b : [0, T] \times C([-\delta, 0]^n) \to \mathbb{R}^n,
\]
\[
\sigma : [0, T] \times C([-\delta, 0]^n) \to \mathbb{R}^{n \times n},
\]
and an $r$-dimensional Brownian motion $W$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the stochastic differential delay equation (SDDE)

\[
dX(t) = b(t, X_t) \, dt + \sigma(t, X_t) \, dW(t).
\]

For such systems we must specify the whole initial path $\varphi(s)$, $-\delta \leq s \leq 0$. That is, we set

\[
X(s, \omega) = \varphi(s, \omega), \quad \text{for } -\delta \leq s \leq 0.
\]

Specifically, we will assume that $\varphi$ is a random variable $\varphi : \Omega \to C([-\delta, 0]^n)$, independent of $W$, with distribution

\[
\mu(B) = \mathbb{P}(\varphi \in B), \quad B \in \mathcal{B}(C([-\delta, 0]^n)).
\]

Let $\mathcal{G}_t := \sigma(\varphi, W(s); 0 \leq s \leq t)$ for $0 \leq t \leq T$. Let $\mathcal{N}$ denote the collection of $\mathbb{P}$-null sets in $\Omega$ and create the augmented filtration

\[
\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}), \quad 0 \leq t \leq T.
\]

Then $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ is a Brownian motion and $\{\mathcal{F}_t\}$ satisfies the usual conditions of completeness and right-continuity.

**Definition 2.1** A strong solution of Eq. (2.1) on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed Brownian motion $W$ and initial path $\varphi$ is a process $X = \{X(t); -\delta \leq t \leq T\}$ with continuous sample paths and the properties

S1 $X(t)$ is $\mathcal{F}_t$-measurable for $0 \leq t \leq T$ and $\mathcal{F}_0$-measurable for $-\delta \leq t \leq 0$,

S2 $\mathbb{P}[X_0 = \varphi] = 1$,

S3 for $1 \leq i \leq n$ and $1 \leq j \leq r$ we have

\[
\mathbb{P} \left[ \int_0^T (|b_i(t, X_t)| + \sigma_{ij}^2(t, X_t)) \, dt < \infty \right] = 1,
\]

S4 the integral version of Eq. (2.1)

\[
X(t) = X_0(0) + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW(s), \quad 0 \leq t \leq T,
\]
\[ X(t) = X_0(t), \quad -\delta \leq t \leq 0. \]

holds almost surely.

**Definition 2.2** Let the functionals \( b(t, \xi) \) and \( \sigma(t, \xi) \) be given. Suppose that whenever \( W \) is an \( r \)-dimensional Brownian motion on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \( \{\mathcal{F}_t\} \) is given by Eq. (2.4), \( \varphi \) is an initial path independent of \( W \) and \( X \), \( \tilde{X} \) are two strong solutions of Eq. (2.1) relative to \( W \) with initial path \( \varphi \), then

\[ \mathbb{P}[X(t) = \tilde{X}(t) \text{ for all } 0 \leq t \leq T] = 1. \]

Under these conditions we say that **strong uniqueness** holds for the pair \((b, \sigma)\).

**Definition 2.3** A **weak solution** of Eq. (2.1) is a triple \((X, W, (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\})\) where

\begin{align*}
W1 \quad & (\Omega, \mathcal{F}, \mathbb{P}) \text{ is a probability space and } \{\mathcal{F}_t\} \text{ is a filtration with } \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq t \leq T, \text{ satisfying the usual assumptions}, \\
W2 \quad & X = \{X(t); -\delta \leq t \leq T\} \text{ is a continuous process and } X(t) \text{ is } \mathcal{F}_t \text{-measurable for } 0 \leq t \leq T \text{ and } \mathcal{F}_0 \text{-measurable for } -\delta \leq t \leq 0, \\
W3 \quad & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq r \text{ we have} \\
\quad & \mathbb{P} \left[ \int_0^T \left( |b_j(t, X_t)| + \sigma_j^2(t, X_t) \right) dt < \infty \right] = 1, \\
W4 \quad & \text{the integral version of Eq. (2.1)} \\
\quad & X(t) = X_0(0) + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW(s), \quad 0 \leq t \leq T, \\
\quad & X(t) = X_0(t), \quad -\delta \leq t \leq 0, \\
\end{align*}

holds almost surely.

When no confusion arises, we sometimes refer to \((X, W)\) or just \(X\) as the weak solution. The probability measure

\[ \mu(B) := \mathbb{P}[X_0 \in B], \quad B \in \mathcal{B}(C[-\delta, 0]^r) \]

is called the initial distribution of the solution.

**Definition 2.4** We say that **weak uniqueness** holds for Eq. (2.1) if, for any two weak solutions \((X, W, (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}) \text{ and } (\tilde{X}, \tilde{W}, (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{\tilde{\mathcal{F}}_t\}) \text{ with} \]
the same initial distribution

\[ P[X_0 \in B] = \hat{P}[\hat{X}_0 \in B] \quad \text{for all } B \in \mathscr{B}(C[-\delta, 0]) , \]

the two processes \( Y := X|_{[0,T]} \) and \( \hat{Y} := \hat{X}|_{[0,T]} \) also have the same law, i.e.

\[ P[Y \in B] = \hat{P}[\hat{Y} \in B] \quad \text{for all } B \in \mathscr{B}(C[0,T]) . \]

**STRONG UNIQUENESS IMPLIES WEAK UNIQUENESS**

It is well known that strong uniqueness implies weak uniqueness for ordinary (no delay) Ito-type stochastic differential equations. This was proved by Yamada and Watanabe in Ref. [18]. See also the survey article [21] by Zvonkin and Krylov for a discussion of weak and strong solutions of ordinary SDEs. We show in this section that the result also holds for the SDDEs in our setup by adapting the method of Yamada and Watanabe as explained by Karatzas and Shreve in Ref. [7] pp. 308–310 to the present situation.

We need the following result about regular conditional probabilities.

**Theorem 3.1** Let \( \Omega \) and \( S \) be complete, separable metric spaces with \( \mathscr{B}(\Omega) \) and \( \mathscr{B}(S) \) the corresponding Borel \( \sigma \)-algebras and let \( \mathbb{P} \) be a probability measure on \( (\Omega, \mathscr{B}(\Omega)) \). Let \( X : (\Omega, \mathscr{B}(\Omega)) \rightarrow (S, \mathscr{B}(S)) \) be a random variable with distribution \( \mathbb{P}^X \) given by

\[ \mathbb{P}^X(B) := \mathbb{P}[\omega \in \Omega; X(\omega) \in B], \quad B \in \mathscr{B}(S) . \]

Then there exists a function \( Q(x; A) : S \times \mathscr{B}(\Omega) \rightarrow [0,1] \) called a regular conditional probability for \( \mathscr{B}(\Omega) \) given \( X \), such that

for fixed \( x \in S \), \( A \mapsto Q(x; A) \) is a probability measure on \( (\Omega, \mathscr{B}(\Omega)) \), \hspace{1cm} (3.1)

for fixed \( A \in \mathscr{B}(\Omega) \), \( x \mapsto Q(x; A) \) is \( \mathscr{B}(S) \) measurable, and, \hspace{1cm} (3.2)

for every \( A \in \mathscr{B}(\Omega) \) and \( B \in \mathscr{B}(S) \), we have

\[ \mathbb{P}[A \cap \{ \omega \in \Omega; X(\omega) \in B \}] = \int_B Q(x; A) \mathbb{P}^X(dx) . \]

(3.3)

If \( Q'(x; A) \) is another such function, then there exists a set \( N \in \mathscr{B}(S) \) with \( \mathbb{P}^X(N) = 0 \) such that \( Q(x; A) = Q'(x; A) \) for all \( A \in \mathscr{B}(\Omega) \) and \( x \in S \setminus N \). This \( N \) can be chosen so that we have the additional property:

\[ Q(x; \{ \omega \in \Omega; X(\omega) \in B \}) = 1_B(x), \quad B \in \mathscr{B}(S), x \in S \setminus N . \]

(3.4)
In particular,
\[ Q(x; \omega \in \Omega; X(\omega) = x) = 1, \quad \mathbb{P}^x - a.e. \quad x \in S. \quad (3.5) \]

**Proof** This follows from Theorem I.3.3 in [6] and the Corollary immediately following it. □

We will show that strong uniqueness implies weak uniqueness by starting off with two weak solutions \( (X^{(i)}, W^{(i)}), (\Omega_j, \mathcal{F}_j, \pi_j), \{\mathcal{F}_j\}; j = 1, 2 \) of Eq. (2.1) with the same initial distribution
\[
\mu(B) = \pi_1[X_0^{(1)} \in B] = \pi_2[X_0^{(2)} \in B], \quad B \in \mathcal{B}(\mathbb{R}^-[\delta, 0]^n). \quad (3.6)
\]

Since these solutions are defined on different probability spaces, we cannot use the assumption of strong uniqueness directly. First, we have to bring the solutions together on the same canonical space \((\Omega, \mathcal{F}, \mathbb{P})\) while preserving their joint distributions. Once this is done, the result follows easily from the strong uniqueness. We proceed as follows.

First set \( Y^{(j)}(t) := X^{(j)}|_{[0,t]}(t) - X^{(j)}(0) \) for \( t \in [0, T] \). Then
\[
X^{(j)}(t) = X_0^{(j)}(0 \wedge t) + Y^{(j)}(0 \vee t) \quad \text{for} \quad t \in [-\delta, T],
\]
and we may regard the \( j \)-th solution as consisting of three parts: \( X_0^{(j)}, W^{(j)}, \) and \( Y^{(j)} \). This triple induces a measure \( P_j \) on
\[
(\Theta, \mathcal{B}(\Theta)) := (C[-\delta, 0]^n \times C[0, T] \times C[0, T]^n, \quad \mathcal{B}(C[-\delta, 0]^n) \otimes \mathcal{B}(C[0, T]) \otimes \mathcal{B}(C[0, T]^n) \quad (3.7)
\]
according to
\[
P_j(A) := \pi_j[(X_0^{(j)}, W^{(j)}, Y^{(j)}(0)) \in A], \quad A \in \mathcal{B}(\Theta), \quad j = 1, 2. \quad (3.8)
\]
We denote by \( \theta = (x, w, y) \) the generic element of \( \Theta \). The marginal of each \( P_j \) on the \( x \)-coordinate is the initial distribution \( \mu \), the marginal on the \( w \)-coordinate is Wiener measure \( \mathbb{P}_w \), and the distribution of the \( (x, w) \) pair is the product measure \( \mu \times \mathbb{P}_w \). This is because \( X_0^{(j)} \) is \( \mathcal{F}_0 \)-measurable (see Ref. [15], Lemma II.2.1) and \( W^{(j)} \) is independent of \( \mathcal{F}_0 \) (see Ref. [7], Problem 2.5.5). Also, under \( P_j \), the initial value of the \( y \)-coordinate is zero, almost surely.

Next we note that on \((\Theta, \mathcal{B}(\Theta), P_j)\) there exists a regular conditional probability for \( \mathcal{B}(\Theta) \) given \((x, w)\). We shall be interested only in conditional probabilities of sets in \( \mathcal{B}(\Theta) \) of the form \( C[-\delta, 0]^n \times C[0, T] \times F \) for \( F \in \mathcal{B}(C[0, T]^n) \). Thus, with a slight abuse of terminology, we speak of
\[
Q_j(x, w; F) : C[-\delta, 0]^n \times C[0, T] \times \mathcal{B}(C[0, T]^n) \rightarrow [0, 1], \quad j = 1, 2,
\]
as the regular conditional probability for $\mathcal{B}(C(0, T]^n)$ given $(x, w)$. According to Theorem 3.1, this regular conditional probability has the following properties:

**RCP1** for each $x \in C[-\delta, 0]^n$, $w \in C[0, T]'$, the mapping $F \mapsto Q_j(x, w; F)$ is a probability measure on $(C[0, T]^n, \mathcal{B}(C(0, T]^n))$.

**RCP2** for each $F \in \mathcal{B}(C(0, T]^n)$, the mapping $(x, w) \mapsto Q_j(x, w; F)$ is $\mathcal{B}(C[-\delta, 0]^n) \otimes \mathcal{B}(C(0, T]')$-measurable, and,

**RCP3** for every $F \in \mathcal{B}(C(0, T]^n)$ and $G \in \mathcal{B}(C[-\delta, 0]^n) \otimes \mathcal{B}(C(0, T]')$, we have

$$P_j(G \times F) = \int_G Q_j(x, w; F) \mu(dx) P_\omega(dw).$$

Next we construct the space $(\Omega, \mathcal{F}, P)$. Put $\Omega := \Theta \otimes C[0, T]^n$ and denote by $\omega = (x, w, y_1, y_2)$ a generic element of $\Omega$. Let $\mathcal{F}$ be the completion of $\mathcal{B}(\Theta) \otimes \mathcal{B}(C[0, T]^n)$ by the collection $\mathcal{N}$ of null sets under the probability measure

$$P(d\omega) := Q_1(x, w; dy_1)Q_2(x, w; dy_2) \mu(dx) P_\omega(dw). \tag{3.9}$$

To endow $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, we take

$\mathcal{F}_0 := \sigma(x(s); -\delta \leq s \leq 0),$

$\mathcal{F}_t := \sigma(\mathcal{F}_0, w(s), y_1(s), y_2(s); 0 \leq s \leq t),$

$\mathcal{F}_t := \sigma(\mathcal{F}_t \cup \mathcal{N}),$

$\mathcal{F} := \bigcap_{t \geq 0} \mathcal{F}_t.$

for $0 \leq t \leq T$.

Now, for any $A \in \mathcal{B}(\Theta)$,

$$P[\omega \in \Omega; (x, w, y_j) \in A]$$

$$= \int_A Q_j(x, w; dy_j) \mu(dx) P_\omega(dw) \quad \text{(by Eq. (3.9))}$$

$$= P_j(A) \quad \text{(by RCP3 and a monotone class argument)}$$

$$= \pi_j[X_0^{(j)}, W^{(j)}; Y^{(j)}] \in A, \quad j = 1, 2. \quad \text{(by Eq. (3.8))} \tag{3.10}$$
This means that the distribution of \((x(0 \land \cdot) + y_1(0 \lor \cdot), w)\) under \(P\) is the same as the distribution of \((X_0^{(j)}(0 \land \cdot) + Y^{(j)}(0 \lor \cdot), W)\) under \(\pi_j\). In particular, the \(w\)-coordinate process \(\{w(t), \mathcal{F}_t; 0 \leq t \leq T\}\) is an \(r\)-dimensional Brownian motion on \((\Omega, \mathcal{F}, P)\) and by Lemma IV.1.2 in Ref. [6], the same is true for \(\{w(t), \mathcal{F}_t; 0 \leq t \leq T\}\).

To sum up, we started from two weak solutions \((X^{(j)}, W^{(j)}), (\Omega_j, \mathcal{F}_j, \pi_j), \{\mathcal{F}_j\}, j = 1, 2\) of Eq. (2.1) having the same initial distribution (i.e. Eq. (3.6) holds). We have constructed, on the probability space \((\Omega, \mathcal{F}, P), \{\mathcal{F}_j\}\), two weak solutions \((x(0 \land \cdot) + y_j(0 \lor \cdot), w), j = 1, 2\) having the same laws as the ones we started with. Moreover, these weak solutions are driven by the same Brownian motion and they have the same initial path. Consequently, we may regard them as two different strong solutions to the SDDE

\[
dX(t) = b(t, X_t) \, dt + \sigma(t, X_t) \, dw(t)
\]

on the space \((\Omega, \mathcal{F}, P), \{\mathcal{F}_j\}\), with initial path \(x\). Then strong uniqueness implies

\[
\mathbb{P}[x(0 \land t) + y_1(0 \lor t) = x(0 \land t) + y_2(0 \lor t); -\delta \leq t \leq T] = 1,
\]

or equivalently,

\[
\mathbb{P}[\omega = (x, w, y_1, y_2) \in \Omega; y_1 = y_2] = 1. \tag{3.11}
\]

From Eqs. (3.10) and (3.11) we see that

\[
\pi_1[(X_0^{(1)}, W^{(1)}, Y^{(1)}) \in A] = \mathbb{P}[\omega \in \Omega; (x, w, y_1) \in A] = \mathbb{P}[\omega \in \Omega; (x, w, y_2) \in A] = \pi_2[(X_0^{(2)}, W^{(2)}, Y^{(2)}) \in A], A \in \mathcal{B}(\Omega),
\]

and this implies weak uniqueness. Thus, we have proved the following theorem.

**Theorem 3.2** Strong uniqueness implies weak uniqueness.

**THE CONTROL PROBLEM**

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t < T}, \mathbb{P})\) and \(W\) be as in section “Notation and Definitions”. We introduce a control \(u\) in the system (2.1) as follows. Let \(U \subset \mathbb{R}^m, m \geq 1\), and let \(u : \Omega \times [0, T] \to U\) be our control process. Then with

\[
b : [0, T] \times C[-\delta, 0]^m \times U \to \mathbb{R}^n,
\]

\[
\sigma : [0, T] \times C[-\delta, 0]^m \times U \to \mathbb{R}^{2m},
\]
we will study the controlled stochastic differential delay system

\[
\begin{align*}
\frac{dX(t)}{dt} &= b(t, X_t, u(t)) \, dt + \sigma(t, X_t, u(t)) \, dW(t), \quad t \in [0, T], \\
X(t) &= \varphi(t), \quad t \in [-\delta, 0],
\end{align*}
\]

(4.1)

where \( \varphi \in C[-\delta, 0]^n \) (deterministic). The cost functional is

\[ J(u) = \mathbb{E} \left[ \int_0^T f(t, X_t, u(t)) \, dt + h(X_T) \right]. \]

(4.2)

Let \( \mathcal{U}[0, T] \) denote the class of \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \)-adapted controls \( u : \Omega \times [0, T] \to U \) such that Eq. (4.1) admits a unique strong solution.

**Problem 4.1** Minimize the cost Eq. (4.2) over the class \( \mathcal{U}[0, T] \) when the state is Eq. (4.1).

This is the strong formulation of the control problem where the noise and the probability space is given. This is the problem that comes from the practical world and is what we ultimately want to solve. We will, however, reformulate the problem in a weak sense as follows.

For any \( (s, \varphi) \in [0, T) \times C[-\delta, 0]^n \) consider the state equation

\[
\begin{align*}
\frac{dX(t)}{dt} &= b(t, X_t, u(t)) \, dt + \sigma(t, X_t, u(t)) \, dW(t), \quad t \in [s, T], \\
X(t) &= \varphi(t - s), \quad t \in [s - \delta, s],
\end{align*}
\]

(4.3)

and the cost functional

\[ J(s, \varphi; u) = \mathbb{E}^{s, \varphi, u} \left[ \int_s^T f(t, X_t, u(t)) \, dt + h(X_T) \right]. \]

(4.4)

**Definition 4.1** The classes \( \mathcal{U}[\cdot, T] \) of admissible controls are defined as follows: For \( s \in [0, T) \) let \( \mathcal{U}[s, T] \) denote the set of all 5-tuples \( (\Omega, \mathcal{F}, \mathbb{P}, W, u) \) satisfying the following:

1. \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a complete probability space.

2. \( \{ W(t) \}_{t \in [s, T]} \) is an \( r \)-dimensional standard Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}) \) over \( [s, T] \) with \( W(s) = 0 \) a.s., and \( \mathcal{F}_t^s = \sigma[W(\tau); s \leq \tau \leq t] \) augmented by the \( \mathbb{P} \)-null sets in \( \mathcal{F} \).

3. \( u : [s, T] \times \Omega \to U \) is an \( \{ \mathcal{F}_t^s \} \)-adapted process on \( (\Omega, \mathcal{F}, \mathbb{P}) \).
For given the control problem as follows:

\[ f(\cdot, X, u(\cdot)) \in L^2_p(0, T; \mathbb{R}) \] and \( h(X_T) \in L^2_p(\Omega; \mathbb{R}) \) where the spaces \( L^2_p(0, T; \mathbb{R}) \) and \( L^2_p(\Omega; \mathbb{R}) \) are defined on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \).

We will write \( u \in \mathcal{U}[s, T] \) instead of \( (\Omega, \mathcal{F}, \mathbb{P}, W, u) \in \mathcal{U}[s, T] \) when the context is clear. Notice that the filtration is not part of the control, but restricted to be generated by the Brownian motion. This is in contrast to the definition of weak solutions (Definition 2.3) where the filtration is part of the solution. Now we state the control problem as follows:

**Problem 4.2** For given \( (s, \varphi) \in [0, T] \times C[-\delta, 0]^n \), find a 5-tuple \( \bar{u} = (\Omega, \mathcal{F}, \bar{P}, \bar{W}, \bar{u}) \in \mathcal{U}[s, T] \) such that

\[ J(s, \varphi; \bar{u}) = \inf_{u \in \mathcal{U}[s, T]} J(s, \varphi; u) \quad (4.5) \]

The notions of weak and strong formulations of the control problem are adapted from Yong and Zhou [19]. The reason for introducing the weak formulation is the following. In stochastic control, the objective is to minimize the expectation of a random variable that depends only on the distribution of the processes involved. So, if the solutions of Eq. (4.3) in different probability spaces have the same distribution of the processes involved. We will see that this is precisely what is needed in the proof of our main result. A similar approach was also used by Zhu in Ref. [20] where he (among other things) proved a DPP for singular control problems without delay.

We now introduce the following assumptions.

**A1** The maps \( b(t, \eta, u) \) and \( \sigma(t, \eta, u) \) are Lipschitz on bounded sets in \( C[-\delta, 0]^n \) uniformly in the second variable: For each integer \( m \geq 1 \) there is a constant \( L_m > 0 \) (independent of \( 0 \leq t \leq T \)) such that

\[
|b(t, \eta, u) - b(t, \tilde{\eta}, u)|_{L^2} + |\sigma(t, \eta, u) - \sigma(t, \tilde{\eta}, u)|_{L^{\infty}} \\
\leq L_m \|\eta - \tilde{\eta}\|_{C[-\delta, 0]^n}
\]

for all \( 0 \leq t \leq T, \eta, \tilde{\eta} \in C[-\delta, 0]^n \) with \( \|\eta\|_{C[-\delta, 0]^n} \leq m, \|\tilde{\eta}\|_{C[-\delta, 0]^n} \leq m \) and \( u \in U \).

**A2** There is a constant \( K > 0 \) such that

\[
|b(t, \eta, u)|_{L^2} + |\sigma(t, \eta, u)|_{L^{\infty}} \leq K(1 + \|\eta\|_{C[-\delta, 0]^n})
\]

for all \( 0 \leq t \leq T, \eta \in C[-\delta, 0]^n \) and \( u \in U \).
A3 The initial path process $\varphi : \Omega \to C[-\delta, 0]^n$ is $\mathcal{F}_t^\omega$-measurable and belongs to the space $L^2(\Omega; C[-\delta, 0]^n)$, that is,

$$\|\varphi\|^2_{L^2(\Omega; C[-\delta, 0]^n)} := E[\|\varphi(\omega)\|^2_{C[-\delta, 0]^n}] < \infty.$$ 

A4 The maps $f : [0, T] \times C[-\delta, 0]^n \times U \to \mathbb{R}$ and $h : C[-\delta, 0]^n \to \mathbb{R}$ are uniformly continuous, and there exists a constant $L > 0$ such that

$$|f(t, \eta, u) - f(t, \eta', u)| + |h(\eta) - h(\eta')| \leq L\|\eta - \eta'\|,$$

for all $t \in [0, T]$, $\eta, \eta' \in C[-\delta, 0]^n$, $u \in U$, and

$$|f(t, 0, u) + |h(0)| \leq L, \quad \text{for all } (t, u) \in [0, T] \times U.$$

**Theorem 4.1** Under the assumptions A1–A3, for any $(s, \varphi) \in [0, T) \times L^2(\Omega; C[-\delta, 0]^n)$ and $\varphi \in \mathcal{P}_s$, Eq. (4.3) admits a unique strong solution $X(\cdot) = X^{s,\varphi}(\cdot) \in L^2(\Omega; C[-\delta, T]^n)$.

**Proof** This is due to Mohammed and follows from Theorem I.2 in Ref. [16] or Theorem 2.1 in Ref. [15].

Under the assumption A4, the cost functional (4.4) is well defined and we can define the following function:

$$V(s, \varphi) = \inf_{u \in \mathcal{U}_s} J(s, \varphi; u), \quad (s, \varphi) \in [0, T) \times C[-\delta, 0]^n,$$

$$V(T, \varphi) = h(\varphi(0)), \quad \varphi \in C[-\delta, 0]^n,$$

which is called the value function of the original Problem 4.1.

Before we state and prove the DPP we need three lemmas. The first is a result about the value function.

**Lemma 4.1** Let A1–A4 hold. Then the value function satisfies

$$|V(s, \varphi) - V(s, \tilde{\varphi})| \leq \|\varphi - \tilde{\varphi}\|_{C[-\delta, 0]^n}, \quad \text{for all } \varphi, \tilde{\varphi} \in C[-\delta, 0]^n.$$

**Proof** By Theorem II.3.1 in Ref. [15] there is a constant $K > 0$ such that

$$\|X^{s,\varphi}_t - X^{s,\tilde{\varphi}}_t\|_{L^2(\Omega; C[-\delta, 0]^n)} \leq K\|\varphi - \tilde{\varphi}\|_{L^2(\Omega; C[-\delta, 0]^n)}, \quad (4.7)$$
for all \( t \in [s, T] \), \( \varphi, \hat{\varphi} \in L^2(\Omega; C[-\delta, 0]) \). We use this in the case when \( \varphi \) and \( \hat{\varphi} \) are deterministic to get an estimate on \( |J(s, \varphi; u) - J(s, \hat{\varphi}; u)| \) as follows:

\[
|J(s, \varphi; u) - J(s, \hat{\varphi}; u)| \\
\leq \mathbb{E} \left[ \int_s^T \left| f(t, X_t^\varphi, u(t)) - f(t, X_t^{\hat{\varphi}}, u(t)) \right| dt + |h(X_T^\varphi) - h(X_T^{\hat{\varphi}})| \right] \\
\leq \mathbb{K} \mathbb{E} \left[ \int_s^T \|X_t^\varphi - X_t^{\hat{\varphi}}\| dt + \|X_T^\varphi - X_T^{\hat{\varphi}}\| \right] \quad \text{(by A4)} \\
= \mathbb{K} \mathbb{E} \left[ \int_s^T \|X_t^\varphi - X_t^{\hat{\varphi}}\| dt + \mathbb{K} \mathbb{E} \left[ \|X_T^\varphi - X_T^{\hat{\varphi}}\| \right] \right] \\
\leq \mathbb{K} \int_s^T \left( \mathbb{E} \left[ \|X_t^\varphi - X_t^{\hat{\varphi}}\|^2 \right] \right)^{1/2} dt + \mathbb{K} \left( \mathbb{E} \left[ \|X_T^\varphi - X_T^{\hat{\varphi}}\|^2 \right] \right)^{1/2} \quad \text{(by Jensen)} \\
\leq \mathbb{K} \int_s^T \left( \mathbb{E} \left[ \|\varphi - \hat{\varphi}\|^2 \right] \right)^{1/2} dt + \mathbb{K} \left( \mathbb{E} \left[ \|\varphi - \hat{\varphi}\|^2 \right] \right)^{1/2} \quad \text{(by Eq. (4.7))} \\
= \mathbb{K} \|\varphi - \hat{\varphi}\| \quad \text{for some constant } \mathbb{K}.
\]

The estimate holds for all \( \varphi, \hat{\varphi} \in C[-\delta, 0] \) and \( u \in \mathbb{U}[s, T] \). It follows that

\[
\sup_{u \in \mathbb{U}[s, T]} |J(s, \varphi; u) - J(s, \hat{\varphi}; u)| \leq \mathbb{K} \|\varphi - \hat{\varphi}\|. \quad (4.8)
\]

Then

\[
|V(s, \varphi) - V(s, \hat{\varphi})| = |\inf_{u \in \mathbb{U}[s, T]} J(s, \varphi; u) - \inf_{u \in \mathbb{U}[s, T]} J(s, \hat{\varphi}; u)| \\
= | - \sup_{u \in \mathbb{U}[s, T]} (J(s, \varphi; u)) + \sup_{u \in \mathbb{U}[s, T]} (J(s, \hat{\varphi}; u))| \\
= |\sup_{u \in \mathbb{U}[s, T]} (J(s, \varphi; u)) - \inf_{u \in \mathbb{U}[s, T]} (J(s, \hat{\varphi}; u))| \\
\leq |\sup_{u \in \mathbb{U}[s, T]} (J(s, \varphi; u) - J(s, \hat{\varphi}; u))| \\
\leq \sup_{u \in \mathbb{U}[s, T]} |J(s, \varphi; u) - J(s, \hat{\varphi}; u)| \leq \mathbb{K} \|\varphi - \hat{\varphi}\| \quad \text{(by Eq. (4.8))}
\]

which is the conclusion. (The inf’s and sup’s are taken over \( u \in \mathbb{U}[s, T] \)) \( \square \)
For the second lemma we need to introduce some notation. Define
\[ C_t(0, T) := \{ \theta(\cdot \wedge t); \theta \in C[0, T] \}, \quad t \in [0, T], \]
\[ \mathcal{B}_t(C[0, T]) := \sigma(\mathcal{B}(C[0, T])), \quad t \in [0, T], \]
\[ \mathcal{B}_t(C[0, T]) := \bigcap_{\tau > t} \mathcal{B}_\tau(C[0, T]), \quad t \in [0, T], \]
and let \( \mathcal{A}_t(V) \) be the set of all \( \{ \mathcal{B}_t(C[0, T]) \}_{t \geq 0} \)-progressively measurable processes
\[ \eta : [0, T] \times C[0, T] \to V \]
where \( V \) is a Polish space.

**Lemma 4.2** Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and \( V \) a Polish space.
Let \( \xi : [0, T] \times \Omega \to \mathbb{R}^d \) be a continuous process and \( \mathcal{F}_t^\xi = \sigma(\xi(s); 0 \leq s \leq t) \).
Then \( \varphi : [0, T] \times \Omega \to V \) is \( \{ \mathcal{F}_t^\xi \} \)-adapted if and only if there exists an \( \eta \in \mathcal{A}_t(V) \) such that
\[ \varphi(t, \omega) = \eta(t, \xi(\cdot \wedge t, \omega)), \quad t \in [0, T], \quad P - \text{a.s.} \omega \in \Omega. \]

**Proof** See Theorem 1.2.10 in Ref. [19].

The advantage of the weak formulation of the control problem will be apparent in the third lemma and its use in the proof of the DPP. Let \( s \in [0, T] \) and \((\Omega, \mathcal{F}, P, W, u) \in \mathcal{U}[s, T] \). Then under A1–A3, for any \( \mathcal{F}_t \)-stopping time \( \hat{s} \in [s, T) \) and \( \mathcal{F}_s \)-measurable random variable \( \xi : \Omega \to C[-\delta, 0]^n \), we can solve the SDDE
\[
\begin{aligned}
dX(t) &= b(t, X_t, u(t)) \, dt + \sigma(t, X_t, u(t)) \, dW(t), \quad t \in [\hat{s}, T], \\
X_{\hat{s}} &= \xi(\cdot - \hat{s}).
\end{aligned}
\tag{4.9}
\]

**Lemma 4.3** Let \( s \in [0, T] \) and \((\Omega, \mathcal{F}, P, W, u) \in \mathcal{U}[s, T] \). Then for any \( \mathcal{F}_t \)-stopping time \( \hat{s} \in [s, T) \) and \( \mathcal{F}_s \)-measurable random variable \( \xi : \Omega \to C[-\delta, 0]^n \),
\[ J(\hat{s}, \xi(\cdot); u) = \mathbb{E} \left[ \int_{\hat{s}}^T f(t, X_t^\xi, u(t)) \, dt + h(X_T^\xi) \right] \mathcal{F}_s(\xi), \quad P - \text{a.s.} \omega. \quad (4.10) \]

**Proof** Since \( u \) is \( \{ \mathcal{F}_t \}_{t \in [s, T]} \)-adapted with \( \mathcal{F}_t = \sigma(W(\tau); s \leq \tau \leq t) \) by Lemma 4.2 there is a function \( \psi \in \mathcal{A}_t(U) \) such that
\[ u(t, \omega) = \psi(t, W(\cdot \wedge t, \omega)), \quad P - \text{a.s.} \omega \in \Omega, \quad \forall t \in [s, T]. \]
Then Eq. (4.9) may be written as
\[
\begin{aligned}
dX(t) &= b(t, X_t, \psi(t, W(\cdot \wedge t))) \, dt \\
&\quad + \sigma(t, X_t, \psi(t, W(\cdot \wedge t))) \, dW(t), \quad t \in [\tilde{s}, T], \\
X_{\tilde{s}} &= \xi(\cdot - \tilde{s}).
\end{aligned}
\] (4.11)

Due to A1–A3, Theorem 4.1 and Theorem 3.2 this equation has a strongly unique strong solution and weak uniqueness holds. In addition, we may write, for \( t \geq \tilde{s} \)
\[
u(t, \omega) = \psi(t, W(\cdot \wedge t, \omega)) = \psi(t, \tilde{W}(\cdot \wedge t, \omega) + W(\tilde{s}, \omega))
\]
where
\[
\tilde{W}(t) = W(t) - W(\tilde{s}).
\]
Since \( \tilde{s} \) is random, \( \tilde{W} \) is not a Brownian motion under \( \mathbb{P} \). However, we may under the weak formulation of the control problem change the measure as follows. Note first that
\[
\mathbb{P}\{\omega: \tilde{s}(\omega) = \tilde{s}(\omega)\} \mathbb{P}_1^{f}(\omega) = E[1_{\{w: \tilde{\sigma}(\omega) = \tilde{s}(\omega)\}} | \mathbb{F}_t^{f}](\omega) = 1, \quad \mathbb{P} - a.s. \omega \in \Omega.
\]
This means that there is an \( \tilde{\Omega}_0 \in \mathbb{F} \) with \( \mathbb{P}(\tilde{\Omega}_0) = 1 \), so that for any fixed \( \omega_0 \in \tilde{\Omega}_0, \tilde{s} \) becomes a deterministic time \( \tilde{s}(\omega_0) \), i.e. \( \tilde{s} = \tilde{s}(\omega_0) \) almost surely in the new probability space \( (\tilde{\Omega}, \mathbb{F}, \mathbb{P}(\cdot | \mathbb{F}_t^{f}(\omega_0))) \). A similar argument shows that \( W(\tilde{s}) \) almost surely equals a constant \( W(\tilde{s}(\omega_0), \omega_0) \) and also that \( \xi \) almost surely equals a constant \( \xi(\omega_0) \) when we work in the probability space \( (\Omega, \mathbb{F}, \mathbb{P}(\cdot | \mathbb{F}_t^{f}(\omega_0))) \). So under the measure \( \mathbb{P}(\cdot | \mathbb{F}_t^{f}(\omega_0)) \), for \( t \geq \tilde{s}(\omega_0) \), \( \tilde{W} \) will be a standard Brownian motion
\[
\tilde{W}(t) = W(t) - W(\tilde{s}(\omega_0)),
\]
and for any \( t \geq \tilde{s}(\omega_0) \),
\[
u(t, \omega) = \psi(t, \tilde{W}(\cdot \wedge t, \omega) + W(\tilde{s}(\omega_0), \omega)).
\]
It follows then that \( \nu(t) \) is adapted to the filtration \( \{\mathbb{F}_t^{(\omega_0)}\} \) generated by the standard Brownian motion \( \tilde{W}(t) \) for \( t \geq \tilde{s}(\omega_0) \). Hence, by the definition of admissible controls,
\[
(\Omega, \mathbb{F}, \mathbb{P}(\cdot | \mathbb{F}_t^{f}(\omega_0)), \tilde{W}, \nu|_{[\tilde{s}(\omega_0), T]} \in \mathbb{F}[\tilde{s}(\omega_0), T].
\]
Note that for \( A \in \mathbb{B}(C[0, T]) \),
\[
\mathbb{P}[\xi \in A | \mathbb{F}_t^{f}(\omega_0)] = E[1_{(\xi \in A)} | \mathbb{F}_t^{f}(\omega_0)] = E[1_{(\xi \in A)}] = \mathbb{P}[\xi \in A].
\]
This means that the two weak solutions

\[(X, W), (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}) \] and \[(X, \tilde{W}), (\Omega, \mathcal{F}, \mathbb{P}(\cdot | \mathcal{F}_{\tilde{t}})(\omega_t)), \{\mathcal{F}_{\tilde{t}}(\omega_t)\}\]

of Eq. (4.11) have the same initial distribution. Then by weak uniqueness (The dynamic programming principle). Let A1–A4 hold. Then

**Theorem 4.2**

for any proof

Denote the right hand side of Eq. (4.12) by \(V(s, \phi; u)\) and \(V(\tilde{s}, X^{s, \phi, u}_t)\). Given any \(\epsilon > 0\) there exists an \((\Omega, \mathcal{F}, \mathbb{P}, W, u) \equiv \mathcal{F}[s, T]\) such that \(J(s, \phi, u) \leq V(s, \phi, u) + \epsilon\), or equivalently,

\[V(s, \phi) + \epsilon > J(s, \phi, u)\]

\[= \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + h(X^{s, \phi, u}_T) \right] \]

\[= \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + \int_s^T f(t, X^{s, \phi, u}_t) \, dt + h(X^{s, \phi, u}_T) \right] \]

\[= \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + \mathbb{E} \left( \int_s^T f(t, X^{s, \phi, u}_t) \, dt + h(X^{s, \phi, u}_T) \right) \right] \]

\[= \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + \mathbb{E} \left( \int_s^T f(t, X^{s, \phi, u}_t) \, dt + h(X^{s, \phi, u}_T) \right) \right] \]

\[= \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + J(\tilde{s}, X^{s, \phi, u}_s; u) \right] \quad \text{(by Lemma (4.3))} \]

\[\geq \mathbb{E} \left[ \int_s^T f(t, X^{s, \phi, u}_t) \, dt + V(\tilde{s}, X^{s, \phi, u}_s) \right]. \]
so by taking the infimum over \( u \in U[s, T] \) we get

\[
V(s, \varphi) + \epsilon > \tilde{V}(s, \varphi) \quad \text{for all } \epsilon > 0.
\]

(4.13)

Conversely, for any \( \epsilon > 0 \), by Lemma 4.1 and its proof, there is a \( \delta = \delta(\epsilon) \) such that whenever \( \|\varphi - \tilde{\varphi}\|_{C[-\delta, 0]^n} < \delta \),

\[
|J(\hat{s}, \varphi; u) - J(\hat{s}, \tilde{\varphi}; u)| + |V(\hat{s}, \varphi) - V(\hat{s}, \tilde{\varphi})| \leq \epsilon, \quad \text{for all } u \in U[\hat{s}, T].
\]

(4.14)

Now let \( \{D_j\}_{j \geq 1} \) be Borel partition of \( C[-\delta, 0]^n \). This means that \( D_j \in \mathcal{B}(C[-\delta, 0]^n) \) for each \( j \), \( \bigcup_{j \geq 1} D_j = C[-\delta, 0]^n \), and \( D_i \cap D_j = \emptyset \) if \( i \neq j \). We also assume that the \( D_j \) are chosen so that \( \|\varphi - \tilde{\varphi}\|_{C[-\delta, 0]^n} < \delta \) whenever \( \varphi \) and \( \tilde{\varphi} \) are both in \( D_j \). Choose \( \varphi_j \in D_j \). For each \( j \), there exists an \( (\Omega_j, \mathcal{F}_j, P_j, W_j, u_j) \in U[\hat{s}, T] \) such that

\[
J(\hat{s}, \varphi_j; u_j) \leq V(\hat{s}, \varphi) + \epsilon.
\]

(4.15)

For any \( \varphi \in D_j \), the inequality (4.14) implies in particular that

\[
J(\hat{s}, \varphi; u_j) \leq J(\hat{s}, \varphi_j; u_j) + \epsilon \quad \text{and} \quad V(\hat{s}, \varphi) \leq V(\hat{s}, \varphi_j) + \epsilon.
\]

(4.16)

Combining Eqs. (4.15) and (4.16) we see that

\[
J(\hat{s}, \varphi; u_j) \leq J(\hat{s}, \varphi_j; u_j) + \epsilon \leq V(\hat{s}, \varphi_j) + 2\epsilon \leq V(\hat{s}, \varphi) + 3\epsilon.
\]

(4.17)

By the definition of the 5-tuple \( (\Omega_j, \mathcal{F}_j, P_j, W_j, u_j) \in U[\hat{s}, T] \), there is a function \( \psi_j \in \mathcal{A}_f(U) \) such that

\[
u_j(t, \omega) = \psi_j(t, W_j(\cdot \wedge t, \omega)), \quad \mathbb{P}_j - \text{a.s.} \quad \omega \in \Omega_j, \quad \text{for all } t \in [\hat{s}, T].
\]

Now let \( (\Omega, \mathcal{F}, P, W, u) \in U[s, T] \) be arbitrary. Define the new control

\[
\tilde{u}(t, \omega) = \begin{cases} 
  u(t, \omega), & \text{if } t \in [s, \hat{s}), \\
  \psi_j(t, W(\cdot \wedge t, \omega) - W(\hat{s}, \omega)), & \text{if } t \in [\hat{s}, T] \text{ and } X^{s, \tilde{u}}(\omega) \in D_j.
\end{cases}
\]
Then \((\Omega, \mathcal{F}, P, W, \tilde{u}) \in \mathcal{H}[s, T]\). Thus

\[
V(s, \varphi) \leq J(s, \varphi; u)
\]

\[
= E \left[ \int_s^T f(t, X_t^{s, \varphi, u}, \tilde{u}(t)) \, dt + h(X_T^{s, \varphi, u}) \right]
\]

\[
= E \left[ \int_s^T f(t, X_t^{s, \varphi, u}, u(t)) \, dt + E \left( \int_s^T f(t, X_t^{s, \varphi, u}, \tilde{u}(t)) \, dt + h(X_T^{s, \varphi, u}) \right) \right]
\]

\[
= E \left[ \int_s^T f(t, X_t^{s, \varphi, u}, u(t)) \, dt + J(\tilde{u}, X_t^{s, \varphi, u}; \tilde{u}) \right] \quad \text{(by Lemma (4.3))}
\]

\[
\leq E \left[ \int_s^T f(t, X_t^{s, \varphi, u}, u(t)) \, dt + V(\tilde{u}, X_t^{s, \varphi, u}) \right] + 3\epsilon. \quad \text{(by Eq. (4.17))}
\]

Since this holds for arbitrary \((\Omega, \mathcal{F}, P, W, u) \in \mathcal{H}[s, T]\), by taking the infimum over \(\mathcal{H}[s, T]\) we obtain

\[
V(s, \varphi) \leq \bar{V}(s, \varphi) + 3\epsilon \quad \text{for all } \epsilon > 0.
\]

The conclusion (4.12) follows from the inequalities (4.13) and (4.18). \(\square\)

**AN APPLICATION: THE HAMILTON–JACOBI–BELLMAN EQUATION**

In this section we look at a one-dimensional system where the dependence of the past is of a particularly simple form. Let \(\lambda \in \mathbb{R}\) be a constant and define for \(\xi \in C[-\delta, 0]\) the linear functionals

\[
x(\xi) := \xi(0), \quad y(\xi) := \int_{-\delta}^0 e^{\lambda s} \xi(s) \, ds, \quad z(\xi) := \xi(-\delta).
\]

For any \(X \in C[-\delta, T]\) we write for \(t \in [0, T]\),

\[
X(t) = x(X_t),
\]

\[
Y(t) := y(X_t) = \int_{-\delta}^0 e^{\lambda s} X(t + s) \, ds,
\]

\[
Z(t) := z(X_t) = X(t - \delta).
\]
Consider the system
\[
\begin{align*}
    dX(t) &= b(t, X(t), Y(t), Z(t), u(t)) \, dt \\
    &\quad + \sigma(t, X(t), Y(t), Z(t), u(t)) \, dW(t), \quad t \in [s, T] \\
    X(t) &= \varphi(t - s), \quad t \in [s - \delta, s], \quad \varphi \in C[-\delta, 0],
\end{align*}
\]
where \( b : \mathbb{R}^4 \times U \to \mathbb{R} \) and \( \sigma : \mathbb{R}^4 \times U \to \mathbb{R} \) are given functions and \( u \) is the control. This type of delayed control system has been treated by many authors, see e.g. Refs. [4,8-10,13,14]. Systems with this type of past dependence has also been studied in Ref. [3] where a singular control was used, and in Ref. [2] where optimal stopping problems and impulse control problems were treated. For systems like Eq. (5.5) there exists an Ito formula. Let
\[
g(t, x(X_t), y(X_t))
\]
and define
\[
G(t) = g(t, x(X_t), y(X_t)).
\]

**Lemma 5.1** (The Ito formula).
\[
dG(t) = \mathcal{L}u \, dt + \frac{\partial g}{\partial x} \cdot \sigma(t, x, y, z, u) \, dW(t)
\]
where
\[
\mathcal{L}u = \mathcal{L}u(t, x, y, z)
\]
\[
= \frac{\partial g}{\partial t} + b(t, x, y, z, u) \frac{\partial g}{\partial x} + \frac{1}{2} \sigma^2(t, x, y, z, u) \frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial y} \cdot [x - e^{-\lambda z} - \lambda y]
\]
which is evaluated at
\[
x = x(X_t) = X(t), \quad y = y(X_t), \quad z = z(X_t).
\]

**Proof** See Lemma 2.1 in [4] □

The cost functional is
\[
J(x, \varphi; u) = \mathbb{E}^{x, \varphi, u} \left[ \int_s^T f(t, X(t), Y(t), u(t)) \, dt + h(X(T), Y(T)) \right]
\]
and the value function \( V(x, \varphi) \) is defined as in Eq. (4.6). In general the value function may depend on the initial path \( \varphi \in C[-\delta, 0] \) in a complicated way. In Ref. [4] it is shown that for a certain class of systems of the form (5.5), the value function depends on the initial path only through the functionals \( x(\varphi) \) and \( y(\varphi) \). Let us therefore, make this assumption in the following, i.e. we assume that
\[
V(x, \varphi) = V(x, x(\varphi), y(\varphi)) = V(s, x, y).
\]

\[\text{(5.10)}\]
Then the DPP takes the form

$$V(s, x, y) = \inf_{u \in [0, T]} \mathbb{E}^{s, u} \left[ \int_s^t f(t, X(t), Y(t), u(t)) \, dt + V(\tilde{s}, X(\tilde{s}), Y(\tilde{s})) \right]$$  \hspace{1cm} (5.11)

for all $\mathcal{F}_s^T$-stopping times $\tilde{s} \in [s, T)$ and $(x, y) \in \mathbb{R}^2$ where $\phi \in C[-\delta, 0]$ is such that $x = x(\phi) = X(s)$ and $y = y(\phi) = Y(s)$.

The following is a converse of Theorem 2.1 in Ref. [4].

**Theorem 5.1** If we assume that Eq. (5.10) holds and that $V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R}^2)$ then $V(s, x, y)$ solves the following Hamilton–Jacobi–Bellman partial differential equation

$$\inf_{u \in U} \{ \mathcal{L}^u V(s, x, y, z) + f(s, x, y, u) \} = 0 \quad \text{for all } z \in \mathbb{R},$$  \hspace{1cm} (5.12)

with boundary condition

$$V(T, x, y) = h(x, y).$$  \hspace{1cm} (5.13)

**Proof** Fix $(x, y) \in [0, T] \times \mathbb{R}^2$, $\tilde{s} \in (s, T)$ (deterministic), and $u \in U$ ($u$ is a control value). Let $X$ be given by Eq. (5.5) with $u(t) = u$, and fix $\phi \in C[-\delta, 0]$ such that $x = x(\phi) = X(s)$ and $y = y(\phi) = Y(s)$. Then Eq. (5.11) implies that

$$V(s, x, y) \leq \mathbb{E}^{s, u} \left[ \int_s^T f(t, X(t), Y(t), u(t)) \, dt + V(\tilde{s}, X(\tilde{s}), Y(\tilde{s})) \right].$$

Dividing by $\tilde{s} - s$ and rearranging we see that

$$\mathbb{E}^{s, u} \left[ \frac{1}{\tilde{s} - s} (V(\tilde{s}, X(\tilde{s}), Y(\tilde{s})) - V(s, x, y)) + \frac{1}{\tilde{s} - s} \int_s^T f(t, X(t), Y(t), u(t)) \, dt \right] \leq 0.$$

As $\tilde{s} \to s$ we obtain, using the Itô formula (5.7),

$$\mathcal{L}^u V(s, x, y, z) + f(s, x, y, u) \geq 0 \quad \text{for all } z \in \mathbb{R},$$

This holds for any $u \in U$, so

$$\inf_{u \in U} \{ \mathcal{L}^u V(s, x, y, z) + f(s, x, y, u) \} \geq 0 \quad \text{for all } z \in \mathbb{R}.$$  \hspace{1cm} (5.14)

Conversely, for any $\varepsilon > 0$, $0 \leq s < \tilde{s} \leq T$, with $\tilde{s} - s > 0$ small enough, we can find $u(\cdot) = u_{\varepsilon, \tilde{s}}(\cdot) \in \mathcal{U}[s, T]$ such that

$$V(s, x, y) + \varepsilon \cdot (\tilde{s} - s) \geq \mathbb{E}^{s, u_{\varepsilon, \tilde{s}}} \left[ \int_s^T f(t, X(t), Y(t), u(t)) \, dt + V(\tilde{s}, X(\tilde{s}), Y(\tilde{s})) \right].$$
or equivalently,

$$e \geq \mathbb{E}^\pi \left[ \frac{1}{s-s_x} \int_s^T \left\{ f(t, X(t), Y(t), u(t)) + dV(t, X(t), Y(t)) \right\} dt \right]$$

$$= \mathbb{E}^\pi \left[ \frac{1}{s-s_x} \int_s^T \left\{ f(t, X(t), Y(t), u(t)) + \mathcal{L}_u V(t, X(t), Y(t), Z(t)) \right\} dt \right]$$

$$\geq \mathbb{E}^\pi \left[ \frac{1}{s-s_x} \inf_{u \in U} \left\{ f(t, X(t), Y(t), u(t)) + \mathcal{L}_u V(t, X(t), Y(t), Z(t)) \right\} dt \right]$$

$$- \inf_{u \in U} \left\{ \mathcal{L}_u V(s, x, y, z) + f(s, x, y, u) \right\} \quad \text{as } s \to s.$$ 

That is, for all $e > 0$,

$$\inf_{u \in U} \left\{ \mathcal{L}_u V(s, x, y, z) + f(s, x, y, u) \right\} \leq e \quad \text{for all } z \in \mathbb{R}. \quad (5.15)$$

Combining the inequalities (5.14) and (5.15) gives Eq. (5.12). The boundary condition (5.13) follows immediately from the definition of the value function. \qed

**Remark 5.1** To accommodate for an application to a control problem with partial observations, the system studied in Ref. [4] is actually on the slightly more general form

\[
\begin{align*}
\dot{x}(t) &= b_1(t, X(t), Y(t), Z(t), R(t), u(t)) dt \\
&\quad + \sigma_1(t, X(t), Y(t), Z(t), R(t), u(t)) dW(t), \quad t \in [s, T] \\
X(t) &= \varphi(t-s), \quad t \in [s-\delta, s], \quad \varphi \in \mathcal{C}[-\delta, 0], \\
\dot{R}(t) &= b_2(t, R(t)) dt + \sigma_2(t, R(t)) dW(t), \quad R(s) = r,
\end{align*}
\]

(5.16)

while the cost functional is

$$J(s, \varphi; u) = \mathbb{E}^\pi \left[ \int_s^T f(t, X(t), Y(t), R(t), u(t)) dt + h(X(T), Y(T), R(T)) \right]. \quad (5.17)$$

This only amounts to adding an extra dimension without delay to the system. The essential features concerning control of delay systems are present already in the simpler system (5.5) and all the results in this section carry over to the situation where the system and cost are given by Eqs. (5.16) and (5.17).

**Remark 5.2** In Ref. [4] it is shown that if $b$ and $\sigma$ in Eq. (5.5) are linear in $x$, $y$, and $z$, with coefficients satisfying a certain relation, then the system (5.5) may be transformed into a system without delay. If in addition $f$ and $h$ in Eq. (5.9) depend
on x and y in a special way consistent with this relation, the control problem reduces to finite dimensions. The dynamic programming principle for no-delay systems applies, and one may deduce the HJB-equation. In our case, we do not have this extra structure, so to prove Theorem 5.1 we really need the DPP for delay systems (Theorem 4.12). Also see Ref. [12] where conditions are given ensuring that the HJB-equation is finite dimensional when the system (5.5) is nonlinear.

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References


