Analytical Model for Predicting Noise and Vibration in Permanent-Magnet Synchronous Motors

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Abstract—This paper analyzes the noise and vibration in permanent-magnet synchronous motors (PMSMs). Electromagnetic forces have been identified as the main cause of noise and vibration in these machines, rather than the torque ripple and cogging torque. A procedure for calculating the magnetic forces on the stator teeth based on the 2-D finite-element (FE) method is presented first. An analytical model is then developed to predict the radial displacement along the stator teeth. The displacement calculations from the analytical model are validated with structural finite-element analysis (FEA) and experimental data. Finally, the radial displacement is converted into sound power level. Four different PMSM topologies, suitable for the electric power steering application, are compared for their performances with regard to noise and vibration.

Index Terms—Finite-element analysis (FEA), noise and vibration, permanent-magnet synchronous motor (PMSM), permanent-magnet synchronous motors (PMSMs) topologies, radial displacement, radial pressure, sound power level.

NOMENCLATURE

Vibration Parameters

- $\sigma_r$, $\sigma_t$ Radial and tangential component of stress (N/m²).
- $\nu$ Poisson’s ratio for steel material used in stator.
- $P_s$ Sound power radiated (watt).
- $\sigma_{rel}$ Relative sound intensity.
- $c$ Traveling speed of sound in the medium (m/s).
- $P_{sref}$ Sound power level reference ($10^{-12}$ watt).
- $n_r$, $n_m$ Harmonic and circumferential mode number.
- $f_m$ Circumferential mode frequency for $m = 0, 1, 2, 3, \ldots$.

Machine Geometry:

- $y_s$, $w_{bi}$ Stator tooth and yoke height (m).
- $R_{in}$, $R_{out}$ Mean and outer radius of stator yoke (m).
- $W_{st}$, $W_{y}$ Total weight of all the stator teeth and yoke (kg).
- $W_{wi}$, $W_i$ Total weight of the windings and insulation (kg).

Fig. 1. Radial and tangential component of electromagnetic forces in a PMSM structure.

I. INTRODUCTION

The acoustic noise in permanent-magnet synchronous machines (PMSMs) is lower compared to switched reluctance and induction machines; yet quieter performance is desired in automotive and robotics applications. Noise and vibration in PM machines can be classified into three categories based on its source: 1) aerodynamic; 2) mechanical; and 3) electromagnetic. Electromagnetic source is the dominating one in low- to medium-power rated machines. Cogging torque, torque ripple, and magnetic radial forces are the main electromagnetic sources of noise and vibration. According to several researchers, the reduction of cogging torque and torque ripple can significantly reduce the vibration and acoustic noise [1]–[7], [15]–[17]. However, the fundamental relationship between torque ripple or cogging torque with noise and vibration was not established through these research works.

The electromechanical energy conversion due to an interaction between the magnetic fields of PMs and armature conductors takes place in the air gap of a PMSM. The normal or the radial component of the electromagnetic force field between the rotor magnets and the stator teeth causes the radial vibration of the stator structure. This normal force is known as radial force $F_{rad}$, which is shown in Fig. 1. The radial force is undesirable particularly when it leads to unbalanced pull on the stator.

This research is focused on finding the root cause of electromagnetic noise and vibration in 9-slot/6-pole (9s6p), 12-slot/10-pole (12s10p), 12-slot/8-pole (12s8p), and 27-slot/6-pole (27s6p) motors. The intended application is automotive electric power steering with power requirements around 600 W. The ratings and outer dimensions of four designed motors are given in Appendix A. The motors have been designed with very low levels of cogging torque and torque ripple to qualify...
for automotive and servo applications. The design features also minimize their effects on noise and vibration. This paper shows that the radial forces are the major contributors to noise and vibration in these PMSMs, not the torque ripple or the cogging torque.

II. BASICS OF RADIAL FORCE DENSITY

The stress between the PM magnetic field and the stator mmf is measured as radial force density which is commonly known as the radial pressure. Maxwell’s stress tensor method is generally used to calculate this electromagnetic force acting on an object. In case of 2-D finite-element analysis (FEA), a surface containing the object is selected as a line (for example, l), and then, the tangential and the radial components of the force are calculated. These forces are given as

\[ F_{\text{tan}} = \frac{L_{\text{stk}}}{\mu_0} \oint_{l} B_l B_n dl \quad (1) \]

\[ F_{\text{rad}} = \frac{L_{\text{stk}}}{2\mu_0} \oint_{l} (B_n^2 - B_l^2) dl. \quad (2) \]

Here, \( B_l \) and \( B_n \) represent the tangential and normal component of the flux density in the air gap, \( L_{\text{stk}} \) is the stack length of the motor and \( l \) represents the line containing the object where force is to be computed. Once the air-gap flux distribution is known, the magnitude of radial force per unit area at any point of the air gap can be calculated according to (2). Since the magnetic permeability of the ferromagnetic core is much higher than that of the air gap, the magnetic flux lines are practically perpendicular to the stator and rotor cores. This means the tangential component of flux is much smaller than the normal component and can be neglected. Again, the force per unit area is pressure. Therefore, the instantaneous value of radial pressure is given by

\[ P_r(\alpha, t) \approx \frac{F_{\text{rad}}}{L_{\text{stk}} l} = \frac{B_n^2(\alpha, t) - B_l^2(\alpha, t)}{2\mu_0} \quad (3) \]

where \( \alpha \) is the angular distance from a given axis of the measurement point. Here, \( r = (\nu \pm \mu)p = 0, 1, 2, 3, \ldots \) is the mode number of the force wave, \( p \) is the number of rotor pole pairs, and \( \omega_r = \omega_\nu \pm \omega_\mu \) is the angular frequency of the \( r \)th order force. Also, \( \nu \) and \( \mu \) are the numbers of the stator and rotor harmonics, respectively. The radial force circulates around the stator bore with the angular speed \( \omega_r \) and frequency \( f_r = \omega_r/(2\pi r) \).

A. Unbalanced Radial Forces in PMSMs

A perfect motor with balanced stator windings should have a net radial force of zero on the stator structure. However, unbalanced radial force can be present in some machines depending on the slot-pole configurations. The PMSMs with fractional ratio of slot number to pole number can be classified into the following three types.

(a) Type I—Machines having diametrically asymmetric disposition of slots and phase windings [6]–[8]: These machines can have slot/pole combinations of 3s2p, 3s4p; 9s8p, 9s10p; 15s14p, 15s16p, and so on. The unbalanced radial force acts on the rotor of these machine configurations due to the asymmetric magnetic field distribution in the air gap.

(b) Type II—Machines with same phase windings in diametrically opposite slots: These machines are known as modular machines. Modular machines have slot-pole combinations of 6s8p, 12s10p, and 12s14p; these machines can also have asymmetrical distribution of windings. Symmetry is achieved in these machines with the appropriate rearrangement of the winding.

(c) Type III—Machines with symmetrical winding: These machines are known non-modular machines. Examples of non-modular machines are 9s6p, 12s8p, 15s10p, 27s6p, and 18s12p configurations.

In the ideal case, none of the motor types in b and c should have any unbalanced radial force on the stator. The following sections focus on noise and vibration in the modular and nonmodular machines caused by radial forces.

1) Radial Pressure Under No Load: The FEA results for radial pressure distribution around the stator tooth periphery without stator excitation are given in Fig. 2 for 9s6p, 12s10p, 12s8p, and 27s6p machines. The dominant modes of the pressure waves are equal to the number of poles in all cases except for the modular machine. In the modular machine of 12s10p topology, second- and tenth-order pressures and their suborders exist.

The fast Fourier transform (FFT) analysis of pressure distributions identifies the orders in the machines as shown in Fig. 3. The orders in a 12s10p machine are 2, 4, 6, 8, 10, 12, etc., and in a 12s8p machine are 8, 16, 24, and so on.

2) Radial Pressure Under Load: The radial pressure distribution and its frequency contents change with stator excitation. The lower mode frequency appears in the machine only with excitation.

The difference in radial pressure distribution for modular and nonmodular machines has been analyzed with the candidate 12s10p and 9s6p motors representing the types II and III. The changes in radial pressure distribution from no-load to full-load conditions are shown in Fig. 4 for the two machine configurations. Fig. 4(c) shows the change of lower order mode from sixth to third for a change from no-load to full-load in a nonmodular PMSM with 9s6p configuration. On the other hand, for a modular PMSM with 12s10p configuration, the low second-order mode prevails and is amplified under full load along with other even orders, such as fourth, sixth, and eighth [Fig. 4(d)]. Similarly, the lowest orders in 12s8p and 27s6p motors under load are fourth order and third order, respectively. In general, the low orders in nonmodular types are higher than the low orders in modular type machines.

The next section discusses the stator displacement and vibration of the motor due to radial pressure.

III. MODE SHAPES AND RADIAL FORCES

The magnetic radial force, circumferential mode frequencies, radial deflections, stator vibration, and noise power level are all functions of machine geometry, configuration, and material properties.
Fig. 2. Radial pressure variation without stator excitation obtained from FEA (a) 9s6p, (b) 27s6p, (c) 12s10p, and (d) 12s8p PMSM.

A. Common Mode Shapes

The radial forces are transmitted through the teeth from the air gap to the yoke which can cause deformation on the stator rings. Deformations of the stator core for different vibration modes are commonly called “mode shapes” [10]. All the mode shapes have their own natural mode frequency. Any particular mode shape is excited when its natural mode frequency matches with any of the harmonics of the magnetic radial force [9]. From airborne noise point of view, the most important frequencies are those of low mode numbers, i.e., frequencies at $r = 0, 2, \text{ and } 4$.

B. Modes in Modular and Nonmodular Machines

The radial force distributions on the stator teeth for modular and nonmodular machines are shown in the radar plots of Fig. 5. The distributions clearly show a mode frequency of 2, irrespective of stator excitation, in the modular machine and mode frequencies of 6 and 3 in the nonmodular machine with unexcited and excited stator, respectively. The mode shapes and orders of the force waves match the lower order frequency contents of the pressure waves from the previous section.

IV. RADIAL DISPLACEMENT MODEL

An analytical displacement model for the motor structure is developed in this section. The results of radial forces obtained by FEA are then used in the analytical model to determine the displacement on the outer periphery of the stator. The results of the analytical model are also validated using results obtained from the structural FEA software.

A. Analytical Model

The stator can be considered as a cylinder body with restrained ends. The cross-sectional dimensions of the device structure are shown in Fig. 6(a). The inside radius (up to the stator air gap) of the cylinder is $R_{gs}$, and the outside radius is $R_{\text{out}}$. Let the internal pressure in the cylinder be $P_i$ which is equal to the radial force per unit stator area ($P_r$) given by (3); the outside or external pressure is $P_0$. The problem can be solved using cylindrical coordinates. Each ring of unit thickness measured perpendicular to the plane of the paper is stressed alike. An infinitesimal element of unit thickness is defined by
Radial Pressure in 9-slot/6-pole PMSM: @ No-load and @ 90A Max Current

Radial pressure in 12-slot/10-pole PMSM: @ No-load and @ 90A Max Current

FFT of Radial Pressure in 9-slot/6-pole PMSM: @ No-load and @ 90A Max Current

FFT of Radial Pressure in 12-slot/10-pole PMSM: @ No-load and @ 90A Max Current

Fig. 4. Radial pressure: (a) variation at full load versus no load in a 9s6p (b) variation at full load versus no load in a 12s10p (c) FFT of radial pressure for 9s6p (d) FFT of radial pressure for 12s10p.

Fig. 5. Radial force distribution on stator teeth under no load and full load: (a) Nonmodular machine (9s6p) (b) Modular machine (12s10p).

Fig. 6. (a) Cylindrical shape of stator with a cross-sectional view (b) an infinitesimal stator element of unit thickness.

two radii \( r \) and \( r + dr \), and angle \( \theta \), as shown in Fig. 6(b). The stress and strain equations applied to this problem are [11]

\[
\sigma_r = \frac{E}{(1 + v)(1 - 2v)} \left[ (1 - v)\varepsilon_r + v\varepsilon_t \right]
\]

\[
\sigma_t = \frac{E}{(1 + v)(1 - 2v)} \left[ v\varepsilon_r + (1 - v)\varepsilon_t \right]
\]

\[
\varepsilon_r = \frac{1}{E} \left( \sigma_r - \nu \sigma_t - \nu \sigma_x \right)
\]

\[
\varepsilon_t = \frac{1}{E} \left( -\nu \sigma_r + \sigma_t - \nu \sigma_x \right)
\]

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_r - \nu \sigma_t \right).
\]
The axial strain can be neglected under the assumption that the ends are restrained. With $\varepsilon_x = 0$ and combining the expressions in (4) with the strain relationships along the radial and tangential directions $\varepsilon_r = dx/dr$ and $\varepsilon_t = x/r$, the stress equations can be simplified as

$$\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu) \frac{dx}{dr} + \frac{x}{r} \right] \tag{5a}$$
$$\sigma_t = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \frac{dx}{dr} + (1 - \nu) \frac{x}{r} \right] \tag{5b}$$

Our interest is on the normal radial stress $\sigma_r$ and the radial displacement $x$ due to this stress for estimating vibration and noise. The condition of static equilibrium requires that the sum of the forces along a radial line is equal to zero

$$\sum F_r = 0. \tag{6}$$

The force on the element of unit thickness is calculated as the product of stress and the cross-sectional area. Therefore, the equilibrium condition in (6) can be simplified as

$$\sigma_r, \ v, \ d\theta + 2\sigma_t, \ r, \ (\frac{d\theta}{2}) - \left( \sigma_r + \frac{d\sigma_r}{dr} \cdot dr \right) (r + dr) \cdot d\theta = 0 \Rightarrow \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0. \tag{7}$$

Combining the relationships in (5) and (7) and after simplifications, the following differential equation can be obtained to estimate the radial displacement $x$:

$$\frac{d^2x}{dr^2} + \frac{1}{r} \frac{dx}{dr} - \frac{x}{r^2} = 0. \tag{8}$$

The details of this derivation are given in Appendix B. The general form of solution to the differential equation in (8) for radial displacement at any point on the cylinder is

$$x = A_1 r + \frac{A_2}{r} \tag{9}$$

where the constants $A_1$ and $A_2$ can be determined by applying the boundary conditions on the body

$$\sigma_r(R_i) = -P_i \quad \text{and} \quad \sigma_r(R_o) = -P_o.$$ 

For simplicity, the outside pressure on the cylinder $P_o$ is considered negligible compared to the pressure on the inside of the cylinder. The final expressions for the constants are

$$A_1 = \frac{(1 + \nu)(1 - 2\nu)}{E} \cdot \frac{P_i R_i^2}{(R_o^2 - R_i^2)} \quad \text{and} \quad A_2 = \frac{(1 + \nu)}{E} \cdot \frac{P_o R_o^2}{(R_o^2 - R_i^2)^2} \tag{10}.$$ 

The constants $A_1$ and $A_2$, when used in (9), give the radial displacement of any point on the elastic cylinder subjected to the specified pressure.

### B. Estimation and Validation of Radial Displacement

Separate calculations have been carried out to find the displacements on the inner surface of the tooth and the outer surface of the stator cylinder along a radial line according to (9). If the displacement on the inside edge of a tooth is $x_{in}$ and the displacement on the outer edge of the stator ring along that tooth is $x_{out}$, then the difference between these two are the estimate for net displacement along that line

$$x = x_{in} - x_{out}. \tag{11}$$

Any displacement due to stator vibration from the end caps, bearings, gears, etc., have not been considered in this model.

1) Displacement From Analytical Model: The radial pressure obtained from FEA (using Flux 2-D) is used in the analytical model to calculate the radial displacement $x$. Here, the radial pressure at the low-order mode is used as the maximum value of tooth pressure for a particular motor topology. The displacement results for the four different PMSMs are given in Table I.

2) Displacement From Structural FEA: The structural FEA (using ANSYS) is done at several locations on the stator housing of a 12s10p machine to compare with the analytical results. The nodes are taken along the center line of each tooth as shown in Fig. 7, where F1 through F12 represent the tooth forces that vary with phase currents or rotor position. These tooth forces are the oscillating portions of radial forces acting on each tooth in one complete cycle. There are only six variations for 12 tooth forces, since the diagonally opposite tooth pair (such as F1-F7 and F2-F8) go through the same magnetic stress. The tooth pressures in Table II have been used as the input to the structural FEA model in ANSYS to determine the displacements (in mm) at the 12 node points on the stator outer periphery (as marked in Fig. 7). All six columns (case I through case VI) in Table III have been completed with displacement values from this structural FEA.

3) Displacement From Experiments: Four accelerometers have been mounted at four different nodes that are 90° apart on the outer periphery of a 12s10p motor in a dyno stand. These nodes are marked as 1592, 1457, 1503, and 1458 in Fig. 8 to replicate some of the nodes in Fig. 7. The stator will go through some form of ovulation since second order is the lowest mode expected in the 12s10p motor. A comparison among the analytical model, structural FEA, and the accelerometer test data is given in Table IV.

### V. Vibration and Noise Due to Radial Displacement

The bending or the torsion of the machine stator structure can be neglected, since the ends of these machines are restrained. It is then the circumferential mode that dictates the mode
frequency and vibration due to radial stress. This natural mode frequency and the excitation frequency establish the condition for resonance.

A. Definition of Noise and Vibration

Vibration is a limited reciprocating particle motion of an elastic body or medium in alternately opposite directions from its equilibrium position when that equilibrium has been disturbed. The amplitude of vibration is the maximum displacement of a vibrating particle or body from its position of rest.

Sound is defined as vibrations transmitted through an elastic solid, liquid, or gas. Noise is disagreeable or unwanted sound. The frequency of interest for vibrations is generally within 1000 Hz, and for noise, it is over 1000 Hz.

The radial vibration of stator of a PMSM is initiated when the magnetic radial pressure causes radial displacement of the stator structure. The machine vibration leads to airborne or structure-borne sound depending on the vibration transmitting medium; when the power level of these sounds exceed certain threshold, it becomes noise.

B. Condition of Maximum Vibration

The maximum vibration occurs at resonance condition when the magnetic radial pressure causes radial displacement of the stator structure. The machine vibration leads to airborne or structure-borne sound depending on the vibration transmitting medium; when the power level of these sounds exceed certain threshold, it becomes noise.

\[ f_{\text{exc}}(n) = n f_p = \frac{n \omega_r N_p}{120} \quad n = 1, 2, 3 \]  

where \( N_p \) is the number of rotor poles. The natural mode frequencies \( f_m \) are a function of stator dimensions [14].
C. Calculation of Natural Mode Frequency

The basic methods of calculating the natural mode frequencies \( f_m \) can be found in [10], [12], and [15]–[17]. First, the stator is considered as a freely vibrating ring; then, stator teeth and winding masses are considered. Finally, the effects of shear and rotary inertia are considered to develop the formulas for estimating the mechanical mode frequencies [14]. The mode zero frequency is given by

\[
f_{(m=0)} = \frac{1}{2\pi R_m} \sqrt{\frac{E}{\rho_s}}.
\]  

(13)

Here, \( R_m = R \) is the Young’s modulus of elasticity of stator material in N/m². \( \rho_s \) is the density of the material in Kg/m³. \( W_t \) is the total weight of the stator comprised of tooth, windings, and insulation weights.

The frequency of mode 1 is given by

\[
f_{(m=1)} = f_{(m=0)} \sqrt{\frac{2}{1 + i^2 \Delta/\Delta_m}}.
\]  

(14)

Here, \( \Delta = 1 + (1.91N_s A_s P d_s^2 W_t / R_m L_{st} y_s^3 W_p) \) \( [(1/3) + (y_s/2.d_s) + (y_s/2.d_s)^2]; \)

\( i = (1/2.\sqrt{3}).(y_s/R_m) \)

The frequencies for mode 2 and higher are given by

\[
f_{(m=2)} = \frac{f_{(m=0)} i^m (m^2 - 1)}{\sqrt{((m^2 + i^2)(m^2 - 1)(4m^2 + m^2 \Delta_m/\Delta + 3))}}.
\]  

(15)

D. Acoustic Noise and Sound Power Level

The analytical values of maximum radial displacements when the corresponding lower order modes are excited for the motors of interest in this research are given in Table I. Radiated noise can be estimated from these displacement results using their relationship with sound energy. The sound power in dB can be obtained from [16], [17]

\[
L_w = 10. \log \left( \frac{2 P_s}{P_{sref}} \right).
\]  

(16)

Here, \( P_s \) is the sound power in watts radiated by an electric machine for a particular excitation frequency and given by

\[
P_s = 4 \sigma_{rel} \rho_c \pi^3 f_{exc}^2 x^2 R_{out} L_{stk}.
\]  

(17)

The description of parameters in (13)–(15) is given in the “Nomenclatures” at the beginning of this paper.

The excitation frequencies for all four motor configurations have been calculated for several rotor speeds and are listed in Table V. The dominant mode numbers are 3, 2, 4, and 3 for the 9s6p, 12s10p, 12s8p, and 27s6p motors, respectively. The corresponding mode frequencies are given in Table VI. For the 12s10p motor, the second mode frequency (393.47 Hz) matches with about five times the excitation frequency at 1000 r/min (83.33 Hz), about two times at 2500 r/min (228.33 Hz), and about one time at 4000 r/min (353.28 Hz). Whereas, for 9s6p, 27s6p, and 12s8p motors, the mode frequencies are too high to match with the excitation frequency at the operating speed of these machines. The numbers show that machines with lower order mode are easily susceptible to resonance.

```
<table>
<thead>
<tr>
<th>Rotational speed, rpm</th>
<th>Rotational freq for different Motor configurations (f_m, Hz)</th>
<th>Excitation freq, Hz (f_m)</th>
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</thead>
<tbody>
<tr>
<td>9s6p</td>
<td>12s10p 12s8p 27s6p</td>
<td>f_m = m f_p</td>
</tr>
<tr>
<td>500</td>
<td>25 41.66 33.33 25</td>
<td>n=1,2,3,...</td>
</tr>
<tr>
<td>1000</td>
<td>50 83.33 66.67 50</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>100 176.64 133.34 100</td>
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</tr>
<tr>
<td>4000</td>
<td>200 353.28 266.68 200</td>
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```

VI. CONCLUSION

An analytical model has been developed to determine radial deformation using radial pressure as the input. The analytical model is validated by the structural FEA and by accelerometer tests predicting radial displacement. The radial displacement thus calculated is used to estimate noise and vibration in different PMSM configurations. This research showed that the root cause of noise and vibration is the radial forces not the torque ripples. Previous research showed that a 12s10p modular PMSM has superior cogging torque and torque ripple performance compared to 9s6p nonmodular PMSM [13]. However, this research proved that the modular machine is prone to higher levels of vibration than the nonmodular machines due to the presence of low-order modes.
**TABLE VIII**

<table>
<thead>
<tr>
<th>Motor configurations</th>
<th>12s10p</th>
<th>12s6p</th>
<th>9s6p</th>
<th>12s6p</th>
<th>units</th>
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<tr>
<td>Shaft torque</td>
<td>6.07</td>
<td>6.04</td>
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<td>5.72</td>
<td>Nm</td>
</tr>
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<td>633</td>
<td>631</td>
<td>599</td>
<td>Watts</td>
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<td>Base speed</td>
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<td>RPM</td>
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<td>RPM</td>
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<td>Stack length, Lstk</td>
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<td>60</td>
<td>60</td>
<td>mm</td>
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<td>Outer Radius, Rout</td>
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<td>47.5</td>
<td>47.5</td>
<td>47.5</td>
<td>mm</td>
</tr>
<tr>
<td>Cogging, P-P</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>mN·m</td>
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<tr>
<td>Torque ripple, P-P</td>
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<td>4.0%</td>
<td>4.0%</td>
<td>4.0%</td>
<td>% of Avg</td>
</tr>
<tr>
<td>BEMF harmonics, 5th, 7th, 11th etc.</td>
<td>0.5%</td>
<td>0.5%</td>
<td>0.5%</td>
<td>0.5%</td>
<td>% of 1st</td>
</tr>
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</table>

**APPENDIX A**

Table VIII shows a summary of motor ratings and other dimensions used in this study.

**APPENDIX B**

**DIFFERENTIAL EQUATION OF VIBRATION MODEL**

From static equilibrium, we have

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0.$$ (D1)

Since $\varepsilon_x = 0$, $\sigma_x = \nu(\sigma_r + \sigma_t)$,

$$\varepsilon_r = \frac{1}{E} \left( \sigma_r - \nu \sigma_t - \nu \sigma_x \right)$$

$$= \frac{1}{E} \left[ \sigma_r - \nu \sigma_t - \nu^2 (\sigma_r + \sigma_t) \right] = \frac{dx}{dr}$$ (D2)

$$\varepsilon_t = \frac{1}{E} \left( -\nu \sigma_r + \sigma_t - \nu \sigma_x \right)$$

$$= \frac{1}{E} \left[ -\nu \sigma_r + \sigma_t - \nu^2 (\sigma_r + \sigma_t) \right] = \frac{x}{r}.$$ (D3)

From (D2) and (D3), we have

$$(1 - \nu)\sigma_r - \nu \sigma_t = \frac{E}{(1 + \nu)} \frac{dx}{dr}$$ (D4)

$$- \nu \sigma_r + (1 - \nu)\sigma_t = \frac{E}{(1 + \nu)} \frac{x}{r}.$$ (D5)

Solving the above equations for $\sigma_r$ and $\sigma_t$,

$$\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu) \frac{dx}{dr} + \frac{x}{r} \right]$$

$$\sigma_t = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \nu \frac{dx}{dr} + (1 - \nu) \frac{x}{r} \right].$$

Therefore,

$$\frac{d\sigma_r}{dr} = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \frac{d^2x}{dr^2} + \frac{E\nu}{r(1 + \nu)(1 - 2\nu)} \frac{dx}{dr}$$

$$- \frac{E\nu}{r^2(1 + \nu)(1 - 2\nu)} x.$$ (D6)

$$\frac{\sigma_r - \sigma_t}{r} = \frac{E}{r(1 + \nu)} \frac{dx}{dr} - \frac{E}{r^2(1 + \nu)} x.$$ (D7)

Combining (D6) and (D7) in (D1),

$$E(1 - \nu) \frac{d^2x}{dr^2} + E(\nu + 1 - 2\nu) \frac{dx}{dr} (1 + \nu)(1 - 2\nu) - \frac{E(1 - \nu)}{r^2(1 + \nu)(1 - 2\nu)} x = 0.$$ (D8)

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**REFERENCES**


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