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Three-dimensional strain localization of water-saturated clay and numerical simulation using an elasto-viscoplastic model

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Since strain localization is a precursor of failure, it is an important subject to address in the field of geomechanics. Strain localization has been analysed for geomaterials by several researchers. Many of the studies, however, treated the problems brought about by strain localization as two-dimensional problems, although the phenomena are generally three-dimensional. In the present study, undrained triaxial compression tests using rectangular specimens and their numerical simulation are conducted in order to investigate the strain localization behaviour of geomaterials under three-dimensional conditions. In the experiments, both normally consolidated and over-consolidated clay samples are tested with different strain rates. Using the distribution of shear strain obtained by an image analysis of digital photographs taken during deformation, the effects of the strain rates, the dilation, and the over-consolidation on strain localization are studied in detail. The analysis method used in the numerical simulation is a coupled fluid-structure finite element method. The method is based on the finite deformation theory, in which an elasto-viscoplastic model for water-saturated clay, which can consider structural changes, is adopted. The results of the simulation include not only the distribution of shear strain on the surfaces of the specimens, but also the distributions of strain, stress, and pore water pressure inside the specimens. Through a comparison of the experimental results and the simulation results, the mechanisms of strain localization are studied under three-dimensional conditions.

1. Introduction

It is a well-known fact that the strain localization of geomaterials brings about important problems like slope failure in which deformation occurs in a narrow zone. The problem of strain localization in geomaterials such as soil and...
rock has been studied within the context of experimental, theoretical, and numerical approaches over the last three decades [1–4]. It has been theoretically found that the onset conditions for strain localization can be captured by a bifurcation analysis [5–7].

Experimental studies on strain localization have been done with some visualization methods for local strain, e.g. the calculation of the displacements of the lattice point printed on a membrane [8, 9], an image analysis of photographs taken during the tests [10–12] and X-rays or X-ray CTs [13–17]. As for numerical studies, many researchers have conducted numerical simulations of strain localization using particular constitutive models in the context of quasi-static strain localization and dynamic strain localization [18–21]. However, it is mainly strain localization problems under plane strain conditions that have been discussed.

In the natural ground, strain localization phenomena such as landslides involve three-dimensional shear bands. In the case of triaxial compression tests using cylindrical soil specimens, we can observe specific three-dimensional shear bands. In this way, the strain localization of geomaterials should be treated as a three-dimensional problem. Various researchers have studied strain localization under triaxial conditions [22, 23] and Asaoka et al. [24] experimentally and numerically investigated the three-dimensional deformation behaviour of rectangular clay specimens. To the author’s knowledge, however, experimental and numerical studies on strain localization under three-dimensional conditions are lacking.

In the present paper, we study strain localization problems under three-dimensional conditions through experiments and their numerical simulation. Triaxial compression tests using rectangular clay specimens and a detailed observation of the shear banding process by an image analysis were conducted, in the same way as by Kodaka et al. [25], in order to investigate the three-dimensional strain localization behaviour of clay. It is easy to observe strain localization with rectangular specimens, since the transverse section of these specimens have a lower symmetry than cylindrical specimens. An image analysis of the digital photographs showing two surfaces of the rectangular specimens during shear deformation provided the distributions of shear strain. By using these strain distributions, the generation of three-dimensional shear bands in the specimens can be very well observed.

A series of triaxial compression tests for the rectangular clay specimens was numerically simulated by the finite element method using an elasto-viscoplastic constitutive equation. The analysis method was a soil–water coupled three-dimensional finite element method based on the finite deformation theory. Biot’s [26] two-phase mixture theory was adopted to give the governing equations for the soil–water coupling problem. An updated Lagrangian method with the objective Jaumann rate of Cauchy stress was used for the weak form of the rate type of equilibrium equations. The constitutive model used for simulating clay has previously been presented in Kimoto et al. [27]. This model can reproduce the behaviour of both normally consolidated (NC) clay and overconsolidated (OC) clay, and can address the material instability induced by structural changes in the clay.
The simulation results yield much information on the response characteristics inside the specimens, e.g. the distributions of strain, the stress levels, and the pore water pressure. This information is difficult to obtain from the present experiments, although it is possible to detect certain internal specimen characteristics by X-ray computer tomography [14]. All of the test cases with different strain rates were simulated. Then, by comparing the results of the experiments and the simulation, we discuss the mechanisms of strain localization.

2. Undrained triaxial compression tests for clay using rectangular specimens

2.1. Clay samples

The clay used in the experiments is Fukakusa clay which is Pleistocene marine clay produced in the south-eastern part of the Kyoto Basin. The liquid limit, \( w_L = 62\% \), the plasticity index \( I_p = 33 \), and the density of soil solid \( \rho_s = 2.69 \text{ g cm}^{-3} \). Reconstituted clay samples were prepared by remoulding them in slurry and then pre-consolidating them. The specimens were consolidated one-dimensionally at a pre-consolidation pressure of 98 kPa. The pre-consolidated specimens were covered with paraffin. The scale of the transverse section was \( 4 \times 4 \text{ cm} \) and the height was 8 cm (see figure 1).

2.2. Testing programme

The test cases are listed in table 1. All the specimens used in the present study were saturated by the double vacuum method and were subjected to a back pressure of 200 kPa. The NC clay specimens were isotropically consolidated to 200 kPa. The OC clay specimens were isotropically consolidated to 300 kPa, and then partially unloaded from 300 to 50 kPa. Therefore, the overconsolidation ratio (OCR) was 6. After the consolidation or the unloading procedure, axial pressure was applied under undrained conditions by an axial loading device with an axial strain or displacement control system. The three axial strain rates monotonically applied in the tests were 1, 0.1, and 0.01%/min. The tests were stopped at an axial strain of 20%.

![Figure 1. Specimen size.](image)
2.3. Image analysis

We drew 2 mm square meshes on the rubber membranes covering the specimens. A digital camera was used to take photographs of two surfaces of the specimens during the tests. Figure 2 shows a sample of the digital photographs taken through the triaxial cell and a schematic figure of the photography, respectively. We took pictures from the front and the side angles. Since the pictures are expanded due to the refraction of the lucid cylindrical cell made of an acrylic tube, the distortion has been corrected such that the enlargement factor was measured as 1.21348 in the present analysis. After correcting the distortion due to the acrylic cell and the water inside the cell, we digitized the nodal coordinates of the meshes. Using the coordinates at the initial state, i.e. before the undrained loading, and those at each axial strain level, the nodal displacements were calculated. Adopting the B matrix for a four-node isoparametric finite element provided the strain of each square (see figure 3).

\[
\{\varepsilon\} = [B]\{u\}
\]
in which

\[ \{\varepsilon\}^T = \{\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy}\}, \quad \{u\}^T = \{u_1^x, u_1^y, u_2^x, u_2^y, u_3^x, u_3^y, u_4^x, u_4^y\} \] (2)

\[ [B] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\
0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x}
\end{bmatrix} \] (3)

where \(\varepsilon_{ij}\) is the strain tensor \((i,j = x,y)\), \(u\) is the displacement, subscripts \(x\) and \(y\) denote horizontal and vertical directions, the superscripted numbers \(i\) \((i = 1, 4)\) indicate the nodal numbers of a four-node isoparametric elements, and \(N_i\) \((i = 1, 4)\) is a shape function. The deviatoric strain is given as

\[ \{\varepsilon\}^T = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\} = \left\{\frac{\varepsilon_{xx} - \varepsilon_y}{2}, \frac{\varepsilon_{yy} - \varepsilon_y}{2}, \varepsilon_{xy}\right\} \] (4)

where \(\varepsilon_y(=\varepsilon_{xx} + \varepsilon_{yy})\) represents the sum of the normal strain components under plane-strain conditions. Finally, we obtain the second invariant of deviatoric strain, \(\gamma\), as follows:

\[ \gamma = \sqrt{\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2\varepsilon_{xy}^2} \] (5)

The contours of \(\gamma\) were drawn for each axial strain, and then the strain localization during deformation was analysed. In the following, ‘shear strain’ indicates the second invariant of deviatoric strain \(\gamma\).
3. Three-dimensional coupled fluid-structure finite element analysis method

We formulated a three-dimensional finite element method, based on Biot’s two-phase mixture theory and the finite deformation theory [28, 29], to simulate the three-dimensional strain localization tests for the rectangular shaped clay specimens. For simplified and practical formulations, both the grain particles and the fluid are assumed to be incompressible.

The strain localization phenomenon is a geometrically nonlinear problem since the deformation in shear bands is large. In addition, the constitutive equation for clay used in this study is nonlinear and is defined in an incremental form. In order to deal with such a large nonlinear deformation problem, using an incremental constitutive model, an updated Lagrangian method is used. Since the reference configuration is updated at each iterative step, it is necessary to use an objective stress rate. Hence, in this study, the objective Jaumann rate of Cauchy stress is adopted.

As for the element type, a twenty-node quadrilateral isoparametric element with a reduced Gaussian eight-point integration is used in order to eliminate shear locking as well as to reduce the appearance of a spurious hourglass mode. The pore water pressure is defined by an eight-node quadrilateral isoparametric element (see figure 4).

We used direct notation for the vectors and the tensors. A dot denotes a contraction of the inner indices, e.g. \( a_i b_i \equiv a \cdot b \) so that \( A_{ij} B_{ij} \equiv A : B \).

3.1. Definition of the effective stress of the fluid–solid mixture theory

Based on Terzaghi’s concept of effective stress, the total stress tensor and the time rate of stress are given as

\[
\mathbf{T} = \mathbf{T}^f + u_w \mathbf{I} \tag{6}
\]

\[
\mathbf{T} = \mathbf{T}^f + \dot{u}_w \mathbf{I} \tag{7}
\]
in which $T$ denotes the total Cauchy stress tensor, $T'$ denotes the effective Cauchy stress tensor, $u_w$ denotes the pore water pressure, $I$ is the second-order identity tensor, and the superimposed dots indicate time differentiation.

### 3.2. Equilibrium equations

When we consider an arbitrary domain, $V$, with boundary $\partial V$, the conservation of linear momentum for the whole fluid–solid mixture in the current configuration is given by the following equation:

$$
\frac{D}{Dt} \int_V \rho \mathbf{v} \, dv = \int_{\partial V} \mathbf{t} \, ds + \int_V \rho \mathbf{b} \, dv
$$

(8)

in which $D/Dt$ is the material time derivative, $\rho$ is the mass density, $\mathbf{v}$ is the velocity vector, $\mathbf{t}$ is the surface traction vector, and $\mathbf{b}$ is the body force vector.

In this study, since we deal with quasistatic problems, the acceleration and the body force can be assumed to be zero. Consequently, this assumption provides the equilibrium equation resulting from equation (8):

$$
\int_{\partial V} \mathbf{t} \, ds = 0
$$

(9)

Thus, the rate type of equilibrium equation is expressed as follows:

$$
\frac{D}{Dt} \int_{\partial V} \mathbf{t} \, ds = 0.
$$

(10)

The material time derivative of the equilibrium equations is given as

$$
\int_V \text{div} \mathbf{S} \, dv = 0
$$

(11)

in which $\mathbf{S}$ is the total nominal stress rate defined as

$$
\mathbf{S} = \mathbf{S}_0 + \mathbf{u}_w I + \text{tr} \mathbf{L} u_w I - u_w \mathbf{I} \mathbf{L}^T.
$$

(12)

The derivation of equation (12) is given by Yatomi et al. [30].

The effective nominal stress rate tensor $\mathbf{S}_e$ is given by the following equation:

$$
\mathbf{S}_e = \mathbf{S}_e' + \mathbf{u}_w I + \text{tr} \mathbf{L} u_w I - u_w \mathbf{I} \mathbf{L}^T.
$$

(13)

To obtain the relation between $\mathbf{S}$ and $\mathbf{S}_e$, we substitute the definition of the effective Cauchy stress and the effective Cauchy stress rate, equations (6) and (7), respectively, into equation (12), namely,

$$
\mathbf{S}_e = \mathbf{S}_e' + \mathbf{u}_w I + \text{tr} \mathbf{L} u_w I - u_w \mathbf{I} \mathbf{L}^T.
$$

(14)

By letting $U = \text{tr} \mathbf{L} u_w I - u_w \mathbf{I} \mathbf{L}^T$, equation (14) becomes

$$
\mathbf{S}_e = \mathbf{S}_e' + \mathbf{u}_w I + U
$$

(15)
When we consider closed domain, \( V \), the weak form of the rate type of equilibrium equation is given as follows:

\[
\int_V \text{div} \, \mathbf{S}_t \cdot \delta \mathbf{v} \, dv = 0
\]  \hspace{1cm} (16)

in which \( \delta \mathbf{v} \) is the virtual velocity vector.

The boundary of the domain, \( V \), is composed of a displacement boundary and a traction boundary. A displacement boundary is denoted by \( \partial V_u \) if the displacement is prescribed; \( \partial V_t \) denotes a traction boundary if the traction is prescribed.

\[
v = \bar{\mathbf{v}} \quad \text{on} \quad \partial V_u
\]  \hspace{1cm} (17)

\[
\mathbf{S}_t \cdot \mathbf{n} = \bar{\mathbf{s}}_t \quad \text{on} \quad \partial V_t
\]  \hspace{1cm} (18)

in which \( \mathbf{v} \) is the velocity vector, \( \mathbf{n} \) indicates the unit normal to the body, \( \mathbf{S}_t \) is the nominal traction rate vector (see Yatomi et al. [30]), and the specified values are designated by a superposed bar.

By taking the Gauss theorem and the compatibility condition, i.e. \( \text{grad}(\delta \mathbf{v}) = \delta \mathbf{L} \), equation (16) can be written as

\[
\int_{\partial V} (\mathbf{S}_t \, \delta \mathbf{v}) \cdot \mathbf{n} \, ds - \int_V \mathbf{S}_t : \delta \mathbf{L} \, dv = 0
\]  \hspace{1cm} (19)

Substituting equations (13) and (14) into the second term of equation (19) and transform of the first term by equation (18) yields

\[
\int_V \mathbf{T}' : \delta \mathbf{D} \, dv + \int_V (\mathbf{T}' \text{tr} \mathbf{D}) : \delta \mathbf{L} \, dv - \int_V (\mathbf{T}' \mathbf{L}^T) : \delta \mathbf{L} \, dv + \int_V \dot{\mathbf{u}}_w \, \text{tr} \delta \mathbf{D} \, dv
\]

\[
+ \int_V \mathbf{U} : \delta \mathbf{L} \, dv - \int_{\partial V_t} \mathbf{s}_t \cdot \delta \mathbf{v} \, ds = 0
\]  \hspace{1cm} (20)

in which \( \mathbf{D} \) is the stretching tensor and the following relations are used:

\[
\text{tr} \delta \mathbf{L} = \text{tr} \delta \mathbf{D}
\]  \hspace{1cm} (21)

\[
\mathbf{T}' : \delta \mathbf{L} = \frac{1}{2} \mathbf{T}' : (\delta \mathbf{L} + \delta \mathbf{L}^T) = \mathbf{T}' : \delta \mathbf{D}
\]  \hspace{1cm} (22)

Equation (22) is obtained from the symmetry of the effective Cauchy stress.

An updated Lagrangian method, based on the finite deformation theory, is used in this formulation. Thus, the Jaumann rate of effective Cauchy stress tensor, \( \mathbf{T}' \), is employed as an objective stress rate, which leads to

\[
\mathbf{T}' = \mathbf{T}' - WT' + T'W
\]  \hspace{1cm} (23)

where \( \mathbf{W} \) is the spin tensor.

The elasto-viscoplastic constitutive model can be written as

\[
\hat{\mathbf{T}}' = \mathbf{C} : \mathbf{D} - \mathbf{C} : \mathbf{D}^{vp}
\]  \hspace{1cm} (24)

where \( \mathbf{D} \) is the total stretching tensor, \( \mathbf{D}^{vp} \) is the viscoplastic stretching tensor and \( \mathbf{C} \) is the elastic stiffness tensor of the fourth order.

### 3.3. Continuity equation

For describing the motion of pore water, a Biot’s type of two-phase mixture theory [26] is used in the analysis with a \( \mathbf{v}^f \) (velocity) – \( u_w \) (pore pressure) formulation.
The Darcy law and the conservation of mass for the mixture give the continuity equation as

\[
\frac{k}{\gamma_w} \nabla^2 u_w + \text{tr } D = 0
\]  

(25)

where \( k \) is the coefficient of permeability, \( \gamma_w \) is the unit weight of the pore water, and \( D \) is the stretching tensor.

Considering the test function, \( \hat{u}_w \), we can obtain the weak form of the continuity equation as:

\[
\frac{k}{\gamma_w} \int_v \nabla^2 u_w \hat{u}_w \, dv + \int_v \text{tr } D \hat{u}_w \, dv = 0.
\]

(26)

\( \partial V_p \), the boundary at which the pore pressure is specified, and \( \partial V_v \), the boundary at which the flow of water is specified, are given by:

\[
u_w = \bar{u}_w \quad \text{on} \quad \partial V_p
\]

(27)

\[
\frac{k}{\gamma_w} \nabla u_w = \bar{v}^f \quad \text{on} \quad \partial V_v
\]

(28)

in which the specified values are designated by a superposed bar and \( \bar{v}^f \) is the velocity of the pore water through the boundary surface.

By applying the Gauss theorem and substituting equation (28) into equation (26), the following equation can be obtained:

\[
- \frac{k}{\gamma_w} \int_v \nabla \hat{u}_w \cdot \nabla u_w \, dv + \int_v \hat{u}_w \text{tr } D \, dv + \int_{\partial V_v} \hat{u}_w \bar{v}^f \cdot n \, ds = 0
\]

(29)

in which \( n \) is the unit normal vector of the body.

3.4. Discretization of the weak forms of the equilibrium equation and the continuity equation

For the discretization of the weak form of the equilibrium equation, the following relations are defined:

\[
v = [N]\{v^*\}, \quad \delta v = [N]\{\delta v^*\}
\]

(30)

in which \( \{v\} \) is the velocity vector in an element, \( \{v^*\} \) is the nodal velocity vector, and \( [N] \) is a shape function of the eight-node quadrilateral element.

It should be noted that the square brackets \( [ \] \) denote the matrix and the curly brackets \( \{ \} \) denote the vector, respectively. Thus:

\[
D = \{D\} = [B]\{v^*\}, \quad \delta D = \{\delta D\} = [B]\{\delta v^*\}
\]

(31)

in which \( [B] \) is the matrix which transforms the nodal velocity vector to the vector form of stretching tensor \( \{D\} \);

\[
L = \{L\} = [B_M]\{v^*\}, \quad \delta L = \{\delta L\} = [B_M]\{\delta v^*\}
\]

(32)
where \([BM]\) is the matrix which transforms the nodal velocity vector into the vector form of velocity gradient vector \([L]\);

\[
\text{tr} \, D = (B_v)^T \{v^*\}, \quad \text{tr} \, \delta D = (B_v)^T \{\delta v^*\}
\]

(33)

where \([B_v]\) is the vector which transforms the nodal velocity into the trace of \(D\); and

\[
\dot{u}_w = \{N_h\}[\dot{u}_w^*]
\]

(34)

in which \([\dot{u}_w]\) represents the pore pressure rate, \([\dot{u}_w^*]\) represents the nodal pore pressure rate vector, and \([N_h]\) represents the four-node quadrilateral element shape function.

In the present analysis, the tangent modulus method for rate dependent material [31] is adopted in order to evaluate viscoplastic stretching tensor, \(D^{vp}\) (see Oka et al. [20, 32] and appendix 1). This method can be classified as a forward gradient method and has been well adopted in the viscoplastic analysis. It is known that the method is stable and yields accurate results for time steps larger than those for the Euler explicit method [31]. This method is identical to the tangent modulus of the semi-implicit backward Euler scheme on the first iteration [33]. In the formulation, the trapezoidal rule is adopted to evaluate the value of the viscoplastic strain, in which the value of trapezoidal parameter \(\theta\) [31] is taken as 0.5 in the present analysis. \(\theta = 1\) corresponds to the usual tangent modulus.

From appendix 1, the relation between the rate of effective stress and the stretching tensor can be written in matrix form, as shown in the following equation:

\[
\{\dot{T}'\} = [C][D] - \{Q\}
\]

(35)

where \([C]\) is the elasto-viscoplastic tangential stiffness matrix and \([Q]\) can be called the relaxation stress vector.

Substituting equation (23) into equation (35) yields:

\[
\{\dot{T}'\} = [C][D] - \{Q\} + \{W^*\}
\]

(36)

where \([W^*]\) is the column vector related to the spin tensor and the components are computed via \(W_{ij}^* = W_{jk} T_{kj}' - T_{jk}' W_{kj}\).

By all the matrix and vector relations obtained previously, and based on the theory of virtual displacement, we have obtained the following relation considering the arbitrariness of the unconstrained virtual nodal velocities:

\[
[K]\{v^*\} - \int_D [B]^T \{Q\} \, dv + \int_D [B]^T \{W^*\} \, dv + [K_L]\{v^*\} + [K_v][\dot{u}_w^*] = \{\dot{F}\}
\]

(37)
in which

\[
[K] = \int_D [B]^T [C] [B] \, dv
\]

\[
\]

\[
[K_i] = \int_D \{B_v\} \{N_h\} \, dv
\]

\[
\{\vec{F}\} = \int_{\partial D_i} [N]^T [\hat{s}] \, ds.
\]

In the above equations, the matrix forms of \((-T'L)^T\) and \(U\) are given by

\[
-T'L^T = [D'][B_M][v^*]
\]

\[
U \equiv u_w I(\text{tr} \, L) - u_{wL}^T = [U][B_M][v^*]
\]

where \([B_M]\) is the nodal velocity \(-L^T\) matrix and \([D']\) is the matrix that expresses the matrix form of \((-T'L)^T\) by \(L^T\). \([\hat{s}]\) is the traction vector at the boundary. \([D']\) and \([U]\) are described in appendix 2.

The relation between nodal velocity vector \(\{v^*\}\) and nodal displacement increment vector \(\{\Delta u^*\}\) can be obtained by using Euler’s approximation as

\[
\{v^*\} \approx \frac{\{\Delta u^*\}}{\Delta t}
\]

Similarly, the pore water pressure can be obtained as

\[
\{\hat{u}_w^*\} \approx \frac{u_w^*_{t+\Delta t} - u_w^*_{t}}{\Delta t}
\]

Substituting equations (44) and (45) into equation (37), the weak form of the equilibrium equations is obtained, that is,

\[
[[K] + [K_L]] [\Delta u^*] + [K_i] \{u^*_w\}_{t+\Delta t} = \Delta t \{\vec{F}\} + [K_i] \{u^*_w\}_t + \Delta t \int_D [B]^T \{Q\} \, dv - \Delta t \int_D [B]^T \{W^*\} \, dv
\]

For the discretization of the continuity equation, equation (29), the following vectors and matrices are used:

\[
u_w = \{N_h\} \{u_w^*\}, \quad \hat{u}_w = \{N_h\} \{\hat{u}_w^*\}
\]

where \(\{u_w^*\}\) is the nodal pore pressure vector and \(\{N_h\}\) is the shape function of the four-node quadrilateral element.

The spatial gradient of the pore water pressure \(u_w\) is discretized by

\[
\nabla u_w = \nabla \{N_h\} \{u_w^*\} = [B_h] \{u_w^*\}
\]
in which \([B_h]\) is the matrix which transforms the nodal pore pressure into the spatial derivative of the pore pressure.

Substituting equations (47), (48), and (28) into equation (29), the following relation is given considering the arbitrariness of the unconstrained virtual nodal pore water pressures:

\[
-\frac{k}{\gamma_w} \int_D [B_h]^T [B_h] \, dv(u_w^*) + \int_D [N_h]^T [B_v]^T \, dv(v^*) + \int_{\partial D_v} [N_h]^T \left\{ \vec{v}^f \right\}^T \{n\} \, ds = 0
\] (49)

Using equation (44), the discretization of the continuity equation is obtained as follows:

\[
[K_v]^T \{\Delta u^*\} - \Delta t [K_h] \{u_w^*\}_{t+\Delta t} = \Delta t [V]
\] (50)

where

\[
[K_h] = \frac{k}{\gamma_w} \int_D [B_h]^T [B_h] \, dv
\] (51)

\[
\] (52)

\[
[V] = -\int_{\partial D_v} [N_h]^T \left\{ \vec{v}^f \right\}^T \{n\} \, ds
\] (53)

Finally, the weak form of the rate-type of equilibrium equation, equation (46), and the weak form of the continuity equation, equation (50), are given in the following matrix form:

\[
\begin{bmatrix}
[K] + [K_L] + [K_v]^T \\
[K_v] & -\Delta t [K_h]
\end{bmatrix}
\begin{bmatrix}
\{\Delta u^*\} \\
\{u_w^*\}_{t+\Delta t}
\end{bmatrix}
= 
\begin{bmatrix}
\Delta t \{\vec{F}\} + [K_v]^T \{u_w^*\}_{t} + \Delta t \int_v [B]^T \{Q\} \, dv - \Delta t \int_v [B]^T \{W^*\} \, dv \\
\Delta t \{V\}
\end{bmatrix}
\] (54)

In the formulation of equation (54), the incremental governing equations are evaluated at \(t + \Delta t\). As shown in equation (50), the pore water pressure is implicitly evaluated at \(t + \Delta t\).

4. Elasto-viscoplastic constitutive model considering structural changes

For the constitutive equation of clay, an elasto-viscoplastic constitutive model considering structural changes was used. The model is given as an extension of the Adachi–Oka elasto-viscoplastic model [34, 35] and is based on an overstress type of viscoplasticity [36]. A characterization of the material instability is proposed considering the microstructural changes in the geomaterials [27, 37]. The structural changes are phenomenologically described by the shrinking of both the overconsolidation boundary surface and the static yield surface with the evolution of viscoplastic strain. In the following, the constitutive model will be introduced.
4.1. Overconsolidation boundary surface

The overconsolidation boundary surface defines the NC region and the OC region and controls the shape of the plastic potential function (see figure 5).

\[ f_b = \tilde{\eta} + M_m^* \ln \frac{\sigma'_m}{\sigma_{mb}} = 0 \]  

where \( \tilde{\eta} \) is an invariant form of the relative stress ratio \( \eta_{ij}^* = S_{ij}/\sigma_m' \) defined as:

\[ \tilde{\eta}^* = \left\{ (\eta_{ij} - \eta_{ij(0)}) (\eta_{ij} - \eta_{ij(0)}) \right\}^{1/2} \]  

\[ \eta_{ij} = \frac{S_{ij}}{\sigma_m}, \quad S_{ij} = \sigma'_{ij} - \sigma'_m \delta_{ij}, \quad \sigma'_m = \frac{1}{3} \sigma'_{kk} \]  

in which \( \sigma'_{ij} \) is Terzaghi’s effective stress tensor, \( \eta_{ij}^* \) is the stress ratio tensor, \( S_{ij} \) is the deviatoric stress tensor, \( \sigma'_m \) is the mean effective stress, \( \delta_{ij} \) is Kronecker’s delta, and subscript \( (0) \) indicates the initial state, corresponding to the end of the pre-consolidation. \( M_m^* \) is the value of \( \sqrt{\eta_{ij}^* \eta_{ij}^*} \) at the maximum compression and \( \sigma_{mb} \) is the parameter which controls the size of the surface.

4.2. Structural changes

Originally, the hardening rule for the overconsolidation boundary surface was defined with respect to viscoplastic volumetric strain \( \varepsilon_{kk}^{vp} \) as:

\[ \sigma'_{mb} = \sigma_{mbi} \exp \left( \frac{1 + \varepsilon^{vp}}{\lambda - \kappa} \varepsilon_{kk} \right) \]  

where \( \lambda \) is the compression index, \( \kappa \) is the swelling index, \( e \) is the void ratio, and \( \sigma_{mbi} \) is the initial value of \( \sigma'_{mb} \), which is defined as the isotropic consolidation yield stress.

In order to describe the degradation of the material caused by structural changes, strain softening is described through the following expression:

\[ \sigma'_{ma} = \sigma'_{maf} + (\sigma'_{mai} - \sigma'_{maf}) \exp(-\beta z) \]
in which \( \sigma'_{ma} \) and \( \sigma'_{maf} \) are the initial and the final values of \( \sigma'_{ma} \), \( \beta \) is a parameter which denotes the degradation rate of \( \sigma'_{ma} \), and \( z \) is the second invariant of accumulated viscoplastic strain rate, \( \dot{\varepsilon}^{vp}_{ij} \), given by:

\[
z = \int_0^t \dot{z} \, dt, \quad \dot{z} = \left( \frac{\dot{\varepsilon}^{vp}_{ij}}{\varepsilon_{ij}^{vp}} \right)^{1/2}.
\]  

(60)

Using \( \sigma'_{ma} \) in equation (58) instead of \( \sigma'_{mbi} \), \( \sigma'_{mb} \) is defined by the hardening rule with respect to the viscoplastic volumetric strain and by the softening rule with respect to structural changes as

\[
\sigma'_{mb} = \sigma'_{ma} \exp \left( \frac{1 + e}{\lambda - \kappa} \varepsilon^{vp}_{kk} \right)
\]  

(61)

Since \( z \) is equal to 0 at the initial state, the relation of \( \sigma'_{ma} = \sigma'_{ma} = \sigma'_{mbi} \) is derived. Hence, two independent parameters, \( \beta \) and \( \sigma'_{maf} \), are introduced to describe the soil structures. The ratio of \( \sigma'_{maf} \) to \( \sigma'_{ma} \), namely, \( n = \sigma'_{maf} / \sigma'_{ma} \), provides the degree for a possible collapse of the structure at the initial state; \( n \) satisfies the condition of \( 0 < n \leq 1 \), in which \( n = 1 \) indicates the soil which loses its structure perfectly. In addition, the ratio of \( \sigma'_{ma} \) to \( \sigma'_{ma} \) is defined by equation (59) as

\[
N_a = \frac{\sigma'_{ma}}{\sigma'_{m}^{*}} = n + (1 - n) \exp(-\beta z).
\]  

(62)

When viscoplastic straining does not occur, i.e. \( z = 0 \), the value of \( N_a \) is equal to 1. On the other hand, when the value of \( z \) becomes large enough, \( N_a \) approaches \( n \) and the softening converges due to structural changes.

4.3. **Viscoplastic potential function**

Viscoplastic potential function \( f_p \) is given as

\[
f_p = \eta^* + \tilde{M}^* \ln \frac{\sigma'_{m}}{\sigma'_{mp}} = 0
\]  

(63)

where \( \sigma'_{mp} \) is the value of \( \sigma'_{m} \) at \( \eta^* = 0 \) and \( \tilde{M}^* \) is constant in the NC region, while the value of \( \tilde{M}^* \) in the OC region depends on the current stress and \( \sigma'_{mc} \) as:

\[
\tilde{M}^* = \begin{cases} 
M^*_m : & f_b \geq 0 \\
-\frac{\sqrt{n_f n_j}}{\ln(\sigma'_{m}/\sigma'_{mc})} : & f_b < 0
\end{cases}
\]  

(64)

where \( \sigma'_{mc} \) denotes the mean effective stress at the intersection of the overconsolidation boundary surface and the \( \sigma'_{m} \) axis as

\[
\sigma'_{mc} = \sigma'_{mb} \exp \frac{\sqrt{n_f(0)n_j(0)}}{M^*_m}
\]  

(65)

In the case of isotropic consolidation, \( \sigma'_{mc} \) equals \( \sigma'_{mb} \).
4.4. Static yield function

Adachi and Oka [35] assumed a Cam-clay type of static yield function to describe the mechanical behaviour of clay at its static equilibrium state, namely,

\[
f_y = \eta^* + \tilde{M} \ln \frac{\sigma'_{my}}{\sigma_{my}^{(s)}} = 0 \tag{66}
\]

\[
\sigma_{my}^{(s)} = \sigma_{myi}^{(s)} \exp \left( \frac{1 + e}{\lambda - \kappa} \varepsilon_{kk}^p \right) \tag{67}
\]

where \(\sigma_{my}^{(s)}\) is the static-hardening parameter, \(\sigma_{myi}^{(s)}\) is the initial value of \(\sigma_{my}^{(s)}\), and \(f_y = 0\) represents the static state in which no viscoplastic deformation occurs.

In a similar manner to overconsolidation boundary surface, \(f_b\), strain softening is defined in order to express the effect of a structural collapse through changes in \(\sigma_{my}^{(s)}\) with the viscoplastic strain. Inserting equation (62) into equation (67) yields:

\[
\sigma_{my}^{(s)} = \frac{\sigma'_{ma}}{\sigma_{myi}^{(s)}} \sigma_{myi}^{(s)} \exp \left( \frac{1 + e}{\lambda - \kappa} \varepsilon_{kk}^p \right) = \{n + (1 - n) \exp(-\beta z)\} \sigma_{myi}^{(s)} \exp \left( \frac{1 + e}{\lambda - \kappa} \varepsilon_{kk}^p \right) \tag{68}
\]

The decrease in \(\sigma_{my}^{(s)}\), defined by equation (68), leads to the shrinkage of the static yield function according to the structural collapse.

The overconsolidation boundary, \(f_b\), static yield function, \(f_y\), and potential function, \(f_p\), at \(\eta^{(s)}_{ij}(0) = 0\) for the NC region and the OC region are demonstrated in the \(\sigma'_m - \sqrt{S_{ij} S_{ij}}\) space in figure 5a and b, respectively. \(\sigma'_{mb}\) and \(\sigma_{my}^{(s)}\) become small as the accumulation of viscoplastic strain increases. Since the increments in viscoplastic strain for the overstress type of model are determined by the difference between the current stress state and the static state, the shrinkage of the static yield surface due to the degradation of the soil structure increases the viscoplastic strain increments. The static yield function, as well as the potential surface, is transformed smoothly between the OC and the NC regions, so that calculations can be conducted without interruption.

4.5. Viscoplastic flow rule

The viscoplastic stretching tensor, \(D_{ij}^p\), based on an overstress type of viscoplasticity theory, is given by:

\[
D_{ij}^p = \gamma \Phi_1(f_y) \frac{\partial f_p}{\partial \sigma_{ij}^p} \tag{69}
\]

\[
\{\Phi_1(f_y)\} = \begin{cases} 
\Phi_1(f_y); & f_y > 0 \\
0; & f_y \leq 0
\end{cases} \tag{70}
\]

where \(\Phi_1\) indicates strain rate sensitivity and \(\gamma\) is a viscous parameter. Based on the experimental data of the strain-rate constant triaxial tests, the material function,
\[ \gamma \Phi_1, \text{ has the following form} \ [27, 35]: \]

\[ \gamma \Phi_1(f_y) = C_0 \sigma'_m \exp \left[ m' \left( \bar{n}^* + \bar{M}^* \ln \frac{\sigma'_m}{\sigma_{mb}/\sigma_{b}} \right) \right] \]

\[ = C \sigma'_m \exp \left[ m' \left( \bar{n}^* + \bar{M}^* \ln \frac{\sigma'_m}{\sigma_{mb}/\sigma_{b}} \right) \right] \quad (71) \]

\[ C = C_0 \exp \left( m' \bar{M}^* \ln \frac{\sigma'_{ma}}{\sigma'_{mb}/\sigma'_{b}} \right) \quad (72) \]

in which equations (61), (62) and (68) are used, and \( m' \) and \( C \) are viscoplastic parameters. \( m' \) can be determined by triaxial compression tests with different strain rates, and an explanation of \( m' \) will be given in section 5. Parameter \( C \) is related to the initial viscoplastic volumetric stretching, i.e. the initial volumetric strain rate when the mean effective stress is taken to the initial value of \( \sigma'_{mb} \) in the NC region, in other words,

\[ C = \frac{D_{kk(0)}^{vp}}{M^*} \quad (73) \]

4.6. Elastic stretching

An additive decomposition of total stretching tensor, \( D_{ij} \), into elastic stretching, \( D_{ij}^e \), and viscoplastic stretching, \( D_{ij}^{vp} \), is assumed such that

\[ D_{ij} = D_{ij}^e + D_{ij}^{vp} \quad (74) \]

Elastic stretching \( D_{ij}^e \) is given by a generalized Hooke type of law, namely,

\[ D_{ij}^e = \frac{1}{2G} S_{ij} + \frac{\kappa}{3(1+e)\sigma'_m} \delta_{ij} \quad (75) \]

where \( S_{ij} \) is the deviatoric stress rate tensor, \( \sigma'_m \) is the mean effective stress, \( G \) is the elastic shear modulus, \( e \) is the void ratio, \( \kappa \) is the swelling index, and the superimposed dot denotes the time differentiation.

5. Numerical simulation of triaxial tests for rectangular specimens

5.1. Determination of the material parameters

The material parameters required by the constitutive model introduced in the last section are listed in table 2. We determined \( \lambda \) to be 0.191 and \( \kappa \) to be 0.043 using the isotropic consolidation and the swelling test results for Fukakusa clay. For the initial void ratio, \( \varepsilon_0 \), we used the average taken from each test, i.e. 1.10 for NC clay and 1.11 for OC clay. The initial elastic shear modulus, \( G_0 \), is determined by the initial slope of the undrained triaxial compression tests, giving \( G_0 = \Delta q / (3 \Delta \varepsilon_{11}) \), in which \( \Delta q \) is the increment in deviator stress and \( \Delta \varepsilon_{11} \) is the increment in axial strain. In this study, \( \Delta \varepsilon_{11} \) was determined to be 0.1\%.
Since the viscoelastic properties of clay are not considered in the present model, the strain rate dependent values of $G_0$ are used for simplicity.

The compressive yield stress, $\sigma_{mbi}$, is assumed to be equal to the pre-consolidation stress. Therefore, that of NC clay is 200 kPa and that of OC clay is 300 kPa. The stress ratio at maximum compression, $M_m^*$, is determined from the stress ratio at the residual state in the undrained triaxial compression tests. The viscoplastic parameters $m'$ and $C$ can be determined from undrained triaxial tests with different strain rates. Adachi and Oka [35] noted that the viscoplastic parameter $m'$ is estimated from the slope of the relation between the stress ratio and the logarithm of the strain rate. In order to estimate viscoplastic parameter, $m'$, we applied the test results using rectangular specimens. We have determined $m'$ directly from the triaxial tests on the rectangular specimens with different strain rates.

Figure 6 shows the relations between the applied strain rates and stress ratio $q/\sigma_m'$ in which $q$ is the deviator stress and $\sigma_m'$ is the mean effective stress. For NC clay, we plot the stress ratios at $\sigma_m' = 0.7\sigma_{mb0}'$ and estimate $m'$ to be 24.3. For OC clay, $m'$ is estimated to be 20.5 from the stress ratios at $\sigma_m' = 2.0\sigma_{mb0}'$. After $m'$ is fixed, the viscoplastic parameter, $C$, is determined from the stress–strain curve.

The structural parameter, $\sigma_{maf}'$, can be obtained from the deviator stress at the residual stress state, while $\beta$, which dominates the decreasing rate of deviator stress, is determined by the residual stress state.

The coefficients of permeability for the NC clay and the OC clay were obtained by falling head permeability tests in the triaxial cell. The head difference applied by the air pressure was 1000 cm for NC clay and 250 cm for OC clay. To prevent leakage along the rubber membrane, silicone grease was put in between the specimen and the membrane.
Figure 7a and b show the simulation results of undrained triaxial compression tests for NC clay and OC clay, respectively. These are obtained by a numerical integration of the constitutive equation, using the fourth-order Runge-Kutta method [39], under triaxial stress conditions \[\sigma_{ij}^0 = 0(i \neq j)\]. It is seen that the simulation results represent the strain rate sensitivity, which is consistent with the results of previous tests on Fukakusa clay [35, 40].

The effective stress paths in the case of OC clay (figure 7b) show the increase of mean effective stress. The increase of the mean effective stress is observed as a result of positive dilatancy. Since under undrained conditions the total volumetric strain increment, \(\Delta \varepsilon_{kk} = (\Delta \sigma_{mm}/K) + \Delta \varepsilon_{vp}^{pp} = 0\), the mean effective stress increment is calculated as \(\Delta \sigma_m' = -K \Delta \varepsilon_{kk}^{vp}\) in the case that the elastic strain increment is given by \(\Delta \varepsilon_{kk}^{el} = \Delta \sigma_m'/K\) in which \(\Delta \varepsilon_{kk}^{el}\) is the elastic volumetric strain increment and \(\Delta \varepsilon_{vp}^{pp}\) is the viscoplastic volumetric strain increment, \(\Delta \sigma_m'\) is the increment of mean effective stress and \(K\) is the elastic volumetric modulus. The positive dilatancy results in the negative viscoplastic volumetric strain increment, i.e. \(\Delta \varepsilon_{kk}^{vp} < 0\), when compression is positive. Hence the increment of the mean effective stress is positive, namely, the mean effective stress increases during the undrained compression test. On the contrary, the mean effective stress decreases due to the negative dilatancy; the volumetric viscoplastic strain is positive during the shearing for the NC clay. In the sense mentioned above, figure 7 confirms the dilatancy characteristics for both the OC clay and the NC clay.

The peak stress and the axial strain at the peak stress in the stress–strain curves for both cases are different between the test results and the simulation. This indicates that further research is needed for the constitutive model and the determination of the material parameters, and it reminds us that the behaviour during triaxial tests is not homogeneous.
5.2. Boundary conditions

Figure 8 shows the boundary conditions, which are set up according to the same boundary conditions as those of the undrained triaxial compression tests with displacement control. Although all the boundaries are assumed to be impermeable, the internal redistribution of pore water is allowed. The deformation for each test is applied through a constant strain rate, taken at selected values of 0.01, 0.1, and 1%/min. The time increment is determined by the increment of average strain $\Delta \varepsilon = 0.05\%$. The top and bottom surfaces of the specimen are constrained through prescribing the nodes at these surfaces equal to zero. This constraint to a certain extent mimics the frictional resistance between the test specimen and the top and the bottom platens of the experimental set-up, which triggers strain localization under external loading.

5.3. Comparison between the experimental results and the simulation results

The undrained triaxial compression tests for NC clay and OC clay with different axial strain rates have been simulated. Figure 9a–e show a comparison between
the experimental results and the simulation results for NC clay and OC clay, respectively.

5.3.1. Stress–strain relations and effective stress paths. Figures 9a and 10a illustrate the stress–strain relations for both the simulation and the experiments, in which deviator stress, \( q = \sigma'_{11} - \sigma'_{22} \) where \( \sigma'_{11} \) is the axial component of the stress tensor and \( \sigma'_{22}(= \sigma'_{33}) \) is the lateral component of the stress tensor, mean effective stress, \( p'(= (\sigma'_{11} + 2\sigma'_{33})/3) \), and average axial strain, \( \varepsilon_{11} \), are plotted. In the simulation, the deviator stress, \( q \), and pore water pressure, \( u_w \), are obtained using the average of those nodal values of the top surface.

We can see that the stress–strain relations for both the experiments and the simulation are dependent on the strain rate and the dilatancy characteristics. In the experimental results, we can also observe gradual strain-softening behaviour for both NC clay and OC clay. On the other hand, in the case of the simulation, it is seen that the stress–strain relations for NC clay consistently show strain-hardening behaviour, while those for OC clay show strain-softening behaviour just after the peak stress around an axial strain of 2%, and then they show a gradual hardening.

It is seen in figures 9b and 10b that the mean effective stress decreases for NC clay, and the mean effective stress increases for the OC clay. These changes in the mean effective stress are consistent with the dilatancy as mentioned for the stress paths of figure 7.

We can see from the stress paths for OC clay that the mean effective stress decreases from the initial value of 50 kPa in the early stage of loading, e.g. the mean effective stress for case OC-1 at an axial strain of 0.5% is 31 kPa. Figure 11 shows the stress path and the distribution of mean effective stress, \( \sigma_m' \), and pore water pressure, \( u_w \), for case OC-1 at an axial strain of 0.5%, in which \( \sigma_m' = (\sigma'_{11} + \sigma'_{22} + \sigma'_{33})/3 \) and \( u_w \), calculated at each Gaussian integration point, are plotted. It can be seen in this figure, however, that the mean effective stress of almost
the entire specimen increases from the initial value of 50 kPa, and the pore water pressure at the top edge is much higher than that at the other parts of the specimen. The mean effective stress displayed in the stress path is calculated by the pore water pressure of the top edge in order to make a comparison under exactly the same conditions as those in the experiment. The reason for the decrease in mean effective stress in the stress paths is that the higher pore water pressure at the top edge leads to a lower mean effective stress.

We can say that the stress–strain curves and the effective stress paths for NC clay are well reproduced by the presented analysis method. However, there are some
differences between the experiments and the simulation for both cases. The other possible method for obtaining a good agreement between them is to perform an inverse finite element analysis of triaxial tests to accurately determine the material parameters starting with the parameters from the average stress–strain curves, since the behaviour of the soil during the triaxial tests is not homogeneous. In addition, we need to further develop the constitutive model to improve the difference between the experiments and the simulated results.

### 5.3.2. Distribution of shear strain $\gamma$

Figures 9c and 10c show the distributions of shear strain, $\gamma$, for the experiments, and figures 9e and 10e show those for the
simulation. Figures 9d and 10d are the pictures taken after the tests were completed at an axial strain of 20%, and Figures 9f and 10f indicate the deformed meshes at an axial strain of 20%. In these figures, ‘s’ and ‘f’ indicate the ‘side surface’ and the ‘front surface’ of the specimens, respectively (see figure 1).

In order to obtain the distributions of shear strain, \( \gamma' \), on the surfaces of the specimens for the simulation results, we used the same method as the image analysis for the experiments introduced in section 3. Figure 12 displays the procedure for obtaining shear strain, \( \gamma' \). After completing the calculations, we picked up the coordinates of the nodes on the \( xz \)-surface and the \( yz \)-surface. Consequently, we used eight-node isoparametric elements to calculate strain values \( \gamma' \) of the \( xz \)-surface and the \( yz \)-surface, as has been explained in section 2.3.

We can see from the experimental results in figures 9c and 10c that strain localization starts at an axial strain of 8% and shear bands are clearly observed at an axial strain of 12%. The shear bands develop from the edges of the top and the bottom of the specimens since the friction force generated between the specimen and the top cap or the pedestal acts as a trigger of strain localization. As the axial strain becomes large, clear shear bands appear on the side surface and the front surface. The thickness of each shear band increases with deformation. As shown in figures 9e and 10e, the simulation results can well reproduce the strain localization behaviour observed in the experiments. Although homogeneous deformations can be seen until an axial strain of 4% is reached, the strain starts to localize at an axial strain of 8%, and then either two, or four shear bands appear at an axial strain of 12% and develop with an increased thickness on both surfaces. The development of the shear bands is well simulated for both the NC clay and the OC clay.

5.3.3. Strain localization pattern. In the experiments and the simulation, we can see a deformation pattern in which either two, or four shear bands develop from the edges of the top and the bottom of the specimens. This mode is due to the material instability induced by the frictional boundary conditions between the clay specimens and the top caps and the pedestals. The two shear bands intercrossing each other are just like an “X”, thus, we call it the “X” mode. Figure 13 shows the schematics
of the estimated process generating the “X” mode. The four shear bands generated finally develop two clear, thick shear bands. In the case of the finite element analysis, we depict the distribution of the second invariant of accumulated viscoplastic deviatoric strain $\gamma_p^{\text{vis}}$ for all cases, namely,

$$
\gamma_p^{\text{vis}} = \int dy_p^{\text{vis}}, \quad dy_p^{\text{vis}} = \sqrt{\text{dev}_{ij}^{\text{vis}} \text{dev}_{ij}^{\text{vis}}} 
$$

(76)

where $\text{dev}_{ij}^{\text{vis}}$ is the deviatory part of the viscoplastic strain increment tensor. $\gamma_p^{\text{vis}}$ is calculated at each Gaussian integration point. In the distributions, by disregarding the smaller values of $\gamma_p^{\text{vis}}$, we can see the localized strain, i.e. the three-dimensional shear bands.

5.3.4. Three-dimensional shear bands. The estimated shear bands for case NC-2 (NC clay, 0.1% min), tested and simulated, are depicted in figure 14. In the contour for the simulation results, the local value of the accumulation of the second invariant of viscoplastic deviatoric strain rate, $\gamma_p^\text{vis}$, is illustrated if the local value of $\gamma_p^{\text{vis}}$ is more than 0.32. Since the shear bands observed on the front surface are clearer
than those on the side surface, two shear planes are estimated. However, it is found that the “X” mode appears just on the surface and that higher levels of shear strain, \( \gamma \), are distributed in the central part of the specimen.

5.4. Effects of the strain rate

5.4.1. Strain rate sensitivity. It is well known that clay exhibits strain rate sensitivity. Oka et al. [40] reported the strain rate sensitivity of Fukakusa clay
under undrained triaxial compression conditions. As shown in figure 9a–e, the strain rate sensitivity of the stress–strain relations and the stress paths for both NC clay and OC clay can be observed. It can be seen that the strain rate sensitivity observed in the experimental results is smaller than that found in the simulation results.

5.4.2. Strain localization pattern. The shear–strain distributions and the inclination angles of the shear bands in both the simulation and the experiments, with different strain rates at an axial strain of 20%, are shown in figure 15. The numerical simulation reproduces the experimental results very well with respect not only to the “X” mode, but also to the effects of the strain rates on the strain localization pattern. Shear bands develop from the top and the bottom edges in the case of higher strain rates, while those with lower strain rates develop under the top and the bottom edges. Due to this tendency, the angles of the shear bands become smaller as the strain rate decreases. Oka et al. [41] investigated the effect

![Figure 15. Comparison of between simulation results and experimental results of the distributions of shear strain, $\gamma$, the inclination angles of the shear bands, and the maximum thickness of the shear bands, at the selected strain rates 1, 0.1 and 0.01%/min (axial strain: 20%, front surface); (a) shear bands develop from the top edge, (b) and (c) shear bands develop beneath the top edge.](image-url)
of permeability on the angles of shear bands under plane strain conditions. If we could assume that the effect of the strain rate is equal to the effect of permeability, i.e. a higher strain rate sensitivity corresponds to a lower permeability level, and vice versa, we would then obtain a different trend from the one in the present study, i.e. angles with higher strain rates are smaller than those with lower strain rates. Note that the constitutive equation used in Oka et al. [41] is slightly different from the one used in the present study. In addition, the maximum thickness of shear bands with lower strain rates is larger than that of shear bands with higher strain rates. These types of behavior are seen more clearly in the case of OC clay.

5.5. Effects of dilatancy

As for the simulation results, it is seen in the stress–strain curves for the simulation shown in figures 9a and 10a that the axial strain of the peak stress for OC clay is smaller than that for NC clay. In addition, it is seen in figure 16 that an accumulation of $\gamma^p$ at an axial strain of 5% for OC clay is larger than that for NC clay. Additionally, the thickness of the shear bands in OC clay is thinner than that in NC clay. As a consequence, the NC clay shows a more ductile stress–strain response than the OC clay. Furthermore, Oka et al. [41] and Kimoto et al. [27] have concluded that OC clay is more unstable than NC clay by an instability analysis and a strain localization analysis under plane strain conditions. These experimental observations are in good agreement with the test results reported in Suiker et al. [42]. In this work, coarse-grained (gravel) and fine-grained (sand) granular specimens were initially subjected to cyclic loading with specific amplitude, and subsequently failed under static loading. It was shown that cyclic loading with higher loading amplitude provided the specimen with a higher consolidation level (or compaction level). Furthermore, a higher level of consolidation induced a narrower failure zone and a less ductile stress–strain response during static failure of the specimens. It was also reported that a specimen with a higher level of consolidation reaches the peak strength at lower deformation, and, under subsequent deformation, shows a steeper softening branch and thus a higher level of instability. These results are all consistent with the present test results on clay. The experimental observations mentioned above are in good agreement with the experimental findings on the static failure of ballast and sub-ballast material (sand and gravel) with different cyclic strain histories [42].

![Figure 16. Distributions of $\gamma^p$ for cases NC-2 and OC-2 (axial strain: 5%).](image-url)
Suiker et al. [42] showed that the specimen failed more locally when the stress level during the cyclic loading process is higher and the material is more dilatant than the virgin material with less dilatant characteristics.

Figure 17 shows the distributions of mean effective stress, \( \sigma'_{\text{eff}} \), and viscoplastic volumetric strain, \( \varepsilon^{vp}_{\text{vol}} \), for cases NC-2 and OC-2. The mean effective stress, \( \sigma'_{\text{eff}}=(\sigma'_{11}+\sigma'_{22}+\sigma'_{33})/3 \), and viscoplastic volumetric strain, \( \varepsilon^{vp}_{\text{vol}}=(\varepsilon^{vp}_{11}+\varepsilon^{vp}_{22}+\varepsilon^{vp}_{33}) \), are calculated at each Gaussian integration point. In the case of NC clay, the mean effective stress decreases from its initial value of 200 kPa and positive viscoplastic volumetric straining (compression is positive) occurs. In particular, this type of behaviour is typically seen inside the shear bands. On the other hand, the mean effective stress of OC clay increases from its initial value of 50 kPa due to its positive dilatancy characteristics.

In this way, although the typical dilatancy characteristics can be seen in the simulation, different types of behaviour can also be observed locally. Near the top and the bottom edges, the mean effective stress of NC clay increases with positive viscoplastic volumetric strain, and negative viscoplastic volumetric strain can be seen where the shear bands cross each other. On the other hand, the mean effective stress of OC clay decreases around the point where the shear bands cross, and viscoplastic compressive volumetric strain accumulates near both edges in spite of its positive dilatancy characteristics. They are closely related to the pore fluid flow. Figure 18 shows the distribution of pore water pressure for cases NC-2 and OC-2. We can see in this figure that the distribution of pore water pressure is rather homogeneous.
This suggests that the migration of pore fluid leads to the pore water flow in the specimen, irrespective of the dilatancy characteristics. In addition, the increase of the mean effective stress near the plates and the volume change shown in figure 19a and b imply that pore fluid flows toward the central part of the specimen. On the other hand, at the centre of each surface where the shear bands intercross, volumetric expansion occurs in the case of NC clay. This might mean that the volumetric compression due to the negative dilatancy characteristics of NC clay is balanced out, namely, offset by the inflow of pore water as shown in figure 19a and b.

5.6. Local volume changes

Figure 20 depicts the distributions of total volumetric strain, \( \varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \), in the case of a simulation with a strain rate of 0.1%/min. It is seen in all cases that positive volumetric strain (compression) occurs at the centre part of the specimens and near the top and the bottom edges. On the other hand, negative volumetric strain (expansion) occurs near the side surfaces. The distributions of volume expansion and compression are very similar for the NC clay and the
OC clay. Comparing the distributions of shear strain, $\gamma$, and the volumetric strain, we can say that the development of volumetric strain does not occur along the shear bands. We have tried to investigate the local volume changes in the experiments through a measurement of the water contents of the samples divided into some parts after the tests. However, the results were not accurate enough to compare them with the simulation results. Detailed observations of the local density by X-ray CT scanners and a comparison with the simulation results are desired.

### 5.7. Mesh size dependence

Finally, we mention the mesh size dependence of the present analysis. It is well known that the finite element analysis with a strain softening model has inherent mesh size dependence and that many researchers have studied regularization methods. One method is to introduce the rate dependency of the material through the use of an elasto-viscoplastic model [43]. The second method is to introduce higher order strain gradients into the constitutive model [44]. The third approach is to incorporate a Darcy type of soil–fluid interaction which can alleviate the problem of instability by delaying the onset of material instability [45]. Oka et al. [41] discussed the mesh size dependence using a coupled fluid-structure finite element method with an elasto-viscoplastic model under plane strain conditions, as shown in figure 21. In the investigation, they found that the analysis method has no significant mesh size dependence because the well-posedness is maintained.
Figure 21. Mesh size dependence of the analysis method used by Oka et al. [41].
(a) Deformation and distribution of $\gamma^p$ for OC clay (0.1%/min, $1.54 \times 10^{-8}$ m/s).
(b) Average stress–strain relations of OC clay with different mesh size.
In the present study, it can be said that the mesh size dependency is small, since we used a similar elasto-viscoplasticity theory and the same solid–fluid mixture theory as those used in Oka et al. [41].

6. Conclusion

In the present study, undrained triaxial compression tests with rectangular specimens for NC clay and OC clay were firstly carried out. Then, a numerical simulation of the tests was conducted using a coupled fluid-structure finite element analysis based on the finite deformation theory. As for the constitutive equation, an elasto-viscoplastic model for water-saturated clay, considering structural changes, was used. The material parameters used in the simulation were determined by undrained triaxial compression tests and isotropic consolidation tests on Fukakusa clay.

The stress–strain relations and the effective stress paths represented the strain rate sensitivity and the dilatancy characteristics well in the sense of increases and decreases in the mean effective stress. However, there were some differences between the experiments and the simulation. In order to improve the simulated results, the further development of the constitutive model and an accurate determination of the input parameters by an inverse analysis technique, through the finite element analysis of the specimen, are needed. It was seen in the distributions of shear strain, $\gamma$, that the present simulation can efficiently reproduce the development of shear bands in cases of both NC clay and OC clay. In addition, the effects of the strain rates on the strain localization patterns were reproduced by the simulation.

From the simulation, we obtained the three-dimensional distributions of strain, stress, and pore water pressure which are difficult to determine from the present experiments. Through the three-dimensional distributions, we discussed the difference in strain localization behaviour inside the specimens between NC clay and OC clay, e.g. the dilatancy characteristics. We confirmed that the simulation methods in this study can be very effectively applied to the strain localization behaviour of clay under three-dimensional conditions. More detailed experimental information about shear localization is needed to further improve the accuracy of the present model. In addition, analyses of case studies which are related to strain localization, such as slope failure and excavations, are necessary in order to practically verify the proposed method.

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Tangent modulus method [31]

In the present model, the viscoplastic tensor, $D_{ij}^{vp}$, is given as shown in equation (69), namely,

$$D_{ij}^{vp} = \gamma \Phi \frac{\partial f_p}{\partial \sigma'_{ij}}$$  \hspace{1cm} (A1)

in which $\langle \Phi_i(f_j) \rangle$ is expressed as $\Phi$ for simplicity. We define the increment of viscoplastic strain, $\Delta D_{ij}^{vp}$, by

$$\Delta D_{ij}^{vp} = \gamma [(1 - \theta)\Phi_n + \theta \Phi_{n+1}] \frac{\partial f_p}{\partial \sigma'_{ij}} \Delta t$$  \hspace{1cm} (A2)

in which subscripts $(n)$ and $(n + 1)$ denote step $n$ and step $n + 1$, respectively.

Using Taylor series expansion for $\Phi_{n+1}$ provides

$$\Phi = (1 - \theta)\Phi_n + \theta \Phi_{n+1}$$  \hspace{1cm} (A3)

Since the Jaumann rate of effective Cauchy stress tensor, $\dot{T}_{kl}$, is given by a fourth-order elastic tensor, $C_{ijkl}^e$, as

$$\dot{T}_{ij} = C_{ijmn}^e (D_{mn} - D_{mn}^{vp})$$  \hspace{1cm} (A4)

by Taylor series expansion, we have

$$\Phi_{n+1} = \Phi_n + \left( \frac{\partial \Phi}{\partial \sigma'^{ij}_{kl}} \dot{T}_{kl} + \frac{\partial \Phi}{\partial \varphi} \dot{\varphi} \right) \Delta t$$  \hspace{1cm} (A5)

Using equation (A4), equation (A5) yields

$$\Phi_{n+1} = \Phi_n + \left[ \frac{\partial \Phi}{\partial \sigma'^{ij}_{kl}} C_{klnn}^e (D_{mn} - D_{mn}^{vp}) + \frac{\partial \Phi}{\partial \varphi} \dot{\varphi} \right] \Delta t$$

$$= \Phi_n + \left[ \frac{\partial \Phi}{\partial \sigma'^{ij}_{kl}} C_{klnn}^e (D_{mn} - \gamma \Phi \frac{\partial f_p}{\partial \sigma'_{mn}}) + \frac{\partial \Phi}{\partial \varphi} \gamma \Phi \frac{\partial f_p}{\partial \sigma'_{kk}} \right] \Delta t$$  \hspace{1cm} (A6)

in which equation (A1) is adopted. From equations (A6) and (A3) we obtain

$$\Phi = \frac{1}{1 + \xi} \left\{ \Phi_n + \frac{\partial \Phi}{\partial \sigma'^{ij}_{pq}} C_{pqrs}^e D_{rs}(\theta \Delta t) \right\}$$  \hspace{1cm} (A7)

where

$$\xi = \gamma \left( \frac{\partial \Phi}{\partial \sigma'^{ij}_{kl}} C_{klnn}^e - \frac{\partial \Phi}{\partial \varphi} \frac{\partial f_p}{\partial \sigma'_{kk}} \right) (\theta \Delta t).$$  \hspace{1cm} (A8)

Finally, combining equations (A1), (A4), (A7) and (A8), we obtain a constitutive relation using a tangential modulus method, shown in equation (35), i.e.

$$\dot{T}_{ij} = C_{ijkl}^{tan} D_{kl} - Q_{ij}$$  \hspace{1cm} (A9)
\[ (\dot{T}') = [C][D] - \{Q \} \text{ in the matrix notation} \]

\[
C_{ijkl}^{\text{tan}} = C_{ijkl}^e - C_{ijmn}^e \frac{\gamma}{1 + \xi} \Phi_{mp} \frac{\partial f_p}{\partial \sigma_{mn}} (\theta \Delta t) \frac{\partial \Phi}{\partial \sigma_{pq}} C_{pqkl}^e
\]

\[
Q_{ij} = C_{ijmn}^e \frac{\gamma}{1 + \xi} \Phi_{mp} \frac{\partial f_p}{\partial \sigma_{mn}}
\]

The above derivation is similar to that performed by Oka et al. [32].

The second term in the right-hand side of equation (A9), \(Q_{ij}\), can be interpreted as a relaxation stress per unit time because \(Q_{ij}\) corresponds to the decrease in stress per unit time when the strain is kept constant, i.e. when the total stretching is zero. From a point of time discretization procedure, \(Q_{ij}\) can be called pseudo-stress which is added to the external force vector in a one-step solution of the incremental weak form [33].

**Appendix 2**

The matrices \([D']\) and \([U]\) in equation (39) are given by:

\[
-T'_{ik} L_{jk} = [D'][L]
\]

\[
[D'] = \begin{bmatrix}
-T'_{11} & 0 & 0 & -T'_{21} & 0 & 0 & 0 & 0 & -T'_{13} \\
0 & -T'_{22} & 0 & 0 & -T'_{23} & 0 & -T'_{21} & 0 & 0 \\
0 & 0 & -T'_{33} & 0 & 0 & -T'_{31} & 0 & -T'_{32} & 0 \\
0 & -T'_{12} & 0 & 0 & -T'_{13} & 0 & -T'_{11} & 0 & 0 \\
0 & 0 & -T'_{23} & 0 & 0 & -T'_{21} & 0 & -T'_{22} & 0 \\
-T'_{31} & 0 & 0 & -T'_{32} & 0 & 0 & 0 & -T'_{33} & 0 \\
-T'_{21} & 0 & 0 & -T'_{22} & 0 & 0 & 0 & 0 & -T'_{23} \\
0 & -T'_{32} & 0 & 0 & -T'_{33} & 0 & -T'_{31} & 0 & 0 \\
0 & 0 & -T'_{13} & 0 & 0 & -T'_{11} & 0 & -T'_{12} & 0
\end{bmatrix}
\]

\[
\{L\} = \begin{bmatrix}
L_{11} \\
L_{22} \\
L_{33} \\
L_{12} \\
L_{23} \\
L_{31} \\
L_{21} \\
L_{32} \\
L_{13}
\end{bmatrix}
\]
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$$u_w \delta_{ij} (\text{tr} L) - u_w \delta_{ik} L_{jk} = [U] \{L\}$$

$$[U] = \begin{bmatrix}
0 & u_w & u_w \\
u_w & 0 & u_w \\
u_w & u_w & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
-u_w & 0 & 0 \\
0 & -u_w & 0 \\
0 & 0 & -u_w
\end{bmatrix}$$

$$\{L\} = \begin{bmatrix}
L_{11} \\
L_{22} \\
L_{33} \\
L_{12} \\
L_{23} \\
L_{31} \\
L_{21} \\
L_{32} \\
L_{13}
\end{bmatrix}$$

References

Three-dimensional strain localization of water-saturated clay
