Fuzzy Robust and Non-fragile Minimax Control of a Trailer-Truck Model

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Abstract—This paper deals with a fuzzy robust and non-fragile minimax control problem of a trailer-truck model. By introducing parametric uncertainty terms into the T-S model for trailer-truck systems, the fuzzy model approaches to the original system more exactly. Existence conditions are derived for the robust and non-fragile minimax control in the sense of Lyapunov asymptotic stability and formulated in the form of Linear Matrix Inequalities (LMIs). The convex optimization algorithm is used to get the minimal upper bound of the performance cost and parameter of the optimal minimax controller. Then the closed-loop system will be asymptotically stable under the condition of the worst disturbance and uncertainty. Finally, an illustrative example is used to demonstrate the better robust and non-fragile performance of the controller design.

I. INTRODUCTION

Most of plants in the industry have severe nonlinearity, which makes the research on nonlinear control systems more practical significance. The past few years have witnessed rapidly growing interest in fuzzy logic control of nonlinear trailer-truck systems. A nonlinear dynamic model of the trailer-truck is represented by a Takagi-Sugeno fuzzy model. Though there have been many successful applications, the heuristics-based approach to fuzzy control lacks the formal and systematic design methodology which guarantees the basic requirement such as stability and acceptable performance. Recently, based on the Takagi–Sugeno (T–S) fuzzy model, researchers have proposed many methods of designing controllers for trailer-truck systems. The main feature of a T–S fuzzy model is to characterize the local dynamics of each fuzzy rule by a linear system model. A complex system is decomposed into several subsystems (fuzzy rules) and a simple control law is employed for each subsystem to emulate the human control strategy[1]. As a common belief, the control technique based on the T–S fuzzy model is conceptually simple and effective for the control of complex systems with nonlinearity. Tanaka [2] studied fuzzy controller and observer design for backing control of a trailer-truck model. Leung et al. [3] studied robust stability for uncertain nonlinear systems on T–S fuzzy model. Tanaka [4] gave a stability condition for designing a fuzzy controller, which involved finding a common positive-definite matrix P for a derived fuzzy model.

With the rapid development of the computer science, the research on discrete system has gained extensive attention.

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Wang et al. [5]and Mahmoud [6] studied robust stability and $H_\infty$ control for uncertain discrete systems. Ma et al. [7] worked on the robust stability of nonlinear discrete systems by fuzzy control method, but the T–S model does not include parametric uncertainty terms. Stability and performance are the most important problems in analysis and synthesis of control systems. Recently the guaranteed cost control is given attention for nonlinear discrete systems on T-S fuzzy model. Shi et al. [8], and Wu et al. [9] studied the fuzzy guaranteed cost control for nonlinear discrete systems with uncertain and time-delay by a linear matrix inequality. However, for the more general guaranteed cost control problem, only the existence of the disturbance is considered, not the degree of the influence of that disturbance. The problem of the minimax control is a special kind of a guaranteed cost control. Under the worst uncertainty or disturbance, considering the bias of state, the resumption of control-energy, the disturbance and the uncertainty in the whole process, we discuss how to control the stability and the performance of the system so that it can gain the best synthetic result. Kogan et al. [10] had developed the sufficient condition of a minimax controller of the linear discrete normal system, but the optimal parameter of the minimax controller and nonlinear factor had not been considered. Yoon et al. [11] studied an optimal minimax control for a linear stochastic systems with uncertainty, but they did not consider the nonlinear factor of the system. So the research of the optimal minimax control problem for the nonlinear discrete system is very necessary. By our approach, we find that the obtained controller is simple and the method used to find required parameters only needs to solve Linear Matrix Inequalities (LMIs), which can easily be solved using the Matlab LMI toolbox.

Recently, it is shown that relatively small perturbations in controller parameters could even destabilize the closed-loop system [12]. Therefore, it is necessary that any controller should be able to tolerate some level of controller gain variations. This raises a new issue: how to design a controller for a given plant with uncertainty so that the controller is non-fragile with regard to its gain variations. More recently, there have been some studies to tackle the non-fragile controller design problem. Yang [13, 14] considered the problem of non-fragile $H_\infty$ control for linear systems under state feedback control gain perturbations. Two classes of perturbations were considered, namely, additive and multiplicative. The non-fragile guaranteed cost control controllers [15-16] are designed for the discrete time-delay systems and the discrete singular systems, respectively. Yang G H et al. [17]-[19] investigated the problem of robust non-
Fragile guaranteed cost control of discrete-time systems with parametric uncertainties.

This paper is organized as follows: Section 2 briefly introduces the construction of T-S fuzzy model for nonlinear systems, and the definition of minimax robust control of uncertain nonlinear systems is proposed. In Section 3 and Section 4, the existence conditions and the design of minimax robust and non-fragile controller with additive and multiplicative gain perturbations are presented, respectively, by the Lyapunov method and linear matrix inequality, which guarantees the asymptotic stability of the system. And the convex optimization algorithm is used to get the minimal upper bound of performance cost and the optimal parameter of minmax non-fragile controller. Simulation of a Trailer-truck system is given in section 5. Finally, Section 6 serves as the conclusion.

II. FUZZY MODELING OF A TRAILER-TRUCK

A nonlinear kinematics model formulated by Tokunaga and Ichihashi is utilized here [2].

$$x_0(k+1) = x_0(k) + \frac{v \cdot \bar{k}}{L} \tan(u(k)) + a(k)x_0(k) \quad (1)$$

$$x_1(k+1) = x_0(k) - x_2(k) + a(k)x_1(k) \quad (2)$$

$$x_2(k+1) = \frac{v \cdot \bar{k}}{L} \sin(x_1(k)) + x_2(k) + a(k)x_2(k) \quad (3)$$

$$x_3(k+1) = x_3(k) + v \cdot \bar{k} \cdot \cos(x_1(k)) \cdot \frac{2 \cdot x_2(k+1) + x_2(k)}{2} + a(k)x_3(k) \quad (4)$$

$$x_4(k+1) = x_4(k) + v \cdot \bar{k} \cdot \cos(x_1(k)) \cdot \frac{2 \cdot x_2(k+1) + x_2(k)}{2} + a(k)x_4(k) \quad (5)$$

where $x_0(k)$ is the angle of truck, $x_1(k)$ is the angle difference between truck and trailer, $x_2(k)$ is the angle of trailer, $x_3(k)$ is the vertical position of rear end of trailer, $x_4(k)$ is the horizontal position of rear end of trailer, $u(k)$ is the steering angle, $l$ is the length of the truck, $L$ is the length of the trailer, $\bar{k}$ is the sampling time, and $v$ is the constant speed of the backward movement, $a(k)$ is parametric uncertainty.

With respect to \(x_1(k), 90^\circ\) and \(-90^\circ\) correspond to the two jack-knife positions. The purpose of the control is to realize the backward movement of the trailer-truck along the straight line ($x_3 = 0$), i.e., $x_1(k) \to 0, x_2(k) \to 0, x_3(k) \to 0$, without forward movements.

The original model (1)-(5) is simplified as

$$x_1(k+1) = (1 - \frac{\bar{k}}{L})x_1(k) + \frac{\bar{k}}{L} u(k) + a(k)x_1(k) \quad (6)$$

$$x_2(k+1) = x_2(k) + \frac{\bar{k}}{L} x_1(k) + a(k)x_2(k) \quad (7)$$

$$x_3(k+1) = x_3(k) + v \cdot \bar{k} \cdot \sin(x_2(k) + v \cdot \bar{k} \cdot \frac{1}{2L} x_1(k)) + a(k)x_3(k) \quad (8)$$

This uncertain nonlinear system that can be described by the following T-S fuzzy model with parametric uncertainties:

$$R^i : \text{If } z_1(k) \text{ is } \bar{F}^i_1, \ z_2(k) \text{ is } \bar{F}^i_2, \ldots, \ z_n(k) \text{ is } \bar{F}^i_n, \text{ then}$$

$$\dot{x}(k) = (A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k)$$

$$y(k) = C_i x(k) \quad i = 1, \ldots, q. \quad (9)$$

where $\bar{F}^i_j$, $(j = 1, \cdots, n)$ is a fuzzy set, $z(k) = [z_1(k), \cdots, \ z_n(k)]^T$ are measurable variables, i.e., the premise variables. $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in R^n$ is the state vector, $u(k) \in R^m$ is the control input vector, and $y(t) \in R^l$ is the output vector, and $A_i \in R^{n \times n}, B_i \in R^{n \times m}$ are system matrix, input matrix, output matrix, respectively, $\Delta A_i$ and $\Delta B_i$ are constant matrices with appropriate dimensions, which represent parametric uncertainties in the plant model, and $q$ is the number of rules of the T-S fuzzy model.

The defuzzified output of this T-S fuzzy system (9) is represented as follows:

$$x(k+1) = \sum_{i=1}^{n} h_i(z(k))[A_i x(k) + B_i u(k)]$$

$$+ \sum_{i=1}^{q} h_i(z(k))\Delta A_i x(k) + \Delta B_i u(k)$$

$$y(k) = \sum_{i=1}^{q} h_i(z(k))C_i x(k) \quad (10)$$

where $h_i(z(k)) = \prod_{j=1}^{n} \bar{F}^i_j(z_j(k)), h_i(z(k)) = \frac{w_i(z(k))}{\sum_{i=1}^{n} w_i(z(k))}, \ i = 1, \cdots, q$, $h_i$ is the grade of membership of $z_j(k) \in \bar{F}^i_j$. Some basic properties of $w_i(z(k))$ are $w_i(k) \geq 0, \sum_{i=1}^{n} w_i(k) > 0, i = 1, 2, \cdots, q$. Obviously, $h_i(z(k)) \geq 0, \sum_{i=1}^{q} h_i(z(k)) = 1, i = 1, \cdots, q$.

We first give the following assumption.

**Assumption 1:** The parameter uncertainties considered here are norm-bounded, in the form

$$[\Delta A_i, \Delta B_i, \Delta T_{i1}, \Delta T_{i2}] = \Delta D_i F_i(k) [E_{i1}, E_{i2}, E_{i3}, E_{i4}]$$

where $D_i, E_{i1}, E_{i2}, E_{i3}$ and $E_{i4}$ are known real constant matrices with appropriate dimension, $F_i(k)$ is an unknown matrix function with Lebesgue-measurable elements, and $F_i^T(k)F_i(k) \leq I$, $I$ is the identity matrix of appropriate dimension.

The controller implemented is assumed to be

$$R^i : \text{If } z_1(k) \text{ is } \bar{F}^i_1, \ z_2(k) \text{ is } \bar{F}^i_2, \ldots, \ z_n(k) \text{ is } \bar{F}^i_n, \text{ then}$$

$$u(k) = (K_i + \Delta K_i)x(k), \ i = 1, \ldots, q. \quad (11)$$

where $K_i \in R^{m \times n}$ is the nominal controller gain, and $\Delta K_i$ represents the gain perturbations. In this paper, the following two classes of perturbations are considered:

(a) $\Delta K_i$ is of the additive form

$$\Delta K_i = \Delta T_{i1}, \ F_i^T F_i \leq I, \ i = 1, \cdots, q. \quad (11)$$

(b) $\Delta K_i$ is of the multiplicative form

$$\Delta K_i = \Delta T_{i2} K_i, \ F_i^T F_i \leq I, \ i = 1, \cdots, q. \quad (12)$$
Then global fuzzy controller is
\[ u(k) = \sum_{i=1}^{q} h_i(z(k))(u_{i0}(k) + \Delta K_i x(k)) \]  
(13)
where \( u_{i0}(k) = K_i x(k) \).

The cost function is defined as follows
\[ J(u, W) = \sum_{0}^{\infty} (x^T(k)Qx(k) + u_{i0}^T(k)Ru_{i0}(k) - \gamma^2 W^T W) \]  
(14)
where \( Q = Q^T > 0 \) and \( R > 0 \) are given weighting matrices, \( W \) is parametric uncertainty and perturbations, let \( x_0 = x(0) \).

**Assumption 2**: The initial value \( x_0 \) of the nonlinear system (10) is the zero mean random variable, satisfying
\[ E \{ x_0, x_0^T \} = I, \]  
where \( E(*) \) denotes the expectation operator.

We suppose that all the theorems of the paper satisfy Assumption 2.

**III. MINIMAX ROBUST CONTROL UNDER ADDITIVE GAIN PERUBTATIONS**

In this section, we consider the minimax control problem under additive gain perturbations of the form (11). By Assumption 1, the global fuzzy closed-loop system is formulated as following
\[ x(k+1) = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(z(k))h_j(z(k))[A_i x(k) + B_i u_{j0} + \bar{D}_i W] \]  
(15)
where
\[ \bar{D}_i = \begin{bmatrix} D_{i1} & D_{i2} & B_{i1} & D_{i4} \end{bmatrix}, \]  
\[ W = \begin{bmatrix} v_{i1}^T & v_{i2}^T & v_{i3}^T & v_{i4}^T \end{bmatrix}^T, \]  
\[ v_{i1} = F_i E_{i1} x(k), \quad v_{i2} = F_i E_{i2} u_{j0}, \quad v_{i3} = \Delta T_{ji} x(k), \quad v_{i4} = F_i E_{i3} \Delta T_{ji} x(k). \]

**Theorem 1**: Considering system (10) and performance cost (14), for a given constant \( \gamma > 0 \), if there exists common symmetric positive definite matrix \( X \) such that
\[ \begin{bmatrix} -\gamma^2 I & \bar{D}_i^T X & -X \\ X \bar{D}_i & -X \end{bmatrix} < 0 \]  
(16)
\[ \begin{bmatrix} -X & X A_i^T \\ A_i X & -(X - \gamma^{-2} \bar{D}_i \bar{D}_i^T + B_i R^{-1} B_i^T) \end{bmatrix} < 0 \]  
(17)
\[ \begin{bmatrix} -X + Q_1 & X A_i^T \\ * & X + Q_1 \\ * & * \end{bmatrix} < 0, \quad i = 1, \cdots, q \]  
(18)
then \( u_{i0}^* = -\sum_{i=1}^{q} \sum_{j=1}^{q} h_i h_j (R + B_i^T \tilde{P} B_i)^{-1} B_i^T \tilde{P} A_i x(k) \) is a minimax robust and non-fragile controller of system (10), by which the closed-loop system will be asymptotically stable, and the corresponding upper bound of performance is:
\[ J^* \leq x^T(0) P x(0) \]  
(19)
where
\[ \tilde{P} = \bar{P} + P \bar{D}_i (\gamma^2 I - \bar{D}_i \bar{D}_i^T + B_i R^{-1} B_i^T), \quad \gamma^{-2} \bar{D}_i \bar{D}_i^T - B_i R^{-1} B_i^T \]  
(20)
where matrix \( \bar{P} > 0 \), along the state trajectory of (15), we have
\[ \Delta V(k) = x^T(k+1) P x(k+1) - x^T(k) P x(k) \]  
(21)
\[ = (A_i x(k) + B_i u_{j0}(k))^T P (A_i x(k) + B_i u_{j0}(k)) - x^T(k) P x(k) \]  
(22)
\[ + 2(A_i x(k) + B_i u_{j0}(k))^T P \bar{D}_i W + u_{j0}^T R u_{j0} \]  
(23)
\[ + W^T (\bar{D}_i^T P \bar{D}_i - \gamma^{-2} I) W \]  
(24)

Then maximizing (22) about \( W \), then
\[ \psi^* = (\gamma^2 I - \bar{D}_i \bar{D}_i^T + B_i R^{-1} B_i^T)^{-1} \tilde{P} (A_i x(k) + B_i u_{j0}(k)) \]  
(25)
\[ + B_i u_{j0}(k) + u_{j0}^T R u_{j0} - x^T(k) P x(k) \]  
(26)

It is easy to know
\[ \frac{\partial^2 \psi^*(x)}{\partial W^2} = R + B_i^T \tilde{P} B_i \geq 0, \]  
so \( \psi^* \) in (25) makes the local check function maximum. Substitute (25) into (24), then
\[ \min_{u_{j0}} \psi^*(k) = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i h_j [x^T(k) A_i T (\tilde{P} - \bar{P} B_i (R + B_i^T \tilde{P} B_i) B_i^T \tilde{P}) A_i x(k) - x^T(k) P x(k)] \]  
(27)

And denote
\[ \min_{u_{j0}} \psi^*(k) = -x^T(k) Q x(k) \]  
(28)

By [9]
\[ \tilde{P} - \bar{P} B_i (R + B_i^T \tilde{P} B_i) B_i^T \tilde{P} = (\bar{P}^{-1} + B_i R^{-1} B_i^T)^{-1} \]  
(29)
\[ P + P \bar{D}_i(\gamma^2 I - \bar{D}_i^T P \bar{D}_i)^{-1} \bar{D}_i^T P = (P^{-1} - \gamma^{-2} \bar{D}_i \bar{D}_i^T)^{-1} \]

If
\[ A_i^T (\bar{P} - \bar{P} B_i (R + B_i^T \bar{P} B_i)^{-1} B_i^T \bar{P} ) A_i - P < 0 \]
then \( Q > 0 \), by Schur complement and Pro- and post multiplying both sides of the above inequality by diag\((P^{-1}, I)\), it follows that
\[
\begin{bmatrix}
-P^{-1} & P^{-1} A_i^T \\
A_i P^{-1} & -(P^{-1} - \gamma^{-2} \bar{D}_i \bar{D}_i^T + B_i R^{-1} B_i^T)
\end{bmatrix} < 0
\]

let \( X = P^{-1} \), the above inequality is equivalent to that inequality (17), and substituting (23) and (25) into (21), by
\[
\bar{P} P^{-1} \bar{P} = P \bar{D}_i(\gamma^2 I - \bar{D}_i^T P \bar{D}_i)^{-1} \bar{D}_i^T P \bar{D}_i(\gamma^2 I - \bar{D}_i^T \bar{P} \bar{D}_i)^{-1} \bar{D}_i^T P + 2 \bar{P} - P
\]
and inequality (17), it follows
\[
\Delta V(k) \leq \begin{bmatrix}
-P^{-1} + Q_1 & P^{-1} A_i^T & P^{-1} A_i^T \\
0 & 0 & 0 \\
0 & 0 & -Z
\end{bmatrix}
\]

where \( \Sigma = P^{-1} - \gamma^{-2} \bar{D}_i \bar{D}_i^T + B_i R^{-1} B_i^T, H = P^{-1} P^{-1}, Z = (\gamma^{-2} \bar{D}_i \bar{D}_i^T - B_i R^{-1} B_i^T)(\gamma^{-2} \bar{D}_i \bar{D}_i^T - B_i R^{-1} B_i^T) - P^{-1}(\gamma^{-2} \bar{D}_i \bar{D}_i^T - B_i R^{-1} B_i^T)(\gamma^{-2} \bar{D}_i \bar{D}_i^T - B_i R^{-1} B_i^T) - P^{-1} \]
\[
Q_1 = \text{diag}[0.1, 0.1, 0.1], \text{let } X = P^{-1}, \text{if inequality (18) holds, then } \Delta V < 0, \text{i.e. the closed-loop system (15) is asymptotically stable, and } V(x(\infty)) = 0. \text{ Now get the integral of (16) after some appropriate transposte, we obtain}
\]
\[
\min_{u \in W} \max_{u_0} J(u, W) = x(0)^T P x(0), \quad i = 1, \ldots, q.
\]

**Remark:** If above existence condition is satisfied, then there exists a minimax robust and non-fragile control law for system (10). From (19), we get to know that the upper bound of performance cost depends on the selection of the minimax control law. So in order to minimize the upper bound of performance of system, it is crucial that how to choose an appropriate minimax control law. By constructing and resolving the convex optimization problem, the optimal parameter of feedback controller and the minimum upper bound of performance cost can be obtained.

**Theorem 2:** Considering the system (10) and performance cost (14), if for the convex optimization problem
\[
\min_{X, \bar{X}} \text{Trace}(\bar{X})
\]
\[ s.t. (16) - (18), \text{and } \begin{bmatrix}
\bar{X} & I \\
I & X
\end{bmatrix} > 0, i = 1, \ldots, q \]
there exist a solution \( X, \bar{X} \), then (25) is the optimal minimax control law of system(10), and the corresponding upper bound of performance is
\[
J = \text{Trace}(\bar{X}) = J^*.
\]

**Proof:** According to the proof of the Theorem 1, Theorem 3 is obvious. By Schur complement (30) is equivalent to \( X^{-1} < \bar{X}, \text{i.e. } P < \bar{X} \). Note that the initial state of system can hardly be accurate measured in fact, so applying Assumption 2, by considering the expected value of performance cost, we obtain
\[
\bar{J} = E \{ J \} = E \{ x_0^T P x_0 \} \leq \text{Trace}(\bar{X}) = J^*.
\]

**IV. MINIMAX ROBUST CONTROL UNDER MULTIPLICATIVE GAIN PERTURBATIONS**

In this section, we consider the minimax control problem under multiplicative gain perturbations of the form (12). We first give the following lemma.

**Lemma 1[17]:** Given matrices \( H, F, \text{and } E \) where \( F^T F \leq I \), and if \( P > 0 \), for arbitrary \( \varepsilon > 0 \), such that \( P^{-1} - \varepsilon H H^T > 0 \), then \( (A + H F E)^T P (A + H F E) \leq A^T (P^{-1} - \varepsilon H H^T)^{-1} A + \varepsilon^{-1} E^T E \).

By Assumption 1, the global fuzzy closed-loop system is formulated as follows
\[
x(k + 1) = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(z(k)) h_j(z(k)) [A_i x(k) + (B_i + B_i \Delta T_{2j}) u_{j0} + \bar{D}_i W]
\]

where \( \bar{D}_i = \begin{bmatrix} D_i & D_i & D_i \end{bmatrix}, W = \begin{bmatrix} v_{i1}^T & v_{i2}^T & v_{i3}^T \end{bmatrix}^T, v_{i1} = F_i E_{i1} x(k), v_{i2} = F_i E_{i2} u_{j0}(k), v_{i3} = F_i E_{i3} \Delta T_{2j} u_{j0}(k) \).

**Theorem 3:** Considering system (10) and performance cost (14), for a given constant \( \gamma > 0 \) and \( \varepsilon > 0 \), if there exists common symmetric matrix \( Y \) such that
\[
\begin{bmatrix}
-\gamma^2 I & \bar{D}_i^T \\
* & -Y + \varepsilon B_i \bar{D}_i \bar{D}_i^T B_i^T
\end{bmatrix} < 0
\]
\[
\begin{bmatrix}
-Y & Y A_i^T \\
A_i Y & -(Y - M_i)
\end{bmatrix} < 0
\]
\[
\begin{bmatrix}
-Y - Q_2 & Y A_i^T & Y A_i^T \\
* & -Y + M_i + I & 0 \\
* & * & -\varepsilon S^{-1}
\end{bmatrix} < 0
\]
then
\[
u_{j0} = -\sum_{i=1}^{q} h_i(\bar{R} + B_i^T \bar{P}_m B_i + \varepsilon^{-1} E_{i3}^T E_{i3})^{-1} B_i^T \bar{P}_m A_i x(k)
\]
is a minimax robust and non-fragile controller of system (10), and the closed-loop system (31) will be asymptotically stable, and the corresponding upper bound of performance is
\[
\min_{u(k)} \max_{u_0} J(u, \omega) = x^T(0) P_m x(0).
\]

where \( P_m = P_m^e + \bar{P}_m \bar{D}_i (\gamma^2 I - \bar{D}_i^T \bar{P}_m \bar{D}_i)^{-1} \bar{D}_i^T \bar{P}_m, \)
\[
M_i = -\gamma^2 \bar{D}_i \bar{D}_i^T - \varepsilon B_i \bar{D}_i \bar{D}_i^T B_i^T + B_i^T (\bar{R} + \varepsilon^{-1} E_{i3}^T E_{i3})^{-1} B_i,
\]
Φ = \Theta \tilde{P}^{-1} S^{-1} B_i + B_i^T S^{-1} \tilde{P}^{-1} \Theta_i + B_i^T B_i, Y = P^{-1}.

Proof: Considering the Lyapunov function candidate

V(k) = x^T(k)P_m x(k) (37)

where matrix \( P_m \) is known, along the state trajectory of (31), by Lemma 1, it follows

\[
\Delta V(k) = x^T(k+1)P_m x(k+1) - x^T(k)P_m x(k) \\
\leq \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(z(k)) h_j(z(k)) [A_i x(k) + B_i u_{j0} + \tilde{D}_i W] \tilde{P}_m \\
[A_i x(k) + B_i u_{j0} + \tilde{D}_i W] + \tilde{\epsilon}^{-1} u_{j0} E_{i3} E_{i3} u_{j0} - x^T(k)P_m x
\]

where \( \tilde{P}_m = (P_m - \tilde{\epsilon} B_i \tilde{D}_i D_i^T B_i)^{-1} \).

Construct the following local check function

\[
\psi(k) = \Delta V(k) + u_{j0}^T (\tilde{D}_i \tilde{P}_m \tilde{D}_i) - \tilde{\epsilon}^2 W^T(k)W(k) (39)
\]

Substituting (38) into (39), and then maximizing of (39) about \( W \)

\[
W^* = (\tilde{\epsilon}^2 I - \tilde{D}_i^T \tilde{P}_m \tilde{D}_i)^{-1} \tilde{D}_i^T \tilde{P}_m (A_i x + B_i u_{j0}) (40)
\]

If inequality (33) holds, then

\[
\frac{\partial^2 \psi(k)}{\partial W^2} = \tilde{D}_i^T \tilde{P}_m \tilde{D}_i - \tilde{\epsilon}^2 I < 0,
\]

i.e. \( W^* \) in (40) makes the local check function maximum. Substituting (40) into (39), and minimizing about \( u_{j0} \), then

\[
u_{j0}^* = - \sum_{i=1}^{q} h_i(z(k)) A_i^T \tilde{P}_m - B_i^T \tilde{P}_m (A_i x + B_i u_{j0}) + \tilde{\epsilon}^{-1} E_{i3} E_{i3}^{-1} B_i^T \tilde{P}_m A_i x(k)
\]

and denote

\[
\min_{u_{j0}} \max_k \psi(k) = - x^T(k)Q x(k) (42)
\]

By Schur complement, if (34) holds, then \( \tilde{Q} > 0 \), and by using condition (35), the rest of the proof is similar to that of Theorem 1, substitute (40) into (38), and Pro- post multiplying both sides of the above inequality by \( [P_m^{-1}, (P_m^{-1} - M_i), (\Theta_i P_m^{-1} + B_i^T)] \), it follows that

\[
\Delta V(k) \leq \Xi + \begin{bmatrix}
-Y - Q_2 & Y A_i^T & Y A_i^T \\
* & -Y + M_i + I & 0 \\
* & * & -\epsilon \Phi_i
\end{bmatrix} (43)
\]

where \( \tilde{P}_m^{-1} = Y - \tilde{\epsilon}^2 D_i \tilde{D}_i^T - \tilde{\epsilon} B_i \tilde{D}_i D_i^T B_i, Y = P^{-1}, \)

\[
\Phi_i = \Theta_i \tilde{P}_m^{-1} S^{-1} B_i + B_i^T S^{-1} \tilde{P}_m^{-1} \Theta_i + B_i^T B_i, \quad Y = P^{-1}, \quad \Xi = \begin{bmatrix}
-Q_2 & 0 & 0 \\
* & -M_i \tilde{P}_m & 0 \\
* & * & -\epsilon \Theta_i \tilde{P}_m^{-1} S^{-1} \tilde{P}_m^{-1} \Theta_i
\end{bmatrix}
\]

It is easy to show that \( \Xi < 0 \) is equivalent to that inequality (35). Thus, from (35) and (36), obvious \( \Delta V < 0 \), i.e. the closed-loop system (31) is asymptotically stable, and \( V(x(\infty)) = 0 \). Now get the integral of (42) after some appropriate transpose, we obtain

\[
\min_{\nu_{j0}} \max_{W} J(u, W) = x(0)^T P_m x(0), \quad i = 1, \ldots, q.
\]

Theorem 3 provides a sufficient condition for the solution to the minimax robust and non-fragile control problem. But it remains unclear as to how one can choose the designing parameter \( P_m \) in order to achieve the minimum upper bound of performance cost.

**Theorem 4:** Considering the system (10) and performance cost (14), if the convex optimization problem

\[
\min_{Y, \tilde{Y}} \text{Trace}(\tilde{Y})
\]

s.t. (33) – (36), \[ \begin{bmatrix} \\ \tilde{Y} & I \\ I & Y \end{bmatrix} \]

there exist a solution \( Y, \tilde{Y} \), then (41) is the optimal minimax robust and non-fragile control law of system (10), and the corresponding upper bound of performance cost is

\[
\tilde{J} \leq \text{Trace}(\tilde{Y}) = J^*.
\]

V. NUMERICAL SIMULATIONS

Considering the additive gain perturbations as an example, the system (6)-(8) can be presented as following T-S model [18]:

\[
R^1: \text{If } z(k) = x_2(k) + \frac{v}{2k} \tilde{x}_1(k) \text{ is about } 0 \text{ [rad], then } x(k+1) = (A_1 + \Delta A_1) x(k) + (B_1 + \Delta B_1) u(k);
\]

\[
R^2: \text{If } z(k) = x_2(k) \pm \frac{v}{2k} \tilde{x}_1(k) \text{ is about } \pm \pi [\text{rad], then } x(k+1) = (A_2 + \Delta A_2) x(k) + (B_2 + \Delta B_2) u(k).
\]

where \( A_i, \) \text{ are matrices corresponding to the } i \text{th subsystem with appropriate dimensions.}

\[
A_1 = \begin{bmatrix} 1 - \frac{v k}{L} & 0 & 0 \\ \frac{v k}{(v k)^2} & 1 & 0 \\ \frac{v k}{(v k)^2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.3846 & 0 & 0 \\ -0.3846 & 1 & 0 \\ 0.009615 & -0.051 & 1 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} 1 - \frac{v k}{L} & 0 & 0 \\ \frac{v k}{(v k)^2} & 1 & 0 \\ \frac{v k}{(v k)^2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.3846 & 0 & 0 \\ -0.3846 & 1 & 0 \\ 0.00032458 & -0.001688 & 0 \end{bmatrix}
\]

\[
\Delta A_1 = \Delta A_2 = \begin{bmatrix} 0.2 \sin(k) & 0 & 0 \\ 0 & 0.2 \sin(k) & 0 \\ 0 & 0 & 0.2 \sin(k) \end{bmatrix}
\]

\[
B_1 = B_2 = \begin{bmatrix} \frac{v k}{L} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.5747 \\ 0 \\ 0 \end{bmatrix}
\]

where \( q = \frac{0.106}{\pi}, \Delta A_i, \text{ and } \Delta B_i \text{ are constant matrices with appropriate dimensions, which represent parametric uncertainties in the plant model, } L = 0.087 [m], \text{ } k = 0.5 [sec], v = -0.10 [m/sec].

Suppose the controller nonentity to gain variety first, i.e. the controller is placed in the ideal appearance. We
can use the Theorem 2 to design an optimal minimax robust control law, and corresponding optimal feedback gain is given by $K_{1R} = \begin{bmatrix} 2.4566 & -0.3167 & 0.0083 \\ 2.4565 & -0.3163 & 0 \end{bmatrix}$; $K_{2R} = \begin{bmatrix} 2.4565 & -0.3163 & 0 \end{bmatrix}$. By Matlab LMI Toolbox, we compute the optimal upper bound of performance cost is $J_R = 15.9600$.

For additive uncertainties of the controller form (12) with $D_1 = D_2 = \begin{bmatrix} 0.1 & -0.01 & 0.1 \end{bmatrix}^T$, $E_1 = E_2 = \begin{bmatrix} 0.3 & 0.15 & -0.2 \end{bmatrix}$, by using Theorem 2, we have that the optimal upper bound of performance cost is $J_N = 15.4703$, and corresponding optimal feedback gain is given by $K_{1N} = \begin{bmatrix} 2.5169 & -0.3316 & 0.0085 \end{bmatrix}$; $K_{2N} = \begin{bmatrix} 2.5168 & -0.3312 & 0 \end{bmatrix}$. Obviously, $J_R > J_N$. It is obvious that under the controller takes to have the uncertainty, performance of closed-loop system is lower.

Give the sine disturbance of vibrant bound with 0.1 to the systems and compare the effects of minimax robust control with minimax robust non-fragile control. The state response of system is shown as follows.

![Fig. 1. State responses of robust control](image1)

![Fig. 2. State responses of robust and non-fragile](image2)

As shown in Fig. 1 and Fig. 2, although the simple minimax robust controller can still make each system state trajectory asymptotically stable usually, there has obvious chatter when there exists some disturbance in the controller, which demonstrates the simple minimax robust controller is fragile. The non-fragile minimax control method designed in this paper is very suitable for these problems of the system stability being destroyed and performance decreasing by apparatuses or sensors, which is viable for the worse engineering environment.

VI. CONCLUSIONS

By introducing parametric uncertainty terms into building the T-S model for trailer-truck systems, the fuzzy model approaches to the original system more exactly. The designed controller can easily minimize the upper bound of performance with few energy expenditure, furthermore, it can guarantee closed-loop system asymptotic stable. Through comparing to the robust control and robust and non-fragile control, we demonstrate the better non-fragile performance of this controller.

REFERENCES
