Bounds on the $ABC$ spectral radius and $ABC$ energy of graphs

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**Abstract**

In this paper, we obtain some bounds for the $ABC$ spectral radius of general graphs. In continuing, we get some properties of the $ABC$ eigenvalues of graphs, and then we establish some new bounds for the $ABC$ energy $E_{ABC}$. Finally, we set up the correlation between the $ABC$ energy and other classes of graph energies.

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1. Introduction

Let $G = (V, E)$ be a simple graph with $n$ vertices, $m$ edges, whose adjacency matrix is $A(G)$. For $v \in V$, the degree of $v$ is denoted by $d_v$ and the set of vertices adjacent to $v$ in $G$ is denoted by $N(v)$. Here the maximum degree and the minimum degree of $G$
are denoted by $\Delta$ and $\delta$, respectively. The graph $G + e$ is constructed from $G$ by adding an edge $e = uv$, where $u, v$ are not adjacent in $G$. The complete graph, the star graph, the cycle graph and the path graph on $n$ vertices are denoted by $K_n$, $S_n$, $C_n$ and $P_n$, respectively. A complete bipartite graph with a bipartition of sizes $n_1$ and $n_2$ is denoted by $K_{n_1,n_2}$, where $n_1 + n_2 = n$.

The eigenvalues of a graph $G$ are the roots of the characteristic polynomial $P_G(\lambda) = \det(\lambda I - A(G))$, where $I$ is the identity matrix of order $n$. The adjacency energy or briefly the energy of $G$ is a graph invariant which was introduced by Ivan Gutman [7]. It is defined as $E_A(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i$’s are eigenvalues of $G$. More on the energy of graphs can be found in [10].

The Randić matrix $R = (r_{ij})$ is defined as $r_{ij} = 1/\sqrt{d_i d_j}$ if the $i$-th and $j$-th vertices are adjacent and $r_{ij} = 0$, otherwise. The Randić energy $E_R(G)$ is the sum of absolute values of the eigenvalues of $R$. The general Randić index or the branching index was defined as $R_{-1}(G) = \sum_{v_i \sim v_j} (1/d_i d_j)$, see [11].

The atom-bond connectivity matrix $ABC = [ABC_{i,j}]_{n \times n}$ is defined as:

$$ABC_{i,j} = \begin{cases} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of this matrix are the $ABC$ eigenvalues of $G$ denoted by $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. A multiset consisting of the $ABC$ eigenvalues of $G$ is called the $ABC$ spectrum of $G$ and we denote it by $\text{Spec}_{ABC}(G)$. If $G$ has exactly $s$ distinct $ABC$ eigenvalues $\nu_1, \nu_2, \ldots, \nu_s$ with multiplicities $t_1, t_2, \ldots, t_s$, respectively, then we write $\text{Spec}_{ABC}(G) = \{[\nu_1]^{t_1}, [\nu_2]^{t_2}, \ldots, [\nu_s]^{t_s}\}$. The $ABC$ spectral radius of $G$ is the largest eigenvalue of the $ABC$ matrix of $G$, which is denoted by $\nu_1(G)$. The $ABC$ energy of a graph $G$ is defined as $E_{ABC}(G) = \sum_{i=1}^{n} |\nu_i|$; see [3,4].

**Lemma 1.1.** [3] For a graph $G$ of order $n \geq 3$ with no isolated vertices, we have

1) $\sum_{i=1}^{n} \nu_i = 0,$

2) $\sum_{i=1}^{n} \nu_i^2 = 2(n - 2R_{-1}(G)).$

2. Bounds for the spectral radius of the $ABC$ matrix

Here we give some bounds for the spectral radius of the $ABC$ matrix.

**Lemma 2.1.** [1] (Perron-Frobenius Theorem) Let $T \geq 0$ be an irreducible matrix with spectral radius $\theta_0$. Suppose $t \in \mathbb{R}$, and $x \geq 0, x \neq 0$. If $T x \leq t x$, then $t \geq \theta_0$. 
Theorem 2.1. An upper bound for the ABC spectral radius of a connected graph \( G \) with \( n \geq 3 \) vertices is

\[ \nu_1(G) \leq \sqrt{2n - 4}, \tag{1} \]

and equality holds if and only if \( G \cong K_n \).

Proof. Set \( x_i = \sqrt{d_i} \). For a connected graph \( G \), clearly we have that for each vertex \( v_i \), \( d_i \leq n - 1 \), and therefore, for any vertex \( v_i \) in \( G \), we yield that

\[
(ABCx)_i = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}} \sqrt{d_j}
\]

\[
= \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_i}} \leq d_i \sqrt{\frac{2n - 4}{d_i}} = \sqrt{2n - 4 \sqrt{d_i}}.
\]

By Lemma 2.1, we have \( \nu_1(G) \leq \sqrt{2n - 4} \). If the equality holds, we obtain that for any pair of vertices \( v_i \) and \( v_j \), \( d_i + d_j = 2n - 2 \). Together with the fact that \( d_i \leq n - 1 \), we know that \( d_i = n - 1 \) (i \( \leq i \leq n \)). Therefore, \( G \) is a complete graph. \( \square \)

Theorem 2.2. Let \( G \) be a connected bipartite graph of order \( n \geq 3 \) vertices. Then

\[ \nu_1(G) \leq \sqrt{n - 2}, \]

and equality holds if and only if \( G \cong K_{n_1,n_2} \), where \( n = n_1 + n_2 \).

Proof. The proof follows directly from that for Theorem 2.1, by substituting \( d_i + d_j \leq 2n - 2 \) with \( d_i + d_j \leq n \) (holding for any pair of adjacent vertices in bipartite graphs). \( \square \)

Theorem 2.3. Let \( G \) be an \( r \)-regular graph. Then

i) \( \sqrt{2r - 2} \) is an ABC eigenvalue of \( G \).

ii) If \( G \) is connected, then the multiplicity of \( \sqrt{2r - 2} \) is one.

iii) For any ABC eigenvalue \( \nu \), we have \( |\nu| \leq \sqrt{2r - 2} \).

Proof. (i) Suppose \( u = [1,1,\ldots,1]^T \). Then we have \( ABCu = \sqrt{2r - 2}u \), which means that \( \sqrt{2r - 2} \) is an ABC eigenvalue of \( G \).

(ii) Let \( x = [x_1,x_2,\ldots,x_n]^T \) be a non-zero vector, where \( ABCx = \sqrt{2r - 2}x \), and suppose that \( x_i \) is an entry of \( x \) having the largest absolute value. Since \( (ABCx)_i = \sqrt{2r - 2}x_i \), we have \( \sum_{v_j \in N(v_i)} x_j = \sqrt{2r - 2}x_i \), where the summation runs over the \( r \) vertices \( v_i \) adjacent to \( v_j \). It is not difficult to see that \( x_i = x_j \) for all these vertices. If \( G \) is a connected graph, we may proceed successively in this way, and eventually show that
all the entries of \( x \) are equal. Thus \( x \) is a multiple of \( u \) and the space of the eigenvectors associated with the eigenvalue \( \sqrt{2r-2} \) has a dimension one.

(iii) Suppose \( ABCy = \nu y, y \neq 0 \), and let \( y_i \) denote an entry of \( y \) which is the largest in absolute value. By the same argument as in (ii), we have \( \sum_{j \in N(v_i)} y_j = \nu y_i \), and so

\[
|\nu||y_j| = \left| \sum_{j \in N(v_i)} y_j \right| \leq \sum_{j \in N(v_i)} |y_i| \leq \sqrt{2r-2}|y_i|.
\]

Thus \( |\nu| \leq \sqrt{2r-2} \), as required. \( \Box \)

In what follows, by \( mG \) we mean the union of \( m \) copies of \( G \), namely \( G \cup \cdots \cup G \) \( m \) times.

A Rayleigh quotient for the \( ABC \) matrix is a scalar of the form \( \frac{x^T ABC x}{x^T x} \) where \( x \) is a non-zero vector in \( \mathbb{R}^n \). The supremum of the set of such scalars is the largest eigenvalue \( \nu_1(G) \) of \( ABC \), or equivalently,

\[
\nu_1(G) = \sup_{x \neq 0} \frac{x^T ABC x}{x^T x} = \sup_{x \neq 0} \frac{2}{\sum_{u \in V(G)} x_u^2} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.
\]

If \( x_u = \sqrt{d_u} \), then we have

\[
\nu_1(G) \geq \frac{1}{m} \sum_{uv \in E(G)} \sqrt{d_u + d_v - 2} \geq \frac{1}{m} \sqrt{\sum_{uv \in E(G)} (d_u + d_v - 2)} = \frac{1}{m} \sqrt{M_1(G) - 2m},
\]

where \( M_1(G) \) is the first Zagreb index. If \( x_u = d_u \), then we have

\[
\nu_1(G) \geq \frac{2}{M_1(G)} \sum_{uv \in E(G)} \sqrt{(d_u + d_v - 2)d_u d_v},
\]

and by using Theorem 2.3, equality holds if \( G \) is regular. By [3, Lemma 3.10], if \( G \) is a graph of order \( n \geq 3 \) with no isolated vertex, then we have

\[
\nu_1(G) \geq \sqrt{\frac{2}{n} (n - 2R_{-1}(G))},
\]

with equality if and only if \( n \) is even and \( G = (\frac{n}{2})K_2 \).
Theorem 2.4. A lower bound for the ABC spectral radius of a connected graph $G$ with $n \geq 3$ vertices is

$$\nu_1(G) \geq \sqrt{2} \cos \frac{\pi}{n+1},$$

and equality holds for the path $P_n$.

Proof. If $\Delta = 2$, then $G \in \{P_n, C_n\}$ and we have $\nu_1(P_n) = \sqrt{2} \cos \frac{\pi}{n+1} < \sqrt{2} = \nu_1(C_n)$, and the equality holds for the path $P_n$. We will prove that for any graph $G$ with $\Delta \geq 3$, $\nu_1(G) \geq \sqrt{2}$. By Lemma 2.1, if we find a vector $x > 0$ such that $ABCx \geq \sqrt{2}x$, then $\nu_1(G) \geq \sqrt{2}$. We consider the following cases:

Case 1: $\delta \geq 2$. Set $x_i = \sqrt{d_i}$. Therefore, we obtain

$$(ABCx)_i = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}} \geq d_i \geq \sqrt{2} \sqrt{d_i} = \sqrt{2}x_i.$$

So we have $ABCx \geq \sqrt{2}x$.

Case 2: $\delta = 1$. We delete all pendent edges from $G$ until there is no pendent edge, and we delete isolated vertices and get a graph $G'$.

Subcase 2.1: $G'$ is an empty graph, that is to say, $G$ is a tree. Choose a vertex $v$ such that $d_v = \Delta$ as the root of the tree. Let $P = v_1v_2 \ldots v_l(v_l = v)$ be a longest path of length $l - 1$ ending at $v$. If $l = 2$, then $G = S_n$, and $\nu_1(S_n) = \sqrt{n - 2} \geq \sqrt{2}$. So we assume $l \geq 3$. Set $x_{v_i} = i, 1 \leq i \leq l$. Then for any given assigned root vertex $v$ with value $x_v$, and a longest assigned path $P$, we give an algorithm to assign values to other vertices.

Step 1: For all $u \in N(v)$ and $d_u = 1$, set $x_u = \sqrt{\frac{d_u - 2}{\Delta}x_v}$. 

Step 2: For all unassigned vertices $u \in N(v)$ and $d_u \geq 2$, find a longest path $P = u_1u_2 \ldots u_kv(u_k = u, k \geq 2)$ ending at $v$, and set $x_{u_i} = \frac{i}{k+1}x_v(1 \leq i \leq k)$.

Step 3: If there is a vertex $u_i$ with $d_{u_i} \geq 3$, take $u_i$ as an assigned root vertex and choose the longest assigned path ending at $u_i$.

Repeat above procedure until all vertices have their values. Fig. 1 is an example of how this algorithm is implemented. We have

$$\sqrt{\frac{d_v - 1}{d_v}}x_v \geq \frac{2}{3}x_v \geq \frac{1}{2}x_v \geq \frac{2}{3}x_v \cdot \sqrt{\frac{2\Delta - 2}{\Delta^2}},$$

for $x_v \geq 3$ and $d_v \geq 3$. Then for the rooted vertex $v$, we have

$$(ABCx)_v \geq \Delta \left(\frac{2}{3}x_v \cdot \sqrt{\frac{2\Delta - 2}{\Delta^2}}\right) = \sqrt{2} \cdot \frac{2}{3}x_v \cdot \sqrt{\Delta - 1} \geq \sqrt{2}x_v,$$
where $\Delta \geq 4$.

Now we assume $\Delta = 3$. For the rooted vertex $v$, there is actually a neighbor with value $x_v - 1$. So for the other two neighbors $u_1, u_2$, we assume that $d_{u_1} \leq d_{u_2}$. If $1 = d_{u_1} = d_{u_2}$, then we have

$$(ABCx)_v = 2 \cdot \frac{\sqrt{2}}{3} x_v + \frac{2}{3} (x_v - 1) > \sqrt{2} x_v,$$

for $x_v \geq 4$; if $1 = d_{u_1} < d_{u_2} \leq 3$, then

$$(ABCx)_v \geq \frac{\sqrt{2}}{3} x_v + \frac{2}{3} \cdot \frac{2}{3} x_v + \frac{2}{3} (x_v - 1) > \sqrt{2} x_v,$$

for $x_v \geq 4$; if $2 \leq d_{u_1} \leq d_{u_2} \leq 3$, then

$$(ABCx)_v \geq 2 \cdot \frac{2}{3} \cdot \frac{2}{3} x_v + \frac{2}{3} (x_v - 1) > \sqrt{2} x_v,$$

for $x_v \geq 5$. For $\Delta = 3$, $3 \leq x_v \leq 4$, we list all such graphs in Appendix B (Tables 2, 3), it is easy to see that $\nu_1(G) \geq \sqrt{2}$, and so we choose the eigenvector corresponding to the spectral radius as $x$. In conclusion, $(ABCx)_v \geq \sqrt{2} x_v$.

Therefore, we verified that for the root vertex $v$, $(ABCx)_v \geq \sqrt{2} x_v$. Next, we consider other vertices. For vertices with degree 1 and 2, by its definition, $ABCx_u \geq \sqrt{2} x_u$. For vertices with degree 3, there is actually a neighbor with value at least $x_u + 1$. So for the other two neighbors $u_1, u_2$, we assume that $d_{u_1} \leq d_{u_2}$. If $1 = d_{u_1} = d_{u_2}$, then we have

$$(ABCx)_u = 2 \cdot \frac{\sqrt{2}}{3} x_u + \frac{2}{3} (x_u + 1) > \sqrt{2} x_u,$$

for $x_u \geq 1$; if $1 = d_{u_1} < d_{u_2} \leq 3$, then

$$(ABCx)_u \geq \frac{\sqrt{2}}{3} x_u + \frac{2}{3} \cdot \frac{2}{3} x_u + \frac{2}{3} (x_u + 1) > \sqrt{2} x_u,$$
for $x_u \geq 1$; if $2 \leq d_{u_1} \leq d_{u_2} \leq 3$, then

$$(ABCx)_u \geq 2 \cdot \frac{2}{3} \cdot \frac{2}{3} x_u + \frac{2}{3} (x_u + 1) > \sqrt{2} x_u,$$

for $x_u \geq 1$. Then we have $(ABCx)_u \geq \sqrt{2} x_u$. For vertices with degree at least 4, verification of $(ABCx)_u \geq \sqrt{2} x_u$ holding is similar to the rooted vertex. Then we have that for any vertex $v \in G$, $(ABCx)_v \geq \sqrt{2} x_v$. So we have $ABCx \geq \sqrt{2} x$.

**Subcase 2.2:** $G'$ is not an empty graph, that is to say, $\delta(G') \geq 2$.

In this case, we can consider $G$ as attaching some trees to the vertices in $G'$. Recall that the degree of $v$ in $G$ is denoted by $d_v$. First, for any vertex $v$ in $G'$, we choose the longest path $P_{x_v}$ in $G \setminus G'$ ending at $v$, and determinate $t = \max\{x_v/\sqrt{d_v}, v \in G'\}$. Set $x_i = t\sqrt{d_i}$ for all vertices in $G'$. Then use the algorithm in Subcase 2.1 to assign all the vertices in the unassigned trees. It is easy to see that the process of verifying each vertices in $G'$ that attached no tree is the same as Case 1, and vertices in $V(G) \setminus V(G')$ is the same as Subcase 2.1. For all the remaining vertices, let $d = d_v(G) - d_v(G') \leq d_v(G) - 2$, $d_1 = \max\{d_u : u \in V(G) - V(G'), u \in N(v)\}, d_2 = \min\{d_u : u \in G', u \in N(v)\}$. By Case 1 and Subcase 2.1, we have that

$$(ABCx)_v \geq d \cdot \frac{2}{3} t\sqrt{d_v} \cdot \sqrt{\frac{d_v + d_1 - 2}{d_v d_2}} + (d_v - d) \cdot t\sqrt{d_2} \cdot \sqrt{\frac{d_v + d_2 - 2}{d_v d_2}}$$

$$(\geq t(\frac{2}{3} d + d_v - d)) \geq t(\frac{2}{3} d_v + \frac{2}{3}) \geq \sqrt{2} \cdot t\sqrt{d_v} = \sqrt{2} x_v,$$

for $d_v \geq 2$. Hence, we have $(ABCx)_v \geq \sqrt{2} x_v$.

In conclusion, since we found a vector $x > 0$ such that $ABCx \geq \sqrt{2} x$, for any $G$ with $\Delta \geq 3$, then $\nu_1(G) \geq \sqrt{2}$ holds. □

**Conjecture 2.1.** Let $G$ be a unicyclic graph of order $n \geq 4$. Then

$$\sqrt{2} = \nu_1(C_n) \leq \nu_1(G) \leq \nu_1(S_n + e),$$

with equality if and only if $G \cong C_n$ for the lower bound, and if and only if $G \cong S_n + e$ for the upper bound.

3. **Bounds for the $ABC$ energy $E_{ABC}$ of graphs**

The aim of this section is to find some bounds on the $ABC$ energy of graphs. At first, by using the methods of Estrada et al. as given in [5], we can obtain the following formula for the $ABC$ energy of a graph. As usual, the binomial coefficients are defined by $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$, where $n \geq r$. 

\textbf{Theorem 3.1.} Let $G$ be a connected graphs of order $n \geq 3$. Then

$$E_{ABC}(G) = \nu_1 \text{tr} \sum_{i=0}^{\infty} \left( \frac{1}{2i} \right) \left( \frac{ABC}{\nu_1} \right)^{2i}.$$  \hspace{1cm} (3)

\textbf{Proof.} Let $G$ be a connected graph. Suppose that $ABC$ matrix of $G$ is a square, symmetric matrix with spectral decomposition $ABC = QDQ^T$, where $Q = [\psi_1 \cdots \psi_n]$ is the matrix of orthonormalized eigenvectors $\psi_j$ associated with the eigenvalues $\nu_j$, and $D = \text{diag}(\nu_1, \ldots, \nu_n)$. Since every symmetric positive semidefinite matrix has a unique positive semidefinite square root, we yield that $|ABC| = Q|D|Q^T = \sqrt{ABC^2}$.

Let $\nu_1 > 0$ be the largest eigenvalue of $ABC$. We note in passing that since $G$ is connected, $\nu_1$ is a simple eigenvalue. Then, $\frac{ABC}{\nu_1}$ has spectral radius 1, and the matrix $M = \left( \frac{ABC}{\nu_1} \right)^2 - I$ has all its eigenvalues in the interval $[-1, 0]$. Hence, $M$ is negative semidefinite and has spectral radius 1. Let us write

$$|ABC| = \sqrt{ABC^2} = \nu_1 \sqrt{\left( \frac{ABC}{\nu_1} \right)^2} = \nu_1 \sqrt{I + \left( \frac{ABC}{\nu_1} \right)^2 - I}$$

$$= \nu_1 (I + M)^{\frac{1}{2}}. \hspace{1cm} (4)$$

Since, for $-1 \leq x \leq 1$, we have

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} x^3 + \cdots, \hspace{0.5cm} 0 \neq \alpha \in \mathbb{R},$$

Eq. (4) can be reformulated as follows:

$$|ABC| = \nu_1 \left( I + \frac{1}{2} M - \frac{1}{4 \cdot 2} M^2 + \frac{3}{2 \cdot 4 \cdot 6} M^3 + \cdots \right)$$

$$= \nu_1 \sum_{i=0}^{\infty} \left( \frac{1}{2i} \right) \left( \left( \frac{ABC}{\nu_1} \right)^2 - I \right)^i.$$

Therefore,

$$E_{ABC}(G) = \text{tr} |ABC| = \nu_1 \text{tr} \left[ \sum_{i=0}^{\infty} \left( \frac{1}{2i} \right) \left( \left( \frac{ABC}{\nu_1} \right)^2 - I \right)^i \right]$$

$$= \nu_1 \text{tr} \sum_{i=0}^{\infty} \left( \frac{1}{2i} \right) \sum_{j=0}^{\infty} \left( \frac{i}{j} \right) (-1)^j \left( \frac{ABC}{\nu_1} \right)^{2j}. \hspace{1cm} \square$$

Equivalently, the Eq. (3), can be rewritten as follows:

$$E_{ABC}(G) = \nu_1 \sum_{i=0}^{\infty} \left( \frac{2i}{i} \right) \frac{(-1)^{i+1}}{2^{2i}(2i - 1)} \text{tr} \left( \left( \frac{ABC}{\nu_1} \right)^2 - I \right)^i. \hspace{1cm} (5)$$
Applying Eq. (5) yields some upper bounds for $ABC$ energy of graphs as follows.

**Theorem 3.2.** Let $G$ be a connected graph. Then,

$$E_{ABC}(G) \leq \frac{n \nu_1}{2} + \frac{1}{\nu_1}(n - 2R_{-1}(G)).$$

**Proof.** By Eq. (5), we have

$$E_{ABC}(G) = \nu_1 \left[ \text{tr}(I) + \frac{1}{2} \text{tr}\left( \frac{ABC^2}{\nu_1^2} - I \right) - \frac{1}{8} \text{tr}\left( \frac{ABC^2}{\nu_1^2} - I \right)^2 + \cdots \right],$$

which implies that

$$E_{ABC}(G) \leq \nu_1 \text{tr} \left[ I + \frac{1}{2} \left( ABC^2 - I \right) \right]
= \nu_1 \left[ \text{tr}(I) + \frac{1}{2\nu_1^2} \text{tr}(ABC^2) - \frac{1}{2} \text{tr}(I) \right]
= \nu_1 \left[ \frac{n}{2} + \frac{1}{2\nu_1^2}(2n - 4R_{-1}(G)) \right]
= \frac{n \nu_1}{2} + \frac{1}{\nu_1}(n - 2R_{-1}(G)),$$

where $\text{tr}(ABC^2) = \sum_{i=1}^{n} \nu_i^2$. This yields the result. $\square$

**Lemma 3.1.** [2] Let $G$ be a connected graph on $n \geq 3$ vertices. Then

$$R_{-1}(G) \leq \frac{15}{56}(n + 1).$$

(6)

**Theorem 3.3.** Let $G$ be a connected graph on $n \geq 3$ vertices. Then,

$$E_{ABC}(G) \leq \frac{n}{2} \sqrt{2n - 4} + \frac{13n - 15}{28}.$$ 

**Proof.** By Eqs. (1), (2), (5) and (6), we obtain

$$E_{ABC}(G) \leq \frac{n}{2} \sqrt{2n - 4} + \frac{1}{\sqrt{2} \cos \frac{\pi}{n+1}} \left( n - \frac{15}{28}(n + 1) \right).$$

It is clear that $\frac{1}{\sqrt{2} \cos \frac{\pi}{n+1}} \leq 1$ and thus

$$E_{ABC}(G) \leq \frac{n}{2} \sqrt{2n - 4} + \frac{13n - 15}{28},$$

proving the result. $\square$
Theorem 3.4. Let $G$ be a graph of order $n \geq 3$ with no isolated vertex and with $ABC$ eigenvalues $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. Then

1) $E_{ABC}(G) \geq \sqrt{2n(n-2R_{-1}(G)) - \frac{n^2}{4} (|\nu_1| - |\nu_{\text{min}}|)^2}$,

2) $E_{ABC}(G) \geq \frac{2\sqrt{2n|\nu_1||\nu_{\text{min}}|(n-2R_{-1}(G))}}{|\nu_1|+|\nu_{\text{min}}|}$, where $\nu_i \neq 0$ ($1 \leq i \leq n$),

where $|\nu_{\text{min}}| = \min\{|\nu_2|, |\nu_3|, \ldots, |\nu_n|\}$.

Proof. We may assume $n \geq 3$. The Ozeki's inequality [9] states that if $a_i$ and $b_i$, ($1 \leq i \leq n$), are non-negative real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i^2 b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2,$$

where $0 \leq m_1 \leq a_i \leq M_1$ and $0 \leq m_2 \leq b_i \leq M_2$, ($1 \leq i \leq n$). Applying this inequality by substituting $a_i = 1$ and $b_i = |\nu_i|$ ($1 \leq i \leq n$), we thus get

$$n \sum_{i=1}^{n} |\nu_i|^2 - \left( \sum_{i=1}^{n} |\nu_i| \right)^2 \leq \frac{n^2}{4} (|\nu_1| - |\nu_{\text{min}}|)^2,$$

where $|\nu_{\text{min}}| = \min\{|\nu_2|, |\nu_3|, \ldots, |\nu_n|\}$. From Lemma 1.1, one can easily see that

$$E_{ABC}(G) \geq \sqrt{2n(n-2R_{-1}(G)) - \frac{n^2}{4} (|\nu_1| - |\nu_{\text{min}}|)^2}.$$

This completes the proof of the first claim. The proof of the second claim is by Poly-
Szegö inequality [8], i.e., if $a_i$, $b_i$, $M_i$, $m_i$, $1 \leq i \leq n$, are as defined above, then we obtain

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{i=1}^{n} a_i b_i \right)^2.$$

Suppose $a_i = 1$ and $b_i = |\nu_i|$, where $\nu_i \neq 0$ ($1 \leq i \leq n$). Then

$$n \sum_{i=1}^{n} |\nu_i|^2 \leq \frac{1}{4} \left( \sqrt{\frac{|\nu_{\text{min}}|}{|\nu_1|}} + \sqrt{\frac{|\nu_1|}{|\nu_{\text{min}}|}} \right)^2 \left( \sum_{i=1}^{n} |\nu_i| \right)^2$$

and

$$8n(n-2R_{-1}(G)) \leq E_{ABC}(G) \left( \sqrt{\frac{|\nu_{\text{min}}|}{|\nu_1|}} + \sqrt{\frac{|\nu_1|}{|\nu_{\text{min}}|}} \right)^2.$$
Thus

$$E_{ABC}(G) \geq \frac{2\sqrt{2n|\nu_1||\nu_{\text{min}}|(n - 2R_1(G))}}{|\nu_1| + |\nu_{\text{min}}|}. $$

This completes the proof of second claim. □

**Corollary 3.1.** Let $G$ be an $r$-regular graph of order $n \geq 3$ with no isolated vertex and with $ABC$ eigenvalues $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, where $|\nu_{\text{min}}| = \min\{|\nu_2|, |\nu_3|, \ldots, |\nu_n|\}$. Then

1) $E_{ABC}(G) \geq \sqrt{2n(n - 2R_1(G)) - \frac{n^2}{4} \left( \sqrt{2r - 2} - |\nu_{\text{min}}| \right)^2},$

2) $E_{ABC}(G) \geq \frac{2\sqrt{2n\sqrt{2r - 2(n - 2R_1(G))}|\nu_{\text{min}}|}}{\sqrt{2r - 2 + |\nu_{\text{min}}|}},$ where $\nu_i \neq 0$ (1 ≤ $i$ ≤ $n$).

**Proof.** Theorem 2.3 implies that $\nu_1 = \sqrt{2r - 2}$. Then by Theorem 3.4, it is not difficult to see above results. □

**Theorem 3.5.** Let $G$ be a graph of order $n \geq 3$ with no isolated vertex. Then

$$E_{ABC}(G) \geq \sqrt{2(n - 2R_1(G)) + \left( \frac{n}{2} \right) \left( \det(ABC) \right)^{\frac{2}{n}}}. $$

**Proof.** Applying Geometric-Arithmetic mean inequality yields that

$$\left( \sum_{i=1}^{n} |\nu_i| \right)^2 = \sum_{i=1}^{n} |\nu_i|^2 + \sum_{1 \leq i, j \leq n} |\nu_i||\nu_j|$$

$$\geq 2(n - 2R_1(G)) + n(n - 1) \left( \prod_{1 \leq i, j \leq n, i \neq j} |\nu_i||\nu_j| \right)^{\frac{1}{n(n-1)}}$$

$$= 2(n - 2R_1(G)) + 2 \left( \frac{n}{2} \right) \left( \prod_{i=1}^{n} \nu_i \right)^{2(n-1)}^{\frac{1}{n(n-1)}}$$

$$= 2(n - 2R_1(G)) + 2 \left( \frac{n}{2} \right) \left( \prod_{i=1}^{n} \nu_i \right)^{2 \frac{1}{n}}$$

$$= 2(n - 2R_1(G)) + 2 \left( \frac{n}{2} \right) \left( \det(ABC) \right)^{\frac{2}{n}}.$$

Since $E_{ABC}(G) = \sum_{i=1}^{n} |\nu_i|$, we get

$$E_{ABC}(G) \geq \sqrt{2(n - 2R_1(G)) + 2 \left( \frac{n}{2} \right) \left( \det(ABC) \right)^{\frac{2}{n}}}.$$
Table 1
The correlation between the ABC Energy and $E_A, E_R, E_L, E_S, E_L, E_Q$.

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This completes the proof. □

4. Correlation between ABC energy and other classes of graph energy

The symmetric $(0, -1, 1)$-adjacency matrix $S(G) = J - I - 2A(G)$ is called the Seidel matrix of $G$, where $J$ is the matrix with entries 1 in every position. The Seidel energy $E_S(G)$ is the sum of absolute values of the eigenvalues of $S(G)$. As usual, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian and signless Laplacian matrices of a graph $G$, respectively, where $D(G) = [d_{ij}]$ is the diagonal matrix with entries equal to the degree of vertices of $G$, namely, $d_{ii} = d_i$, and $d_{ij} = 0$ for $i \neq j$. If $\mu_1 \geq \cdots \geq \mu_n \geq 0, q_1 \geq q_2 \geq \cdots \geq q_n$ are the Laplacian and the signless Laplacian eigenvalues of $G$, then the quantities $E_L(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|$ and $E_Q(G) = \sum_{i=1}^{n} |q_i - 2m/n|$ are called the Laplacian and the signless Laplacian energies of $G$, respectively. The normalized Laplacian matrix of $G$ is defined as $L = D(G)^{-\frac{1}{2}}LD(G)^{-\frac{1}{2}}$. Then $E_L(G) = \sum_{i=1}^{n} |\delta_i - 1|$ is called normalized Laplacian energy of $G$, where $\delta_i$’s are the normalized Laplacian eigenvalues of $G$.

In [3], Chen conjectured that among all trees of order $n$, the star graph $S_n$ has the minimum $ABC$ energy and Gao et al. in [6] proved this conjecture. In the following, we investigate correlations between the $ABC$ energy and other classes of energy such as adjacency energy, normalized Laplacian energy, Randić energy, Seidel energy, Laplacian and signless Laplacian energy among all trees up to 18 vertices. These values are reported in Table 1. Also, in Appendix A one can see that among all trees up to 18 vertices, the star graph has the minimum $ABC$ energy. The values of $E_{ABC}(S_n)$ are colored by red, see Fig. 2.

Declaration of competing interest

The authors declare no conflict of interest.
Appendix A

Fig. 2. The $E_{ABC}$ of trees on $6 \leq n \leq 18$ vertices versus the respective $E_A$, $E_R$, $E_S$, $E_L$. 
Fig. 2. (continued)
Fig. 2. (continued)
Appendix B. Tables 2, 3

Table 2  
Graphs with $\Delta = 3$, $x_v = 3$.

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References
