3 Stress-Strain Behavior of Jointed Rock

3.1 Introduction

For both deep and shallow underground openings the rock mass normally represents
the main load-carrying structure. Consequently, the stress-strain behavior of the rock
mass is of great importance for the stability and thus the design and construction of
tunnels, caverns and other underground openings in rock. Similarly when concentrat-
ed loads are applied to a rock mass, as in the case of concrete dam foundations and
abutments, the rock represents a critical part of the overall structure. When slopes are
constructed, the rock mass, in combination with potential retaining structures, also has
the task of carrying loads due to self-weight and other impacts.

An investigation of the stability of rock engineering structures therefore requires an
adequate description of the stress-strain behavior of the rock mass, taking into account
the influence of the structure and particularly the influence of discontinuities.

The recommended model for the stress-strain behavior, which is subsequently formu-
lated, is based on structural models as described in Section 2.7.3. It was developed in
the 1970s and, since 1980, successfully applied by WBI, in many cases in combina-
tion with numerical analysis methods, to solve practical problems in rock engineering

3.2 Intact Rock

3.2.1 Elastic Behavior

The elastic behavior of intact rock with random grain structure can be considered as
isotropic. Two elastic constants, Young’s modulus E and Poisson’s ratio ν, defined by
means of an applied uniaxial normal stress $\sigma_z$ and normal strains $\varepsilon_x$ and $\varepsilon_z$ perpendicular and parallel to $\sigma_z$, respectively, are then sufficient to describe the linear elastic stress-
strain behavior (Fig. 3.1). The shear modulus $G$ results from $E$ and $\nu$ as a dependent
elastic constant:

$$G = \frac{E}{2\cdot(1+\nu)}. \quad (3.1)$$

In an arbitrarily oriented Cartesian coordinate system (x,y,z) normal stresses $\sigma$ and
normal strains $\varepsilon$ as well as shear stresses $\tau$ and shear strains $\gamma$ are related to each other
by Hooke’s law:
\[
\sigma_x = \frac{E}{1 - \nu - 2\nu^2} \left( (1 - \nu) \cdot \varepsilon_x + \nu \cdot \varepsilon_y + \nu \cdot \varepsilon_z \right), \\
\sigma_y = \frac{E}{1 - \nu - 2\nu^2} \left[ \nu \cdot \varepsilon_x + (1 - \nu) \cdot \varepsilon_y + \nu \cdot \varepsilon_z \right], \\
\sigma_z = \frac{E}{1 - \nu - 2\nu^2} \left[ \nu \cdot \varepsilon_x + \nu \cdot \varepsilon_y + (1 - \nu) \cdot \varepsilon_z \right], \\
\tau_{xy} = \frac{E}{2 \cdot (1 + \nu)} \cdot \gamma_{xy}, \\
\tau_{yz} = \frac{E}{2 \cdot (1 + \nu)} \cdot \gamma_{yz}, \\
\tau_{zx} = \frac{E}{2 \cdot (1 + \nu)} \cdot \gamma_{zx},
\] (3.2)

\[
\{\sigma\} = [D] \cdot \{\varepsilon\}. \tag{3.3}
\]

In (3.3)
\[
\{\sigma\} = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})^T
\]

and
\[
\{\varepsilon\} = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})^T
\]

are the stress and strain vectors in which \(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\) and
\[
\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}
\]
are the components of the symmetric stress and strain tensors, respectively, and
is called the “elasticity matrix”.

As an example, Fig. 3.2 shows the axial and lateral strains $\varepsilon_x$ and $\varepsilon_z$, respectively, of a granite specimen during loading and unloading, measured under uniaxial normal stress $\sigma_z$ (Walsh 1965c). The stress-strain curves are both more or less linear and reversible up to high stress levels. Deviations from linearity at low stress levels can be attributed to the closing of cracks that are pre-existing in the specimen. The sliding between contacting crack surfaces at the beginning of unloading creates a slight hysteresis loop. From a practical point of view, however, these deviations from a linear elastic stress-strain behavior in most rocks can be neglected.
anisotropic and can be described by transverse isotropy. Such an elastic behavior is specified by five independent elastic constants. Two Young’s moduli $E_1$ and $E_2$ characterize the deformability parallel and perpendicular to the structure planes. Parallel to a structure plane the deformability is assumed to be isotropic. A structure plane is therefore called an “isotropic plane”. The shear modulus $G_2$ describes the deformability for a shear loading parallel to the isotropic plane. Furthermore, two Poisson’s ratios $\nu_1$ and $\nu_2$ are needed.

In Fig. 3.3 the elastic constants of a transversely isotropic rock are defined by means of stresses and strains applied to a cube-shaped specimen using a Cartesian coordinate system ($x',y',z'$) that is related to the isotropic plane. The $z'$ axis coincides with the direction perpendicular to the isotropic plane and the $x'$ and $y'$ axes lie in the isotropic plane.
3.2 Intact Rock

Figure 3.3 defines, in addition to $E_1$, $E_2$, $G_2$, $v_1$ and $v_2$ the elastic constants $G_1$ and $v_3$, which are dependent on $E_1$, $E_2$, $v_1$ and $v_2$.

In the coordinate system $(x',y',z')$ the stress-strain relationship for transverse isotropy can be expressed as:

$$\{\sigma'\} = [D'] \cdot \{\varepsilon'\}$$  \hspace{1cm} (3.4)

with

$$[D'] = \begin{bmatrix}
E_1 \cdot \frac{1-n \cdot v_2^2}{(1+v_1) \cdot m} & E_1 \cdot \frac{1+n \cdot v_2^2}{(1+v_1) \cdot m} & E_1 \cdot \frac{v_2}{m} & 0 & 0 & 0 \\
E_1 \cdot \frac{1+n \cdot v_2^2}{(1+v_1) \cdot m} & E_1 \cdot \frac{1-n \cdot v_2^2}{(1+v_1) \cdot m} & E_1 \cdot \frac{v_2}{m} & 0 & 0 & 0 \\
E_1 \cdot \frac{v_2}{m} & E_1 \cdot \frac{v_2}{m} & E_1 \cdot \frac{1-v_1}{m} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{E_1}{2 \cdot (1+v_1)} & 0 & 0 \\
0 & 0 & 0 & 0 & G_2 & 0 \\
0 & 0 & 0 & 0 & 0 & G_2
\end{bmatrix},$$

where $n = E_1/E_2$ and $m = 1 - v_1 - 2 \cdot n \cdot v_2^2$.

The inverse relation of Equation (3.4) is

$$\{\varepsilon'\} = [D']^{-1} \cdot \{\sigma'\}$$  \hspace{1cm} (3.5)

in which

$$[D']^{-1} = \begin{bmatrix}
\frac{1}{E_1} & \frac{-v_1}{E_1} & \frac{-v_2}{E_2} & 0 & 0 & 0 \\
\frac{v_1}{E_1} & \frac{1}{E_1} & \frac{-v_2}{E_2} & 0 & 0 & 0 \\
\frac{-v_2}{E_2} & \frac{-v_2}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2 \cdot (1+v_1)}{E_1} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_2}
\end{bmatrix}$$

is denoted as the “compliance matrix”.
For stability analyses, a global coordinate system (x,y,z) is needed that is related to the engineering structure’s overall geometry. This usually does not coincide with the coordinate system (x’,y’,z’) introduced above. Linking of the two coordinate systems can be achieved by using two angles α and β describing the direction of the contour line and the inclination of the line of dip of the isotropic plane in relation to the global coordinate system (Fig. 3.4). These angles were introduced in Section 2.7.2 (Fig. 2.25).

\[
\begin{align*}
\{\sigma'\} &= [T] \cdot \{\sigma\}, \\
\{\varepsilon'\} &= [T^*] \cdot \{\varepsilon\}.
\end{align*}
\]

The dependency of [T] and [T*] from the angles α and β is given by the following equations:

\[
[T] = \begin{bmatrix}
\ell_1^2 & m_1^2 & n_1^2 & 2\ell_1m_1 & 2m_1n_1 & 2n_1\ell_1 \\
\ell_2^2 & m_2^2 & n_2^2 & 2\ell_2m_2 & 2m_2n_2 & 2n_2\ell_2 \\
\ell_3^2 & m_3^2 & n_3^2 & 2\ell_3m_3 & 2m_3n_3 & 2n_3\ell_3 \\
\ell_1\ell_2 & m_1m_2 & n_1n_2 & \ell_1m_2 + \ell_2m_1 & m_1n_2 & n_2\ell_1 \\
\ell_2\ell_3 & m_2m_3 & n_2n_3 & \ell_2m_3 + \ell_3m_2 & m_2n_3 + m_3n_2 & n_2\ell_3 + n_3\ell_2 \\
\ell_3\ell_1 & m_3m_1 & n_3n_1 & \ell_1m_3 + \ell_3m_1 & m_1n_3 & n_3\ell_1
\end{bmatrix},
\]
$$\left[ \mathbf{T}^* \right] = \begin{bmatrix}
\ell_1^2 & m_1^2 & n_1^2 & \ell_1m_1 & m_1n_1 & n_1\ell_1 \\
\ell_2^2 & m_2^2 & n_2^2 & \ell_2m_2 & m_2n_2 & n_2\ell_2 \\
\ell_3^2 & m_3^2 & n_3^2 & \ell_3m_3 & m_3n_3 & n_3\ell_3 \\
2\ell_1\ell_2 & 2m_1m_2 & 2n_1n_2 & \ell_1m_2 + \ell_2m_1 & m_1n_2 & n_1\ell_2 \\
2\ell_1\ell_3 & 2m_1m_3 & 2n_1n_3 & \ell_1m_3 + \ell_3m_1 & m_1n_3 & n_1\ell_3 \\
2\ell_2\ell_3 & 2m_2m_3 & 2n_2n_3 & \ell_2m_3 + \ell_3m_2 & m_2n_3 + m_3n_2 & n_2\ell_3 + n_3\ell_2 \\
\end{bmatrix} , \quad (3.9)$$

in which

$$\ell_1 = \sin \alpha, \quad m_1 = \cos \alpha, \quad n_1 = 0,$$

$$\ell_2 = \cos \beta \cos \alpha, \quad m_2 = -\cos \beta \sin \alpha, \quad n_2 = -\sin \beta,$$

$$\ell_3 = -\sin \beta \cos \alpha, \quad m_3 = \sin \beta \sin \alpha, \quad n_3 = -\cos \beta.$$

Insertion of (3.6) and (3.7) into (3.4) and multiplication by $[\mathbf{T}]^{-1}$ yields

$$\{ \sigma \} = [\mathbf{T}]^{-1} \cdot [\mathbf{D}'] \cdot [\mathbf{T}^*] \cdot \{ \varepsilon \}.$$

The calculation of $[\mathbf{T}]^{-1}$ leads to the relationship

$$[\mathbf{T}]^{-1} = [\mathbf{T}^*]^T. \quad (3.10)$$

Thus, (3.4) can be replaced by

$$\{ \sigma \} = [\mathbf{D}] \cdot \{ \varepsilon \} \quad (3.11)$$

with

$$[\mathbf{D}] = [\mathbf{T}^*]^T \cdot [\mathbf{D}'] \cdot [\mathbf{T}^*].$$

Equation (3.11) describes the relation between stresses and strains for a transversely isotropic rock in the global coordinate system.

The inverse relation of (3.11)

$$\{ \varepsilon \} = [\mathbf{D}]^{-1} \cdot \{ \sigma \} \quad (3.12)$$

is obtained by inverting the elasticity matrix $[\mathbf{D}]$, which can be accomplished in consideration of (3.10) as follows:

$$[\mathbf{D}]^{-1} = [\mathbf{T}]^T \cdot [\mathbf{D}']^{-1} \cdot [\mathbf{T}].$$

From the thermodynamic constraint of positive definite elastic strain energy the theory of elasticity provides lower and upper bounds of the elastic constants for
isotropic and transversely isotropic bodies. The following relations are valid for an isotropic elastic body (Love 1927):

\[ E \geq 0 \]  \hspace{1cm} (3.13)

and

\[-1 \leq \nu \leq 0.5. \]  \hspace{1cm} (3.14)

The corresponding restrictions for a transversely isotropic elastic body are (Pickering 1970):

\[ E_1 \geq 0, E_2 \geq 0, G_2 \geq 0 \]  \hspace{1cm} (3.15)

and

\[-1 \leq \nu_1 \leq 1 - 2\nu_2^2 \frac{E_1}{E_2}. \]  \hspace{1cm} (3.16)

Relations (3.14) and (3.16) imply that, in principle, negative values of Poisson’s ratios may also be possible. However, with a few exceptions regarding highly anisotropic rocks only, Poisson’s ratios between 0 and 0.5 are reported in the related literature (Vutukuri et al. 1974, Hatheway & Kiersch 1986, Gercek 2007).

Further restrictions for Poisson’s ratios \( \nu_1 \) and \( \nu_2 \) of transversely isotropic rocks were found by Knops & Payne (1971)

\[-1 < \nu_1 < 0.5 \]  \hspace{1cm} (3.17)

and Amadei (1996)

\[-\sqrt{\frac{(1-\nu_1)\cdot E_2}{2E_1}} < \nu_2 < \sqrt{\frac{(1-\nu_1)\cdot E_2}{2E_1}}. \]  \hspace{1cm} (3.18)

Both inequalities were later derived by Exadaktylos (2001) assuming plane strain conditions.

Not all intact rocks can be described by isotropic or transversely isotropic elastic behavior. The general anisotropic body with no symmetry at all, exhibits 21 elastic constants (Lekhnitskii 1963). However, the effort needed to evaluate more than five elastic constants is unjustifiably high and therefore carried out only in exceptional cases.

### 3.2.2 Strength and Failure Criteria

**Shear and tensile strength of intact rocks with random grain structure**

Experience has shown that intact rocks with random grain structure may be considered as isotropic with regard to their strength. The shear strength of those rocks can be described by an approximation using the Mohr-Coulomb failure criterion:
\[ \tau = \sigma \cdot \tan \varphi_{\text{IR}} + c_{\text{IR}}. \]  

(3.19)

\(\sigma\) is the normal stress acting on the shear plane and \(\tau\) is the absolute value of the corresponding shear stress in the failure state. The shear parameters \(c_{\text{IR}}\) and \(\varphi_{\text{IR}}\) are referred to as “cohesion” and “angle of internal friction” or simply “friction angle”, respectively. This failure criterion is based on the hypothesis that shear failure occurs on a plane inclined at an angle of \(\alpha = 45^\circ + \varphi_{\text{IR}}/2\) to the horizontal. Furthermore, it is assumed that the shear strength is independent of the intermediate principal normal stress \(s_2\). Consequently, the failure stress state can be represented in a \(\tau-\sigma\) diagram in the form of a Mohr’s circle defined by the maximum and minimum principal normal stresses \(\sigma_1\) and \(\sigma_3\). The failure criterion (3.19) represents a straight line in the \(\tau-\sigma\) diagram (Fig. 3.5).
Using the relationships valid for a Mohr’s circle of stress, the failure criterion (3.19) may also be expressed in terms of the maximum and minimum principal normal stresses:

\[
\sigma_1 = \frac{1 + \sin \varphi_{IR}}{1 - \sin \varphi_{IR}} \sigma_3 + \frac{2c_{IR} \cdot \cos \varphi_{IR}}{1 - \sin \varphi_{IR}}.
\] (3.20)

In the \(\sigma_1-\sigma_3\) diagram this equation is represented by a straight line as well (Fig. 3.5). The point of intersection on the \(\sigma_1\) axis \((\sigma_3 = 0)\) represents the uniaxial or unconfined compressive strength \(\sigma_{cIR}\):

\[
\sigma_{cIR} = \frac{2c_{IR} \cdot \cos \varphi_{IR}}{1 - \sin \varphi_{IR}}.
\] (3.21)

The description of the shear strength in the \(\tau-\sigma\) diagram or the \(\sigma_1-\sigma_3\) diagram by means of a straight line at times does not accord with the results of triaxial compression tests with \(\sigma_1 > \sigma_2 = \sigma_3\) on intact rock (Section 14.4.2) carried out with high \(\sigma_3\) values. As an example, Fig. 3.6 illustrates the results of triaxial compression tests on granite (Franklin & Hoek 1970). These were adjusted by a curved failure line in a \(\sigma_1-\sigma_3\) diagram.

Such deviations from linearity have led to the development of a number of nonlinear failure criteria. Among these, the best known criterion is the so-called “Hoek-Brown criterion” (Hoek & Brown 1980a):

\[
\sigma_1 = \sigma_3 + \sigma_{cIR} \cdot \sqrt{m_i \cdot \frac{\sigma_3}{\sigma_{cIR}}} + s,
\] (3.22)

where \(m_i\) and \(s\) are parameters that are determined by matching the theoretical failure curve obtained from (3.22) with the test results.

Critical stress states that may lead to an exceeding of the shear strength of the intact rock occur, if at all, in the area of the unsupported excavation contour of a tunnel, that is, along the circumference of an unsupported tunnel in areas near the surface of a slope or at the ground surface near foundations on rock. In these areas the minimum principal normal stress \(\sigma_3\) is relatively small and the stress state in the rock approaches that of a uniaxial or biaxial stress state. As shown in Wittke (1990) the deviation of the Hoek-Brown criterion (3.22) from a straight line is relatively small in cases where the minimum principal normal stress levels are low to moderate. Thus, the Mohr-Coulomb failure criterion affords sufficient accuracy for practical applications.

To check the hypothesis that the shear failure criterion is independent of the intermediate principal normal stress \(\sigma_2\), tests in which the three principal normal stresses are different from each other, i.e. \(\sigma_1 > \sigma_2 > \sigma_3\), referred to as “true triaxial tests” or “polyaxial tests”, have been carried out on different types of intact rock (Akai & Mori 1970, Mogi 1971, Michelis 1985b, Michelis 1987, Takahashi & Koide 1989, Chang & Haimson 2000, Haimson & Chang 2000, Colmenaries & Zoback 2002, Al-Ajni & Zimmermann 2005, Haimson 2006, You 2009). In some of these tests a marked influence of the intermediate principal normal stress \(\sigma_2\) on shear strength, especially at high minimum principal normal stress levels \(\sigma_3\), was observed. These results and theoretical considerations
brought about the formulation of polyaxial failure criteria that consider the influence of $\sigma_3$ on intact rock failure (Drucker & Prager 1952, Murrell 1963, Mogi 1967, Mogi 1971, Ewy 1999, Haimson & Chang 2000, Kulatilake et al. 2006, You 2009). However, the evaluation of test data with different failure criteria led to the result that polyaxial failure criteria do not, in every case, give a better fit of test data than the Mohr-Coulomb criterion or the Hoek-Brown criterion (Colmenaries & Zoback 2002). Thus, in most cases, the shear strength of isotropic intact rocks at least at low to moderate $\sigma_2$ levels, which are relevant for rock engineering structures, can reasonably be described by failure criteria that are independent of $\sigma_2$.

For tensile failure it is assumed that cracking of isotropic intact rock occurs perpendicularly to the direction of the minimum principal normal stress $\sigma_3$ if the latter exceeds the tensile strength $\sigma_{tIR}$. Thus, the criterion for tensile failure can be represented in the $\tau$-$\sigma$ diagram as well as in the $\sigma_1$-$\sigma_3$ diagram by a vertical line (Fig. 3.5)

$$\sigma_3 = -\sigma_{tIR}.$$ (3.23)
On the right-hand side of this straight line the criterion for shear failure applies while stress states corresponding to a point on or left of this straight line lead to a tensile failure. This combination of criteria for tensile and shear failure is referred to as the “tension cutoff criterion”.

The Hoek-Brown criterion for tensile strength of intact rock is obtained by putting $\sigma_1 = 0$ in (3.22) and solving the resultant equation for $\sigma_3 = -\sigma_{tIR}$ (Hoek & Brown 1980a):

$$\sigma_{tIR} = \frac{\sigma_{eIR}}{2} \cdot \left( \sqrt{m_1^2 + 4s_{\text{IR}}} \right).$$  \hspace{1cm} (3.24)

The tensile strength of intact rock is usually assumed to be in the order of $1/10$ of the unconfined compressive strength. This assumption is motivated by the two-dimensional theory of brittle failure from Griffith (1921) predicting that the unconfined compressive strength is $8$ times the tensile strength, and its three-dimensional extension by Murrell (1963) leads to the result that the unconfined compressive strength is $12$ times the tensile strength.

**Shear and tensile strength of intact rocks with planar grain structure**

The Mohr-Coulomb failure criterion is also used to describe the shear strength on the isotropic plane of intact rocks with planar grain structure. It is formulated for the resultant shear stress $\tau_{\text{res}}$ in this plane and the corresponding normal stress $\sigma_n$ acting on this plane:

$$\tau_{\text{res}} = \sigma_n \cdot \tan \varphi + c.$$  \hspace{1cm} (3.25)

In the general case $\sigma_n$ and $\tau_{\text{res}}$ must be determined from the three-dimensional stress state described in the global coordinate system $(x,y,z)$. To this end a transformation of the stress vector $\{\sigma\}$ in the global coordinate system $(x,y,z)$ into the coordinate system related to the orientation of the isotropic plane $(x',y',z')$ according to (3.6) must be carried out. Thus, $\sigma_n$ and $\tau_{\text{res}}$ are functions of $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$ and $\tau_{zx}$ (Wittke 1990).

In the special case that the two principal normal stresses $\sigma_1$ and $\sigma_3$ are vertically and horizontally oriented, respectively, and lie within the plane perpendicular to the isotropic plane, which is inclined at an angle $\beta$, the stress components $\sigma_n$ and $\tau_{\text{res}}$ are functions of $\sigma_1$, $\sigma_3$ and $\beta$:

$$\sigma_n(\beta) = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} \cdot \cos (180^\circ \cdot 2\beta)$$

$$= \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \cdot \cos 2\beta,$$  \hspace{1cm} (3.26)

$$\tau_{\text{res}}(\beta) = \frac{\sigma_1 - \sigma_3}{2} \cdot \sin (180^\circ \cdot 2\beta)$$

$$= \frac{\sigma_1 - \sigma_3}{2} \cdot \sin 2\beta.$$  \hspace{1cm} (3.27)
It should be noted that the shear strength of intact rocks with planar grain structure is no longer independent of the intermediate principal normal stress \( \sigma_2 \). If, for example, the principal normal stresses \( \sigma_1 \) and \( \sigma_2 \) lie within the plane perpendicular to the isotropic plane failure may also take place in the isotropic plane. In such cases, in (3.26) and (3.27) \( \sigma_3 \) must be replaced by \( \sigma_2 \).

For a tensile failure normal to the isotropic plane the following failure criterion, as in (3.23), is used:

\[
\sigma_n = -\sigma_{ts}.
\] (3.28)

The combination of failure criteria for shear and tension is illustrated in Fig. 3.7.

![Figure 3.7 Criteria for shear and tensile failure in the isotropic plane S of an intact rock with planar grain structure (Wittke et al. 2006)](image)

The anisotropic shear strength of intact rock with planar grain structure can be described by means of Equation (3.20) valid for the intact rock matrix and by transforming \( \tau_{res} \) and \( \sigma_n \) in (3.25) into \( \sigma_1 \) and \( \sigma_3 \) using (3.26) and (3.27) and solving for \( \sigma_1 \). Thus, the maximum principal normal stress at failure is given by:

\[
\sigma_1 = \min \left\{ \frac{\sin(2\beta - \phi_S) + \sin \phi_S \cdot \sigma_3 + 2c_S \cdot \cos \phi_S}{\sin(2\beta - \phi_S) - \sin \phi_S}, \frac{(1 + \sin \phi_{IR}) \cdot \sigma_3 + 2c_{IR} \cdot \cos \phi_{IR}}{1 - \sin \phi_{IR}} \right\}. \tag{3.29}
\]

This model was first proposed by Jaeger (1960) and referred to as “single plane of weakness theory”. This model leads to a marked anisotropy of the unconfined compressive strength of an intact rock with planar grain structure as shown in Fig. 3.8, where a schistose rock is uniaxially loaded by a stress \( \sigma_z \) with different orientations with respect to the schistosity. The angle of inclination of the schistosity to the horizontal is denoted by \( \beta \). Setting \( \sigma_1 = \sigma_z \) and \( \sigma_3 = 0 \) in (3.29) yields the following expression for shear failure on the schistosity:

\[
\sigma_z = \frac{2c_S \cdot \cos \phi_S}{\sin(2\beta - \phi_S) - \sin \phi_S}. \tag{3.30}
\]
Figure 3.8 Uniaxial compressive strength of schistose intact rock as a function of direction of loading with respect to the schistosity (Wittke 1990)

Relationship (3.30) is illustrated in Fig. 3.8 for $\varphi = 30^\circ$ and $c_S = 0.1$ MPa by the red line. Differentiation of (3.30) with respect to $\beta$ reveals that the minimum strength occurs at $\beta = 45^\circ + \varphi/2$. Thus, when $\varphi = 30^\circ$ the minimum strength is obtained at $\beta = 60^\circ$.

$\sigma_z$ increases beyond all limits for angles of $\beta = \varphi$ and $\beta = 90^\circ$ and is negative for angles of $\beta < \varphi$. This means that uniaxial loading in these directions of dip is unable to produce shear failure on the schistosity. The shear strength for these dip directions is equal to the shear strength of the intact rock matrix that is illustrated in Fig. 3.8 by the blue line. This failure criterion in the case of uniaxial loading $\sigma_1 = \sigma_z$ and $\sigma_3 = 0$ in consideration of (3.29) is

$$
\sigma_z = \frac{2c_{IR} \cdot \cos \varphi_{IR}}{1 - \sin \varphi_{IR}}.
$$

In Fig. 3.8 the shear parameters of the intact rock matrix are assumed as $\varphi_{IR} = 45^\circ$ and $c_{IR} = 0.2$ MPa. It should also be noted that for angles $\beta$, which are slightly smaller than $90^\circ$ or are slightly greater than $\varphi$, according to (3.29) shear failure is predicted to occur in the intact rock matrix.

The shear strength of anisotropic intact rocks under uniaxial and triaxial compression has been investigated by a large number of laboratory tests on natural and artificial
3.2 Intact Rock

rocks (Donath 1964, McLamore & Gray 1967, Nova 1980, Niandou et al. 1997, Duveau et al. 1998, Tien & Tsao 2000, Tien et al. 2006). To obtain a better fit of the test results the original single plane of weakness theory was modified (McLamore & Gray 1967, Nova 1980, Duveau & Shao 1998, Tien & Kuo 2001). Recently also the nonlinear Hoek-Brown criterion was modified to apply to anisotropic intact rocks (Saroglou & Tsiambaos 2008). However, the benefit of such refinements is questionable because of the inhomogeneity of test specimens with respect to mineralogical composition, which usually leads to a large scatter of test results.

The tensile strength of intact rock with planar grain structure can be described correspondingly to (3.29) with the aid of Equations (3.23) and (3.26) when setting $s_n(\beta) = -\sigma_{\text{IS}}$ in (3.26) and solving for $\sigma_3$:

$$\sigma_3 = \max \left\{ \frac{2\sigma_{\text{IS}} + \sigma_1 - (1 + \cos 2\beta)}{1 - \cos 2\beta}, -\sigma_{\text{IR}} \right\}$$

Alternative formulations for the tensile strength of anisotropic intact rocks are given, for example, by Barron (1971), Nova & Zaninetti (1990) and Liao et al. (1997).

3.2.3 Post-Failure Behavior

**Intact rocks with random grain structure**

In Fig. 3.9 (upper left) a realistic stress-strain curve for uniaxial compressive loading also referred to as “complete stress-strain curve” is represented. Strength after failure (residual strength) is usually lower than strength at failure (peak strength). If strength after failure drops to very low values or to zero we are talking about brittle behavior. Otherwise we are talking about ductile behavior. Ductile behavior is typical, for example, for argillaceous rocks and salt rocks. Most intact rocks, however, exhibit brittle behavior at low confining stress with a gradual transition to ductile behavior at high confining stresses that virtually eliminates microfracturing.

The stress-strain curve in the model used within this book is idealized by an elastic-viscoplastic stress-strain curve illustrated in Fig. 3.9 (upper right). Stresses $\sigma$ below the unconfined compressive strength of intact rock $\sigma_{\text{cIR}}$ only lead to elastic strains $\varepsilon_{\text{el}}$ that are independent of time and proportional to the stresses (green line in Fig. 3.9). If $\sigma_{\text{cIR}}$ is reached, the stress in case of brittle behavior instantaneously drops to low values or to zero (dashed red line in Fig. 3.9). In case of ductile behavior the stress either maintains its level or drops down to a lower stress level (continuous and dashed-dotted red lines in Fig. 3.9). In the post-failure domain, inelastic irreversible strains $\varepsilon_{\text{vp}}$ occur that increase with time (blue line in Fig. 3.9, lower left) and are referred to as “viscoplastic strains”.

To illustrate elastic-viscoplastic stress-strain behavior, the rheological model represented in Fig. 3.9 (lower right) may be used. This one-dimensional model is composed of a spring, a so-called “Hooke element”, to describe elastic behavior, followed by a dashpot and a sliding element, referred to as the “Newton element” and “Saint-Venant element”, respectively. The latter are arranged in parallel to each other forming a so-called “Bingham body”. The dashpot element describes delayed (viscoplastic) strain. The sliding element consists of two blocks in contact with each other. They can only be displaced relative to each other when the shear stress acting in the contact surface exceeds $\sigma_{\text{cIR}}$. The displacements at the sliding element and the resulting strains are irreversible.

According to the Mohr-Coulomb failure criterion, the residual unconfined compressive strength $\sigma_{\text{cIR}}$ can be expressed as with (3.21):

$$\sigma_{\text{cIR}} = \frac{2c_{\text{IR}}^* \cdot \cos \varphi_{\text{IR}}^*}{1 - \sin \varphi_{\text{IR}}^*},$$  \hspace{1cm} (3.33)

where $c_{\text{IR}}^*$ and $\varphi_{\text{IR}}^*$ are the residual values of cohesion and friction angle. In case of a tensile failure the residual tensile strength $\sigma_{\text{tIR}}$ can normally be assumed to be zero. In
the following we will distinguish between peak strength described by the strength parameters $\varphi_{IR}, c_{IR}$ and $\sigma_{IR}$ and residual strength characterized by $\varphi_{IR}^*, c_{IR}^*$ and $\sigma_{IR}^* = 0$.

When describing intact rock as an isotropic ideally viscoplastic solid the derivations of viscoplastic strain components with respect to time $t$ referred to as “strain rates” $\{\dot{\varepsilon}^{vp}\}$ are defined according to Perzyna (1966) as follows:

$$\left\{ \frac{d\varepsilon^{vp}}{dt} \right\} = \{\dot{\varepsilon}^{vp}\} = \frac{1}{\eta_{IR}} \cdot \langle F_{IR} \rangle \cdot \frac{\partial Q_{IR}}{\partial [\sigma]},$$

(3.34)

where

$$\langle F_{IR} \rangle = \begin{cases} 0 & \text{if } F_{IR} < 0 \\ F_{IR} & \text{if } F_{IR} \geq 0 \end{cases}.$$

$F_{IR}$ and $Q_{IR}$ are the so-called “yield function” and “plastic potential”, respectively, and $\eta_{IR}$ is denoted as the “viscosity”. Equation (3.34) is referred to as the “flow rule”. The yield function $F_{IR}$ is related to the failure criterion. The Mohr-Coulomb failure criterion (3.19), for example, can be formulated by a yield function $F_{IR}$ as follows:

$$F_{IR} = \tau \cdot \sigma \cdot \tan \varphi_{IR} - c_{IR}$$

(3.35)

or alternatively

$$F_{IR} = \frac{1}{2} \cdot \sigma_1 \cdot (1 - \sin \varphi_{IR}) - \frac{1}{2} \cdot \sigma_3 \cdot (1 + \sin \varphi_{IR}) \cdot c_{IR} \cdot \cos \varphi_{IR}.$$

(3.36)

If the peak shear strength is exceeded, the viscoplastic strain rate $\{\dot{\varepsilon}^{vp}\}$ according to (3.34) is defined as:

$$\{\dot{\varepsilon}^{vp}\} = \frac{1}{\eta_{IR}} \cdot F_{IR}^* \cdot \frac{\partial Q_{IR}}{\partial [\sigma]} \quad \text{if } F_{IR} > 0.$$

(3.37)

The yield function $F_{IR}^*$ is formulated as a function of the residual strengths parameters $\varphi_{IR}^*, c_{IR}^*$ and $\sigma_{IR}^*$, instead of the peak strength parameters $c_{IR}, \varphi_{IR}$ and $\sigma_{IR}$. This means that after exceeding the peak strength ($F_{IR} > 0$) an instantaneous reduction of strength down to the residual strength takes place (Fig. 3.9, upper right).

The plastic potential $Q_{IR}$ is defined as with the function $F_{IR}$:

$$Q_{IR} = \frac{1}{2} \cdot \sigma_1 \cdot (1 - \sin \psi_{IR}) - \frac{1}{2} \cdot \sigma_3 \cdot (1 + \sin \psi_{IR}) \cdot c_{IR} \cdot \cos \psi_{IR}.$$

(3.38)

The so-called dilatancy angle $\psi_{IR}$ specifies the volume increase or loosening of the intact rock when the peak shear strength is exceeded. When $\psi_{IR} = \varphi_{IR}$ the plastic potential $Q_{IR}$ is equal to $F_{IR}$. Then we are talking about an “associated flow rule”. When an associated
flow rule is applied, the viscoplastic volumetric strain is often overestimated. Therefore, the dilatancy angle is usually chosen to be smaller than $\phi_{IR}$. Then we are talking about a “non-associated flow rule”. If $\psi_{IR} = 0$ no volume increase of the intact rock occurs after peak shear strength is exceeded. Thus, the dilatancy angle has a marked influence on the viscoplastic strains and also on the stresses if volumetric strain is confined.

The dilatancy angle, among others, is dependent on the stress level and the viscoplastic shear strain (Detournay 1986, Lai (2002), Yuan & Harrison 2004, Alejano & Alonso 2005, Chen et al. 2007). At large shear deformation and high confining stress therefore, as a rule, volumetric viscoplastic strain will be overestimated when using a constant dilatancy angle which is not equal to zero. However, for reasons already discussed in Section 3.2.2 the range of high confining stress is not very relevant for civil engineering structures in rock, and large deformations are usually not admitted in rock engineering. However, if in the surrounding of a tunnel or underground opening the strength of the intact rock is exceeded in a large area, major viscoplastic deformations may occur. In such a case it may be useful to assume zero dilatancy angle after a certain viscoplastic volumetric strain has occurred. Such conditions are referred to as “squeezing rock conditions” (Chapter 4).

If in squeezing rock the rock mass pressure cannot be carried by an immediate installation of the support a yielding support is required (Section 4.3). In such cases large time dependent displacements are allowed to occur and the viscosity $\eta_{IR}$ of the intact rock is of importance for evaluation of the time dependency of the stresses and strains.

Tensile failure in isotropic intact rock is described by an associated flow rule:

$$Q_{IR} = F_{IR} = -\sigma_3 - \sigma_{3IR}. \quad (3.39)$$

**Intact rocks with planar grain structure**

If in case of an intact rock with planar grain structure the peak strength is exceeded on the isotropic plane the viscoplastic strain rate is calculated as

$$\dot{\varepsilon}^{vp} = \frac{1}{\eta_S} \cdot F^*_S \cdot \frac{\partial Q_S}{\partial \{\sigma\}} \quad \text{if} \quad F_S > 0 \quad (3.40)$$

with

$$F^*_S = \tau_{res} \cdot \sigma_n \cdot \tan \varphi_S - c_S^* \quad (3.41)$$

and

$$Q_S = \tau_{res} \cdot \sigma_n \cdot \tan \psi_S - c_S \quad (3.42)$$

in the case of a shear failure and

$$Q_S = F^*_S = -\sigma_n - \sigma_{IS} \quad (3.43)$$

in the case of a tensile failure.
If the intact rock's strength is exceeded in the rock matrix as well as on the isotropic plane the strain rates according to (3.37) and (3.40) have to be superimposed:

\[
\dot{\varepsilon}_{\text{vp}}^1 = \frac{1}{\eta_{IR}} \cdot F_{IR}^* \cdot \frac{\partial Q_{IR}}{\partial \sigma} + \frac{1}{\eta_S} \cdot F_{S}^* \cdot \frac{\partial Q_S}{\partial \sigma}. \tag{3.44}
\]

To calculate \( \dot{\varepsilon}_{\text{vp}}^1 \) in (3.37), (3.40) and (3.44) \( Q_{IR} \) and \( Q_S \) need to be expressed as functions of the stress components \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz} \) and \( \tau_{zx} \) of the global coordinate system (Wittke 1990).

### 3.2.4 Intact Rocks with Deviations from Elastic-Viscoplastic Stress-Strain Behavior

The assumption that the stress-strain behavior of intact rock below strength may be regarded as linear elastic and independent of time of course represents an idealization. However, a very large number of rock types can be described reasonably well by this idealization provided that consideration is restricted to the stresses and temperatures usually arising in rock engineering. Deviation from linear elastic stress-strain behavior that is not time-dependent may nonetheless occur, particularly in weak or soft rocks such as argillaceous rocks. Considerable differences between Young's moduli for initial loading, unloading and reloading may be observed in laboratory tests conducted on such rock types. Also the strains measured during phases of constant load may indicate time-dependent behavior prior to failure (Section 14.4).

Intact rocks, even though brittle, tend towards ductile behavior at high confining stress. When ductile behavior becomes dominant, in some cases the strength increases with increasing strain which is known as “strain hardening” or “work hardening” (Singh 1989). Hardening may also cause an initially isotropic rock to become anisotropic (Celle & Cheatham 1981). However, as already mentioned in Section 3.2.2 the range of high confining stress is usually not too relevant for rock engineering problems.

Deformability and strength of intact rocks are also dependent on stress history. Cyclic loading and unloading often causes intact rock to fail at a stress level lower than its compressive strength determined in static compression tests (Rao et al. 1985, Ray et al. 1999). This lower stress level is often denoted as “fatigue stress” or “long-term strength”. On the other hand, it has been found that Young’s modulus and strength of intact rocks increase considerably with increasing stress rate and strain rate (Cristescu & Hunsche 1998, Ray et al. 1999, Li & Xia 2000). Also the deformation at failure shows a marked dependence on stress rate or strain rate (Cristescu & Hunsche 1998). In addition, it has been observed that intact rock behaves more brittlely at higher strain rates (Ray et al. 1999). However, under quasi-static loading conditions these effects in most cases are negligible.

In certain rock types under constant stress levels below strength time-dependent deformation takes place which is referred to as “creep” (Chapter 5). Another feature of time-dependent behavior is a change of stress state under constant deformation referred to as “relaxation”.
3.3 Discontinuities

3.3.1 Types of Discontinuities

Discontinuities have a considerable influence on the deformability and strength of a rock mass. Fig. 3.10 shows some types of discontinuities that are frequently encountered in nature. Their appearance is described in qualitative terms given in Section 2.7.2 (Fig. 2.31). With regard to the stress-displacement behavior, which will be treated in the following sections, an estimation of the degree of persistence of a discontinuity is also of considerable importance (for a definition, see Section 2.7.2 and Fig. 2.32).

![Discontinuity Types Diagram]

Figure 3.10 Appearance of discontinuities (Wittke 1990)

3.3.2 Stress-Displacement Behavior of Discontinuities Loaded by a Normal Compressive Stress

Largely closed discontinuities do not deform under a compressive stress applied normal to their surfaces regardless of their surface properties, that is, if they are planar, uneven, smooth or rough, and if they are persistent or non-persistent.

Discontinuities with rough or uneven surfaces are normally not fully closed. As a result, an applied normal compressive stress $\sigma_n$ causes localized stress concentrations and plastic zones in the neighboring rock mass (Fig. 3.11, upper). The resulting irreversible deformations lead to a partial closure of the discontinuity. Experimental studies have shown that the relationship between the applied normal compressive stress $\sigma_n$ and the mean closure $\delta_{n,D}$ of such a discontinuity is nonlinear (Goodman 1976, Bandis 1980, Witherspoon et al. 1980, Bandis et al. 1983, Swan 1983, Malama & Kulatilake 2003).
3.3 Discontinuities

Figure 3.11 Idealized stress-displacement behavior of a rough or uneven, not fully closed discontinuity, loaded by a normal compressive stress (Wittke 2003)

The most common description of the closure of rough discontinuities is the hyperbolic approach (Bandis 1980):

\[
\delta_{h,D} = \frac{\sigma_n \cdot \delta_m}{K_{n0} \cdot \delta_m + \sigma_n}
\]  

(3.45)

\(K_{n0}\) is referred to as the initial normal stiffness of the discontinuity and \(\delta_m\) is the maximum potential closure of the discontinuity (Fig. 3.11, lower).

According to Bandis (1980), the behavior of discontinuities during unloading is markedly inelastic and the unloading curve can also be described by a hyperbola. Since the reversible, elastic part of the displacement taking place during unloading and reloading in most cases is small compared to the irreversible part of the displacement, the closure of the discontinuity may be considered as irreversible (Fig. 3.11, lower).
However, natural discontinuities in hard rocks such as granite show nonlinear but elastic stress-displacement behavior under cycling loading (Pyrak-Nolte et al. 1987). This can be explained by purely elastic closure and opening of asperities resulting from an increase and decrease of contact areas during cyclic loading (Myer 2000).

The deformability of a filled discontinuity loaded by a normal compressive stress is determined by the compressibility of the filling material.

### 3.3.3 Strength and Failure Criteria

**Shear strength of discontinuities**

The shear strength of persistent, planar discontinuities can be described exclusively by friction and may be approximated by a linear relationship between $\tau_{\text{res}}$ and $\sigma_n$:

$$\tau_{\text{res}} = \sigma_n \cdot \tan \varphi_D. \quad (3.48)$$

The friction angle $\varphi_D$ may be the same or lower than that of the intact rock due to weathering or greasy films on the discontinuity’s surfaces. The residual friction angle $\varphi_D^*$ of persistent, planar and closed discontinuities can be assumed to be equal to $\varphi_D$.

On uneven discontinuities two modes of shear failure may occur as illustrated in Fig. 3.12 by the example of a regular sawtoothed discontinuity. At low normal compressive stress $\sigma_n \leq \sigma_{ns}$ a sliding on the discontinuity surfaces takes place under the shear loading $\tau_{\text{res}}$ (Fig. 3.12, left upper). Consideration of the ultimate state of equilibrium provides the following peak shear strength:

$$\tau_{\text{res}} = \sigma_n \cdot \tan (\varphi_D' + i) = \sigma_n \cdot \tan \varphi_D \quad (3.49)$$

in which $i$ is the sliding-up angle and $\varphi_D'$ is the friction angle of the discontinuity surfaces (Fig. 3.12). If the normal stress exceeds a certain value $\sigma_{ns}$, instead of a sliding on the discontinuity surfaces, the failure takes place in the intact rock by shearing off the sawtoothed asperities (Fig. 3.12, upper right). Both the friction and cohesion of the intact rock have to be overcome to initiate this shearing process. Thus, the peak shear strength for $\sigma_n > \sigma_{ns}$ is given by

$$\tau_{\text{res}} = \sigma_n \cdot \tan \varphi_{IR'} + c_{IR'}, \quad (3.50)$$

where $\varphi_{IR'}$ and $c_{IR'}$, due to weathering, may take on smaller values than $\varphi_{IR}$ and $c_{IR}$ for the unweathered intact rock. Combination of (3.49) and (3.50) leads to the bilinear failure criterion for peak shear strength illustrated in Fig. 3.12 (lower) as was first suggested by Patton (1966).
As regards the residual strength $\tau_{res}^*$ of uneven or rough persistent discontinuities, a distinction must also be made between shear failure due to sliding and that caused by shearing off the asperities. At low normal stress $\sigma_n < \sigma_{ns}$, no reduction of strength during shearing takes place and the residual friction angle $\phi_D^*$ may be considered to be equal to $\phi_D$. At high normal stress $\sigma_n \geq \sigma_{ns}$, the residual friction angle $\phi_D^*$ can be assumed equal to the residual friction angle $\phi_{IR}^*$ of the sheared off rock material (Fig. 3.12, lower). As a result, the residual shear strength of an uneven or rough discontinuity can be described by a discontinuous failure criterion (Wittke 1990):
Naturally appearing discontinuities with uneven surfaces usually exhibit highly irregular shapes and sliding-up angles which may vary considerably from place to place. Thus, nonlinear failure criteria were formulated (Ladanyi & Archambault 1970, Barton 1973, Schneider 1975, Leichnitz 1981, Erban 1986, Saeb 1989, Maksimovic 1992, Haberfield & Johnston 1994, Kulatilake et al. 1995, Seidel & Haberfield 1995, Maksimovic 1996, Zhao 1997a, Zhao 1997b). Fig. 3.13 (blue line) shows qualitatively such a nonlinear criterion for peak shear strength in which the friction angle $\varphi_D$ is a function of $\sigma_n$:

\[
\tau_{\text{res}} = \sigma_n \cdot \tan \varphi_D (\sigma_n).
\] (3.52)

One of the most often used failure criteria for the shear strength of rough discontinuities is the following approach proposed by Barton (1973):

\[
\tau_{\text{res}} = \sigma_n \cdot \tan \left[ \text{JRC} \cdot \log \left( \frac{\text{JCS}}{\sigma_n} \right) + \varphi_{IR}^* \right].
\] (3.53)

JRC stands for “joint roughness coefficient”, which varies from 0 (completely smooth and planar) to 20 (rough and uneven). A qualitative scale of values for this parameter based on surface profiles suggested in ISRM (1978e) was presented by Barton & Choubey (1977).

JCS referred to as “joint wall compression strength” is defined as the unconfined compressive strength of discontinuity surfaces, depending on their degree of weathering. This parameter controls the stress level $\sigma_n$ on which the asperities on the discontinuity surfaces are sheared off. In the case of a completely smooth and planar discontinuity (JRC = 0) Equation (3.53) takes on the form of the linear $\tau_{\text{res}} - \sigma_n$ relationship (3.48). The residual friction angle of the discontinuity surfaces $\varphi_{IR}^*$, depending on the degree of weathering of the sheared off rock material, is also denoted as the “basic friction angle” (Barton 1973).

The residual shear strength of natural discontinuities with uneven surfaces may also be described by a discontinuous failure criterion as in the case of a regular sawtooth discontinuity (cf. Fig. 3.12). However, because the stress level $\sigma_{ns}$ defining the transition from the sliding mode to the shearing-off mode (blue dashed line in Fig. 3.13) cannot be clearly specified in practice, the residual shear strength is usually described by a linear criterion (red line in Fig. 3.13):

\[
\tau_{\text{res}}^* = \sigma_n \cdot \tan \varphi_{IR}^*.
\] (3.54)
Natural discontinuities in sandstone and slate often exhibit marked anisotropic roughness surfaces, as illustrated in Fig. 3.14. Slickensides, which are planar and smooth in the direction of movement and uneven and rough perpendicular to the same, are another example for anisotropic roughness (cf. Fig. 2.31). As a consequence, the peak shear strength of such discontinuities is dependent on the shearing direction (Huang & Doong 1990, Jing et. al. 1992, Kulatilake et al. 1995).
The shear strength of discontinuities with clayey, silty or sandy fillings whose aperture is considerably greater than the amplitude of their roughness or unevenness is largely determined by the peak and residual shear strength of the filling. These may be described in a simplified manner by Mohr-Coulomb failure criteria (Wittke 1990):

\[
\tau_{\text{res}} = \sigma_n \cdot \tan \varphi_F + c_F, \quad (3.55)
\]

\[
\tau_{\text{res}}^* = \sigma_n \cdot \tan \varphi_F^* + c_F^*, \quad (3.56)
\]

where \( \varphi_F, c_F, \varphi_F^* \) and \( c_F^* \) are the shear parameters for the peak and residual strengths of the discontinuity’s filling, respectively.

Ladanyi & Archambault (1977) found that the shear strength of rough discontinuities filled with clay and sandy silt decreases with increasing thickness of the filling. Not before the thickness exceeds the height of irregularities of the joint surfaces by 50% will the shear strength of the joints approach the shear strength of the filling. Consequently, (3.56) and (3.57) may be considered as lower bounds for the shear strength of rough, filled discontinuities.

In the following, the shear strength of a non-persistent discontinuity, that is, a discontinuity that is interrupted by rock bridges, will be discussed.

Fig. 3.15 (upper) shows a one-dimensional model of a non-persistent discontinuity with an infinite number of regularly ordered, open discontinuity sections separated by rock bridges. The open discontinuity sections are modeled by cracks with elliptic cross-sections with lengths of 2a and a maximum aperture of 2b. The length of the rock bridges is denoted with \( \ell \). To study the shear strength of such a discontinuity it is sufficient to consider only a section of length 2a + \( \ell \), as illustrated in Fig. 3.15 (upper) since the vertical boundaries of this section represent symmetry planes.

Generally the major semi-axes “a” of the cracks are large compared with the minor semi-axes “b”. In the following, the limiting case \( b \to 0 \), which is known as a Griffith crack, will be considered. The Griffith crack is assumed to be located in an infinite disc of an isotropic elastic continuum loaded at its boundaries by a normal stress \( \sigma_n \) and a shear stress \( \tau_{\text{res}} \), respectively. The stress distribution along the axis of the Griffith crack for plane strain conditions is then given by the following analytic solution (Stevenson 1945):

\[
\sigma_n(x) = \frac{\sigma_n}{\sqrt{x^2 - a^2}} \quad \text{if } x \geq a, \quad (3.57)
\]

\[
\tau_{\text{res}}(x) = \frac{\tau_{\text{res}}}{\sqrt{x^2 - a^2}} \quad \text{if } x \geq a. \quad (3.58)
\]

In (3.57) and (3.58) it is assumed that the origin of the coordinate x is located in the crack’s center. Consequently the stresses \( \sigma_n \) and \( \tau_{\text{res}} \) at the tips of the crack (\( x = a \)) are infinitely large (Fig. 3.15).
If the open discontinuity sections in the model represented in Fig. 3.15 are modeled as Griffith cracks, the stress distributions due to $\sigma_{n}$ and $\tau_{res}$ in the area of the rock bridges along the discontinuity’s plane are obtained by the superposition of the stress distributions of two adjacent Griffith cracks with interspace $l$:
\[ \sigma_{n,IR}(x) = \sigma_n \left( \frac{x}{\sqrt{x^2 - a^2}} + \frac{x'}{\sqrt{x'^2 - a^2}} \right) \text{ if } a \geq x \geq a + \ell, \quad (3.59) \]

\[ \tau_{res,IR}(x) = \tau_{res} \left( \frac{x}{\sqrt{x^2 - a^2}} + \frac{x'}{\sqrt{x'^2 - a^2}} \right) \text{ if } a \geq x \geq a + \ell, \quad (3.60) \]

where \( x' = 2a + \ell - x \).

This solution is qualitatively represented in Fig. 3.15. Along the rock bridges between the cracks the stresses \( \sigma_{n,IR} \) and \( \tau_{res,IR} \) are considerably larger than the applied stresses \( \sigma_n \) and \( \tau_{res} \). According to (3.59) and (3.60), in each point \( x \)

\[ \frac{\sigma_{n,IR}(x)}{\tau_{res,IR}(x)} = \frac{\sigma_n}{\tau_{res}} \quad (3.61) \]

applies. If the isotropic shear strength of the intact rock is assumed, the failure criterion for the rock bridges can thus be described by the local stresses \( \sigma_{n,IR} \) and \( \tau_{res,IR} \) as follows:

\[ \tau_{res,IR}(x) = \sigma_{n,IR}(x) \cdot \tan \varphi_{IR} + c_{IR}. \quad (3.62) \]

The maximum tangential force \( T \), which can be carried by a rock bridge, is obtained by integrating the admissible shear strength of the intact rock at failure \( \tau_{res,IR} \) over the length \( \ell \) of a rock bridge. In consideration of (3.61) it follows that:

\[ T = \int_{a}^{a+\ell} \tau_{res,IR}(x) \, dx \]

\[ = \tau_{res} \int_{a}^{a+\ell} \left( \frac{x}{\sqrt{x^2 - a^2}} + \frac{x'}{\sqrt{x'^2 - a^2}} \right) \, dx \]

\[ = \sigma_n \cdot \tan \varphi_{IR} \cdot \int_{a}^{u+\ell} \left( \frac{x}{\sqrt{x^2 - a^2}} + \frac{x'}{\sqrt{x'^2 - a^2}} \right) \, dx + c_{IR} \cdot \ell \quad (3.63) \]

Introducing the mean values of the normal and shear stress in the intact rock along the discontinuity’s plane

\[ \sigma_{n,IR} = \frac{1}{\ell} \int_{a}^{a+\ell} \sigma_{n,IR} \, dx = \sigma_n \int_{a}^{a+\ell} \left( \frac{x}{\sqrt{x^2 - a^2}} + \frac{x'}{\sqrt{x'^2 - a^2}} \right) \, dx \quad (3.64) \]

and
Equation (3.63) can be rewritten as:

\[
\tau = \tau_{\text{res,IR}} \cdot t = \bar{\sigma}_{n,\text{IR}} \cdot t \cdot \tan \phi_{\text{IR}} + c_{\text{IR}} \cdot t.
\]

or

\[
\tau_{\text{res,IR}} = \bar{\sigma}_{n,\text{IR}} \cdot \tan \phi_{\text{IR}} + c_{\text{IR}}. \tag{3.66}
\]

Thus, the local stresses \(\tau_{\text{res,IR}}(x)\) and \(\sigma_{n,\text{IR}}(x)\) in (3.62) can be replaced by the mean stresses \(\bar{\sigma}_{n,\text{IR}}\) and \(\bar{\tau}_{\text{res,IR}}\) existing in the intact rock along the discontinuity’s plane between two neighboring cracks.

To obtain relationships between the mean stresses \(\bar{\sigma}_{n,\text{IR}}\) and \(\bar{\tau}_{\text{res,IR}}\) in the intact rock and the applied mean stresses \(\bar{\sigma}_{n}\) and \(\bar{\tau}_{\text{res}}\) we must account for the fact that shear and normal stresses in the intact rock cannot be transmitted across the open discontinuity sections. It follows that

\[
\bar{\sigma}_{n} = \frac{\ell}{2a + \ell} \cdot \bar{\sigma}_{n,\text{IR}}, \tag{3.67}
\]

\[
\bar{\tau}_{\text{res}} = \frac{\ell}{2a + \ell} \cdot \bar{\tau}_{\text{res,IR}}. \tag{3.68}
\]

Inserting (3.66) and (3.67) into (3.68) yields:

\[
\bar{\tau}_{\text{res}} = \bar{\sigma}_{n} \cdot \tan \phi_{\text{IR}} + c_{\text{IR}} \cdot \frac{\ell}{2a + \ell}. \tag{3.69}
\]

In the case under consideration the linear degree of separation introduced in Section 2.7.2 can be expressed as

\[
k_{\ell} = \frac{2a}{2a + \ell} \tag{3.70}
\]

and thus

\[
1 - k_{\ell} = \frac{\ell}{2a + \ell}. \tag{3.71}
\]

Inserting (3.71) into (3.69) the following peak shear strength criterion for a one-dimensional non-persistent discontinuity formed by regularly ordered Griffith cracks separated by rock bridges is obtained:

\[
\bar{\tau}_{\text{res}} = \bar{\sigma}_{n} \cdot \tan \phi_{\text{IR}} + (1 - k_{\ell}) \cdot c_{\text{IR}}. \tag{3.72}
\]
If the shape of the open discontinuity sections deviates from that of Griffith cracks the normal and shear stress distributions along the discontinuity’s plane can be qualitatively described by the red and orange lines in Fig. 3.16; that is, finite stress values $\sigma_{n,IR}$ and $\tau_{res,IR}$ arise at the edges of the discontinuity sections. Also, the distributions of the local stresses $\sigma_{n,IR}$ and $\tau_{res,IR}$ may be different and Equation (3.61) does not apply anymore. Equations (3.67) and (3.68), however, are still valid. The failure criterion (3.72) is therefore also valid for one-dimensional, non-persistent discontinuities with irregular ordered open sections of differing lengths.

![Normal stress distribution](image1)

![Shear stress distribution](image2)

**Figure 3.16** Qualitative stress distribution along the rock bridges of a one-dimensional non-persistent discontinuity with open discontinuity sections
For a two-dimensional, non-persistent discontinuity with open sections Equation (3.72) can also be applied provided that the shear strength of the discontinuity is isotropic and the linear degree of separation $\kappa_l$ is replaced by the planar degree of separation $\kappa_p$ defined in Section 2.7.2:

$$\tau_{cs} = \bar{\tau}_n \cdot \tan \varphi_{IR} + (1 - \kappa_p) \cdot c_{IR}. \quad (3.73)$$

It has been observed in a large number of direct shear tests carried out on artificial open non-persistent discontinuities produced in specimens composed of both natural rock and model materials such as gypsum that failure in the material bridges occurs in two phases: tensile wing cracks were found to appear first at the tips of the open joints followed by secondary shear cracks propagating towards the adjacent joints (Lajtai 1969a, Lajtai 1969b, Savilahti et al. 1990, Jung et al. 1995, Wong et al. 1999, Gehle 2002, Gehle & Kutter 2003).

As an example, Fig. 3.17 shows these two phases of failure in rock bridges located between horizontal open cracks in a gypsum specimen (Gehle 2002). Thus, crack formation and propagation in the rock bridges generally occurs neither along the discontinuity’s plane nor by means of pure shear failure.

![Forming of wing cracks](image1)

![Failure after forming of secondary shear cracks](image2)

**Figure 3.17** Direct shear test on a gypsum specimen with horizontal open cracks (Gehle 2002)
The formation of wing cracks as well as the formation of secondary shear cracks could also be verified using numerical crack propagation models based on fracture mechanics (Gehle 2002, Gehle & Kutter 2003, Kemeny 2003, Kemeny 2005, Zhang et al. 2006).

Equation (3.73) can be validated by the results of direct shear tests on gypsum specimens with regularly ordered open joints carried out by Gehle (2002). One of these specimens is shown in Fig. 3.17. As can be seen in Fig. 3.18 the influence of both normal stress and the degree of separation on peak shear strength can be reproduced with good agreement, assuming shear parameters of $c_{IR} = 4.7$ MPa and $\phi_{IR} = 32^\circ$ for the gypsum material.

The linear relationship between peak shear strength and the degree of separation of non-persistent open discontinuities has also been proven numerically using the finite element code RFPA based on fracture mechanics (Zhang et al. 2006).

For a non-persistent discontinuity with closed separated discontinuity sections Equations (3.67) and (3.68) need to be replaced by:

$$\bar{\sigma}_{n,D} = \frac{\ell}{2a + \ell} \cdot \bar{\sigma}_{n,IR} + \frac{2a}{2a + \ell} \cdot \bar{\sigma}_{n,D} = (1 - \kappa_f) \cdot \bar{\sigma}_{n,IR} + \kappa_f \cdot \bar{\sigma}_{n,D}, \quad (3.74)$$

$$\bar{\tau}_{res,D} = \frac{\ell}{2a + \ell} \cdot \bar{\tau}_{res,IR} + \frac{2a}{2a + \ell} \cdot \bar{\tau}_{res,D} = (1 - \kappa_f) \cdot \bar{\tau}_{res,IR} + \kappa_f \cdot \bar{\tau}_{res,D}, \quad (3.75)$$

where $\bar{\sigma}_{n,D}$ and $\bar{\tau}_{res,D}$ are the mean values of the local normal and shear stresses $\sigma_{n,D}(x)$ and $\tau_{res,D}(x)$ along the closed separated discontinuity sections.

In this case the mean stresses $\bar{\sigma}_{n,IR}$ and $\bar{\sigma}_{n,D}$ can be considered to be approximately equal:

$$\bar{\sigma}_{n,IR} \approx \bar{\sigma}_{n,D} \approx \bar{\sigma}_n. \quad (3.76)$$

Assuming isotropic shear strength of the intact rock, the failure criterion for the rock bridges can be formulated by the mean stresses $\bar{\sigma}_{n,IR}$ and $\bar{\tau}_{res,IR}$ in accordance with (3.66).

If a Mohr-Coulomb failure criterion is assumed, the shear strength along the separated discontinuity sections that have no cohesion can be described as

$$\bar{\tau}_{res,D} = \bar{\sigma}_{n,D} \cdot \tan \phi_D, \quad (3.77)$$

where $\phi_D$ is the friction angle of the closed separated discontinuity sections. Inserting (3.66) and (3.77) into (3.75) yields:

$$\bar{\tau}_{res} = (1 - \kappa_f) \cdot \bar{\sigma}_{n,IR} \cdot \tan \phi_{IR} + \kappa_f \cdot \bar{\sigma}_{n,D} \cdot \tan \phi_D + (1 - \kappa_f) \cdot c_{IR}, \quad (3.78)$$

and from (3.76):

$$\bar{\tau}_{res} = (1 - \kappa_f) \cdot \bar{\sigma}_n \cdot \tan \phi_{IR} + \kappa_f \cdot \bar{\sigma}_n \cdot \tan \phi_D + (1 - \kappa_f) \cdot c_{IR}. \quad (3.79)$$
Figure 3.18  Peak shear strength of non-persistent open discontinuities
Replacing $\kappa_l$ by $\kappa_p$ yields:

$$\tau_{res} = (1 - \kappa_p) \cdot \sigma_n \cdot \tan \varphi_{IR} + \kappa_p \cdot \sigma_n \cdot \tan \varphi_D + (1 - \kappa_p) \cdot c_{IR}. \quad (3.80)$$

To describe the peak shear strength of non-persistent discontinuities an equation similar to (3.80) has been proposed by Robertson (1970) and Jennings (1970):

$$\tau_{res} = (1 - \kappa_p) \cdot [\sigma_n \cdot \tan \varphi_{IR} + c_{IR}] + \kappa_p \cdot [\sigma_n \cdot \tan \varphi_D + c_D]. \quad (3.81)$$

Relationships (3.80) and (3.81) are identical if $c_D = 0$, as we have presupposed in our derivation.

According to Wittke (1990), the residual shear strength of a non-persistent discontinuity can be described by the mean value of the residual shear resistances of the rock bridges and the persistent sections of the discontinuity weighted with the planar degree of separation $\kappa_p$:

$$\tau_{res}^* = (1 - \kappa_p) \cdot \sigma_n \cdot \tan \varphi_{IR}^* + \kappa_p \cdot \sigma_n \cdot \tan \varphi_D^*. \quad (3.82)$$

**Tensile strength of discontinuities**

The tensile strength of persistent discontinuities is generally zero. If a non-persistent discontinuity is loaded by a normal tensile stress and a shear stress, the strength should conservatively also be considered as negligible. The tensile strength of all types of discontinuities can therefore be assumed to be zero.

**Summary and conclusion**

In Fig. 3.19 the failure criteria for peak shear strength, residual shear strength and tensile strength for discontinuities of different appearance are summarized.

It can be concluded that the strength of discontinuities is significantly smaller than that of the intact rock. As an example, Fig. 3.20 shows the shear strength of a rock block that exhibits two mutually perpendicular discontinuity sets D1 and D2 and is loaded by the principal normal stresses $\sigma_1$ and $\sigma_3$, as a function of the direction of loading with respect to the discontinuities. Of these, $\sigma_3$ is defined by the angles of dip $\beta_1$ and $\beta_2 = 90^\circ - \beta_1$ of D1 and D2, respectively, which are varied between $0^\circ$ and $90^\circ$. The maximum principal normal stress $\sigma_1$, which leads to a shear failure of the intact rock and/or on the discontinuities, is obtained as a function of $\beta_1$, $\beta_2$, the shear parameters of the intact rock ($c_{IR}$ and $\varphi_{IR}$) and those of the two discontinuity sets ($c_{D1}$, $\varphi_{D1}$, $c_{D2}$ and $\varphi_{D2}$) from the following equation (Wittke 1990):
3.3 Discontinuities

\[
\sigma_1 = \text{Minimum} \left[ \frac{\left( \sin(2\beta_1 - \varphi_{D1}) + \sin \varphi_{D1} \right) \cdot \sigma_3 + 2c_{D1} \cdot \cos \varphi_{D1}}{\sin(2\beta_1 - \varphi_{D1}) \cdot \sin \varphi_{D1}} \right] \\
\frac{\left( \sin(2\beta_2 - \varphi_{D2}) + \sin \varphi_{D2} \right) \cdot \sigma_3 + 2c_{D2} \cdot \cos \varphi_{D2}}{\sin(2\beta_2 - \varphi_{D2}) \cdot \sin \varphi_{D2}} \cdot \frac{1 + \sin \varphi_{IR} \sigma_3 + 2c_{IR} \cdot \cos \varphi_{IR}}{1 - \sin \varphi_{IR}}
\]

(3.83)

**Figure 3.19** Failure criteria for discontinuities of different appearance

This relationship is illustrated in Fig. 3.20, for a given set of shear parameters and three different values of \(\sigma_3\), in the form of a polar diagram. The plot clearly indicates that shear failure on intact rock only takes place when \(\beta_1 = 90^\circ - \beta_2\) is either approximately

<table>
<thead>
<tr>
<th>Persistent discontinuities</th>
<th>Non-persistent discontinuities</th>
</tr>
</thead>
<tbody>
<tr>
<td>largely planar</td>
<td>filled</td>
</tr>
<tr>
<td>filled</td>
<td>uneven/rough</td>
</tr>
<tr>
<td></td>
<td>discontinuity sections open</td>
</tr>
<tr>
<td></td>
<td>discontinuity sections closed</td>
</tr>
<tr>
<td></td>
<td>((\varphi_D &lt; \varphi_{IR}))</td>
</tr>
</tbody>
</table>

\[
\sigma_n = \sigma_n \cdot \tan \varphi_D \quad \text{bilinear: if } \sigma_n < \sigma_{n0}:
\]

\[
= \sigma_n \cdot \tan \varphi_D
\]

\[
= \sigma_n \cdot \tan \varphi_{IR} + \sigma_{n0}
\]

\[
= (1 - \kappa_p) \cdot \sigma_n \cdot \tan \varphi_{IR} + \kappa_p \cdot \sigma_n \cdot \tan \varphi_D
\]

\[
\sigma_n = \sigma_n \cdot \tan \varphi_D \quad \text{or nonlinear: if } \sigma_n \approx \sigma_{n0}:
\]

\[
= \sigma_n \cdot \tan \varphi_D
\]

\[
= \sigma_n \cdot \tan \varphi_{IR} + \sigma_{n0}
\]

\[
= (1 - \kappa_p) \cdot \sigma_n \cdot \tan \varphi_{IR} + \kappa_p \cdot \sigma_n \cdot \tan \varphi_D
\]

\[
\sigma_n = \sigma_n \cdot \tan \varphi_D \quad \text{or continuous: if } \sigma_n \approx \sigma_{n0}:
\]

\[
= \sigma_n \cdot \tan \varphi_D
\]

\[
= \sigma_n \cdot \tan \varphi_{IR}
\]

\[
= (1 - \kappa_p) \cdot \sigma_n \cdot \tan \varphi_{IR} + \kappa_p \cdot \sigma_n \cdot \tan \varphi_D
\]

\[
\sigma_n = \sigma_n \quad \text{or continuous: if } \sigma_n \approx \sigma_{n0}:
\]

\[
\sigma_n = \sigma_n
\]

\[
\sigma_n = \sigma_n
\]

\[
= 0
\]

\[
= 0
\]

\[
= 0
\]

\[
= 0
\]

\[
= 0
\]
0° or 90°, that is, when the discontinuities are not subjected to shear stresses. In most other directions of loading, shear failure occurs on discontinuities, resulting in a marked reduction of shear strength.

Figure 3.20 Maximum principal normal stress $\sigma_1$ at failure for a rock mass with two mutually perpendicular discontinuity sets (Wittke et al. 2006)

### 3.3.4 Stress-Displacement Behavior of Discontinuities Loaded by a Normal Stress and a Shear Stress

The stress-displacement behavior of persistent discontinuities loaded by a normal compressive stress and a shear stress according to the results of direct shear tests (Barton 1973, Barton & Choubey 1977, Bandis 1980, Leichnitz 1981, Erban 1986, Hutson 1987, Huang et al. 1993) is characterized by nonlinear relationships that are qualitatively illustrated in Fig. 3.21. Formulations of the stress-displacement behavior of discontinuities are given in Erban (1986), Saeb & Amadei (1992), Barton & Bandis (1982) and Olsson & Barton (2001). The approach of Erban (1986), among others, was applied by Erichsen (1987) and Wittke (2003) to the coupling of stress-strain behavior and seepage flow in discontinuities (Chapter 7). The model of Saeb & Amadei (1992) can be considered as a generalization of earlier models established by Goodman (1976) and Bandis et al. (1983). A stress-displacement relationship that takes into account the anisotropic friction of rough discontinuities and is based on experimental results from Jing et al. (1992) was derived by Jing et al. (1994).
Based on the results of cyclic direct shear tests reported in Huang et al. (1993) a stress-displacement relationship for discontinuities under cyclic loading conditions was derived by Qiu et al. (1993). The model formulated by Saeb & Amadei (1992) was extended by Souley et al. (1995) to include cyclic loading and was validated by experimental data taken from Wibowo et al. (1993).

3.4 Rock Mass

3.4.1 Discrete Model

The stress-strain behavior of jointed rock includes the deformability and strength of both intact rock and discontinuities. Thus, along with the deformability and strength of the intact rock the larger deformability and smaller strength of the discontinuities must be accounted for. This can be accomplished by simulating the intact rock as a continuum, and each discontinuity in the rock mass individually (discretely). In this regard, the reader is referred to numerical procedures that are used for discrete modeling of jointed rock. Beside methods based on discontinuum mechanics such as the distinct element
method (DEM), methods based on continuum mechanics such as the finite element method (FEM, see Chapter 10) are also suitable for that purpose. As an example, Fig. 3.22 illustrates the discrete modeling of discontinuities by finite elements for the case of a sedimentary rock with vertical joints interrupted by horizontal bedding-parallel shear zones filled with clayey-silty soil.

Figure 3.22 Discrete simulation of discontinuities

An accurate discrete modeling of a rock mass with non-persisting discontinuities requires the simulation of the extension of existing discontinuities and the formation of new discontinuities if the strength of the intact rock is exceeded. Fracture mechanics concepts for crack propagation need to be incorporated in order to simulate crack expansion at the tips of non-persistent discontinuities in a discrete model. For this purpose, procedures are required for automatic finite element mesh generation and remeshing.
Examples for the application of computer codes based on fracture mechanics concepts on rock masses with non-persisting discontinuities can be found, for example, in Gehle (2002), Kemeny (2003) and Zhang et al. (2006).

Application of discrete models to the stability analyses for rock engineering structures, in particular in the case of jointed rock with several sets of discontinuities, usually results in a large computational effort. Moreover, it is impossible to completely determine the three-dimensional orientation, location and extent of discontinuities in the ground underneath or around a rock engineering structure. The applicability of discrete models is therefore limited in practice.

### 3.4.2 Homogeneous Model

#### General

The computational effort and the lack of detailed information on discontinuity data associated with discrete models gave impetus to the development of models that can describe the influence of discontinuities with much less effort. In these models referred to as “homogeneous models” or “equivalent continuum models” the intact rock and the discontinuities of a rock mass are treated as parts of a continuum; the deformations of the equivalent continuum include the deformations of the intact rock as well as the deformations resulting from displacements on discontinuities (Wittke 1990). Furthermore, it is assumed in these models that discontinuities with their average orientations and their characteristic parameters are located at each point of the rock mass. The application of a homogeneous model therefore requires the estimation of a so-called “representative elementary volume” (REV). In the REV, the rock mass can be considered to be statistically homogeneous in the sense that an increase of this volume does not change the mean stresses and displacements when describing the rock mass as a continuum.

Without the assumption that in each point of the continuum a discontinuity of each set is present the interaction between individual discontinuities must be taken into account. Homogeneous models in which deformability and strength of the rock mass are expressed as a function of orientation, size and intensity of the discontinuities and were implemented into numerical analysis procedures were formulated, for example, by Cai & Horii (1993) and Yoshida & Horii (2004). However, such models are sophisticated and, like discrete models, require a large amount of input data, including information on not only the orientation of the discontinuities but also other properties such as spacing, degree of separation and size. The computational effort and the lack of detailed information on discontinuity data are the main reasons that these models are not normally applied to practical rock engineering problems.

The structure of the proposed elastic-viscoplastic stress-strain law of jointed rock can be described with the aid of the rheological model represented in Fig. 3.23, which is an extension of the rheological model for intact rock represented in Fig. 3.9 (lower right). It consists of a spring to describe elastic behavior of the rock mass followed by a series arrangement of dashpot and sliding elements that are arranged in parallel to each other, representing the viscoplastic behavior of N discontinuity sets. The spring and each parallel arrangement of dashpot and sliding elements correspond to a strain component.
The elastic strain component $\varepsilon_{el}$ is reversible, while all other strain components $\varepsilon_{IR}^{vp}$, $\varepsilon_{D1}^{vp}$, …, $\varepsilon_{DN}^{vp}$ are irreversible. The total strain $\varepsilon^{tot}$ of jointed rock is obtained by the superposition of all these strain components.

Figure 3.23 Rheological model and strain components of a jointed rock mass with $N$ discontinuity sets

**Elastic behavior**

The elastic constants of the rock mass in a homogeneous model are related to the average stresses and strains. They differ generally from the elastic constants of the intact rock. This particularly applies to Young’s moduli and shear moduli which, in general, are smaller than those of the intact rock due to the higher deformability of the discontinuities.

The influence of discontinuities on the deformability of a rock mass separated by one discontinuity set is demonstrated in Fig. 3.24. If a compressive stress $\sigma_n$ is applied normal to the discontinuities the total normal strain of the rock mass according to the homogeneous model is composed of the normal strain of the intact rock $\varepsilon_{n,IR}$ and the normal strain $\varepsilon_{n,D}$ of the discontinuities leading to

$$\varepsilon_n = \varepsilon_{n,IR} + \varepsilon_{n,D} = \frac{\delta_{n,IR}}{s} + \frac{\delta_{n,D}}{s} = \frac{\sigma_n}{E_n},$$

where $\delta_{n,IR}$ is the compression of the intact rock and $\delta_{n,D}$ is the change of apertures of the discontinuities. $s$ is the spacing of the discontinuities, which is assumed to be a constant and $E_n$ is the so-called “equivalent Young’s modulus”, also referred to as the “deformation modulus”, for a loading normal to the discontinuities. If the rock mass is subjected to a shear stress $\tau_{res}$ parallel to the discontinuities the total shear strain of the rock mass according to the homogeneous model is composed of the shear strain of the intact rock $\gamma_{res,IR}$ and the shear strain $\gamma_{res,D}$ of the discontinuities leading to

$$\gamma_{res} = \gamma_{res,IR} + \gamma_{res,D} = \frac{\delta_{s,IR}}{s} + \frac{\delta_{s,D}}{s} = \frac{\tau_{res}}{G_{res}},$$
where $\delta_{s,IR}$ is the shear deformation of the intact rock and $\delta_{s,D}$ is the shear displacement on the discontinuities. $G_{res}$ is the equivalent shear modulus for a shear loading parallel to the discontinuities. Thus, the deformability of the rock mass in general is anisotropic.

Figure 3.24 Influence of discontinuities on the deformability of a rock mass separated by one discontinuity set (Wittke 2003)

The equivalent elastic constants of a regularly jointed rock mass with isotropic intact rock and three persistent, orthogonal discontinuity sets were derived by Duncan & Goodman (1968) under the assumption that the discontinuities have negligible thicknesses. As a result, such a rock mass can be described by seven independent elastic constants. In the special case that the rock mass is separated by only one set of discontinuities the elastic behavior turns out to be transversely isotropic with only three independent equivalent elastic constants. Gerrard (1982b) provides extended approaches describing the elastic behavior of a rock mass consisting of isotropic intact rock having one, two and three perpendicular sets of joints with not negligible thicknesses. Singh (1973) derived a similar equivalent continuum model describing the anisotropic elastic behavior of a rock mass containing orthogonal joint sets with staggered cross joints.

A model for the elastic behavior of a rock mass which is separated by one set of non-persistent, open discontinuities according to the homogeneous model was devel-
oped by Berry et al. (1974). This approach is based on studies carried out by Walsh (1965a, 1965b, 1965c) and Walsh & Brace (1966) on the effect of cracks on the elastic properties of intact rocks. Accordingly, the rock mass can be described by a special case of transverse isotropy with only three independent equivalent elastic constants. Equations for the equivalent elastic constants of a rock mass that is separated by two sets of non-persistent, open discontinuities, which are perpendicular to each other, are given by Hu & Huang (1993).

The elastic properties of rock masses with discontinuity systems, which are not perpendicular to each other and/or contain non-persistent discontinuities, can be formulated with the aid of a “crack tensor”, also referred to as a “fabric tensor” or “fracture tensor”, introduced by Oda (1982).

However, all these analytic approaches to estimating the elastic properties of jointed rock are not suitable for application because the deformability of discontinuities can hardly be estimated. It is therefore recommended to evaluate the deformability of jointed rock on the basis of site investigation, field testing, back analysis and experience, as described in Section 18.1.

In most cases the elastic rock mass deformability can be related to structural models as shown in Fig. 2.33. Accordingly, in jointed rock, in addition to elastic and transversely isotropic behavior, orthotropic elastic behavior may also be relevant. A regularly jointed rock mass with one to three persistent, mutually perpendicular discontinuity sets can be described by an equivalent orthotropic continuum. This type of anisotropy is characterized by the following compliance matrix:

\[
[D']^T = \begin{bmatrix}
\frac{1}{E_1} & \frac{v_1}{E_1} & \frac{v_2}{E_2} & 0 & 0 & 0 \\
\frac{v_1}{E_1} & \frac{1}{E_1} & \frac{v_3}{E_3} & 0 & 0 & 0 \\
\frac{v_2}{E_2} & \frac{v_3}{E_3} & \frac{1}{E_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The nine independent elastic constants \(E_1, E_2, E_3, G_1, G_2, G_3, v_1, v_2\) and \(v_3\), as well as the three dependent Poisson’s ratios \(v_4, v_5, v_6\) of an orthotropic rock mass, are defined in Fig. 3.25 by means of normal and shear stresses and strains.

A layered rock mass with discontinuities filled with soil or an alternating sequence of different rock types allow a reasonable estimate of the mean thicknesses of the individual rock layers and/or discontinuity fillings so that a reliable estimation of the elastic constants may be possible.
As an example, the elastic behavior of a layered rock mass consisting of isotropic intact rock and one set of discontinuities filled with soil is considered. According to Wittke (1990), the elastic behavior of such a rock mass is transversely isotropic and the elastic
constants can be expressed by the elastic constants \((E_{IR}, v_{IR}, E_F, v_F)\) and the volumetric contents \(\alpha\) and \(\beta = 1 - \alpha\) of the intact rock and the filling, respectively, as follows:

\[
E_1 = \alpha \cdot E_{IR}, \quad (3.87)
\]

\[
\frac{1}{E_2} = \frac{\alpha}{E_{IR}} + \frac{\beta}{E_{oedF}}, \quad (3.88)
\]

\[
\frac{1}{G_2} = \frac{2\alpha \cdot (1 + v_{IR})}{E_{IR}} + \frac{2\beta \cdot (1 + v_F)}{E_F}, \quad (3.89)
\]

\[
v_1 = v_{IR}, \quad (3.90)
\]

\[
v_2 = \frac{E_{oedF} \cdot v_{IR}}{\alpha \cdot E_{oedF} + \beta \cdot E_{IR}}. \quad (3.91)
\]

Equations (3.87) to (3.91) were derived assuming \(\alpha >> \beta\) and \(E_{IR} >> E_F\) and accounting for shear bond between intact rock and filling (Wittke 1990). \(E_{oedF}\) in (3.88) and (3.91) is the oedometric modulus of the discontinuity filling defined as

\[
E_{oedF} = \left(\frac{1 - v_F}{1 - v_F - 2v_F^2}\right) \cdot E_F. \quad (3.92)
\]

The example represented in Fig. 3.26 demonstrates the marked influence of persistent, filled discontinuities on the elastic deformability of a rock mass even when their volumetric content is only 2%.

The elastic constants of a closely spaced alternating sequence of two rock types A and B were determined by Pinto (1966) assuming shear bond between the individual rock layers. The elastic behavior can then also be described as transversely isotropic. The comprehensive formulas for the elastic constants presented by Pinto (1966) can be simplified while retaining sufficient accuracy as follows (Wittke 1990):

\[
E_1 = \alpha \cdot E_A + \beta \cdot E_B, \quad (3.93)
\]

\[
v_1 = \frac{\alpha \cdot E_A \cdot v_A + \beta \cdot E_B \cdot v_B}{\alpha \cdot E_A + \beta \cdot E_B}, \quad (3.94)
\]

\[
\frac{1}{E_2} = \frac{\alpha}{E_A} + \frac{\beta}{E_B} + \frac{\alpha \cdot 2v_A}{1 - v_A} \left(\frac{k \cdot v_A}{E_A}\right),
+ \beta \cdot \frac{2v_B}{1 - v_B} \left(\frac{k \cdot v_B}{E_B}\right). \quad (3.95)
\]
\[
\frac{1}{G_2} = \frac{\alpha}{G_A} + \frac{\beta}{G_B}, \tag{3.97}
\]

where

\[
k = \frac{\alpha \cdot \nu_A + \beta \cdot \nu_B}{1 - \nu_A}, \quad \frac{\alpha \cdot E_A + \beta \cdot E_B}{1 - \nu_A}.
\]

\[
G_A = \frac{E_A}{2 (1 + \nu_A)},
\]

\[
G_B = \frac{E_B}{2 (1 + \nu_B)}
\]

and \(\alpha, E_A, \nu_A, \beta, E_B, \nu_B\) are the volumetric contents and elastic constants of rocks A and B, respectively.

**Figure 3.26** Example of the calculation of elastic constants for a rock mass separated by one set of persistent discontinuities filled with clay (Wittke 1990)
Extended approaches according to the homogeneous model describing the elastic behavior of alternating sequences that are composed of isotropic and anisotropic rocks can be found in Salamon (1968), Wardle & Gerrard (1972) and Gerrard (1982a).

**Strength and failure criteria**

According to the homogeneous model, the strength of jointed rock can be described by the superposition of the intact rock’s strength and the strengths of the different discontinuity sets as illustrated by the example represented in Fig. 3.20 and expressed by Equation (3.83). The validity of this superposition principle was checked by John (1969), Brown (1970), Yang et al. (1998) and Prudencio & Van Sint Jan (2007) by means of compression tests on regularly jointed artificial model materials. In these tests, dependent on the direction of loading, different failure modes were observed, including failure of intact rock, sliding on discontinuities and mixed failure modes. The last may be due to complex failure mechanisms particularly of non-persistent discontinuities explained in Section 3.3.3. However, the results of these tests confirmed to a large extent the superposition principle, so that from a practical point of view this principle is applicable with sufficient accuracy.

Strength criteria for rock masses based on the Hoek-Brown criterion (Hoek 1983, Hoek et al. 1992, Hoek 1994, Hoek et al. 1995, Hoek & Brown 1997, Hoek et al. 2002, Benz et al. 2008) are generally not suitable to be applied to jointed rock because anisotropy is not accounted for, which is a decisive characteristic of jointed rock. To overcome this shortcoming, recently an approach was made to modify the Hoek-Brown criterion for the prediction of strength of transversely isotropic rock masses (Lee & Pietruszczak 2008).

**Post-failure behavior**

According to the homogeneous model, the normal strain $\varepsilon_n$ and the shear strain $\gamma_{res}$ of a rock mass with one discontinuity set $D$ being subjected to a normal stress $\sigma_n$ perpendicular to the discontinuities and a shear stress $\tau_{res}$ parallel to the discontinuities according to (3.84) and (3.85) is obtained by a superposition of the strains of the intact rock and the discontinuities. The latter can be resolved into elastic and viscoplastic parts:

$$\varepsilon_{n,D} = \varepsilon_{n,D}^{el} + \varepsilon_{n,D}^{vp} = \frac{\delta_{n,D}^{el}}{s} + \frac{\delta_{n,D}^{vp}}{s},$$

$$\gamma_{res,D} = \gamma_{res,D}^{el} + \gamma_{res,D}^{vp} = \frac{\delta_{s,D}^{el}}{s} + \frac{\delta_{s,D}^{vp}}{s},$$

where $s$ is the mean spacing of discontinuities. $\delta_{n,D}^{el}$ and $\delta_{s,D}^{el}$ are the elastic displacements which take place before the peak shear strength is reached. Subsequently irreversible, viscoplastic displacements $\delta_{n,D}^{vp}$ and $\delta_{s,D}^{vp}$ occur.

According to the homogeneous model, the stress-displacement behavior of discontinuities can be idealized as shown in Fig. 3.27. The elastic displacements for two normal stress levels $\sigma_{n1}$ and $\sigma_{n2}$ are represented by green lines in the $\tau_{res}$-$\delta_{s,D}$ and $\delta_{n,D}$-$\delta_{s,D}$ diagrams in Fig. 3.27 (upper). After the peak shear strength $\tau_{res}$ is exceeded, a sudden drop
of the shear strength down to the residual strength \( \tau_{\text{res}}^* \) occurs. This is represented in Fig. 3.27 (upper) for normal stress levels \( \sigma_{n1} \) and \( \sigma_{n2} \) (red lines in the \( \tau_{\text{res}}^* - \delta_{s,D} \) diagram).

According to the theory of viscoplasticity, the strain rates \( \dot{\varepsilon}_{n,D}^{vp} \) and \( \dot{\gamma}_{\text{res},D}^{vp} \) can be calculated with the aid of a flow rule:

\[
\dot{\varepsilon}_{n,D}^{vp} = \frac{\dot{\delta}_{n,D}^{vp}}{s} = \frac{1}{\eta_D} \cdot F_D \cdot \frac{\partial Q_D}{\partial \sigma_n}
\]
if \( F_D > 0 \), \hspace{1cm} (3.100)

\[
\dot{\gamma}_{\text{res},D}^{vp} = \frac{\dot{\delta}_{s,D}^{vp}}{s} = \frac{1}{\eta_D} \cdot F_D \cdot \frac{\partial Q_D}{\partial \tau_{\text{res}}}
\]
if \( F_D > 0 \), \hspace{1cm} (3.101)

with \( \eta_D \) as the viscosity of the discontinuities. The yield functions for shear failure and tensile failure are

\[
F_D = \tau_{\text{res}} - \sigma_n \cdot \tan \phi_D - c_D,
\]
(3.102)

\[
F_D^* = \tau_{\text{res}}^* - \sigma_n^* \cdot \tan \phi_D^* - c_D^*.
\]
(3.103)
When the discontinuities are filled, the yield functions need to be formulated in terms of the shear parameters of the filling material \( \varphi_F, c_F, \varphi_F^* \) and \( c_F^* \) (Fig. 3.19, column 2). Rough and uneven discontinuities may be described by a bilinear failure criterion for the peak shear strength (Fig. 3.19, column 3). In this case an adaptation of the corresponding yield function is required. For non-persistent discontinuities the yield functions can be adjusted to the corresponding shear failure criteria as well (Fig. 3.19, columns 4 and 5).

The plastic potential \( Q_D \) for shear failure is obtained by replacing the friction angle \( \varphi_D \) in (3.102) by the dilatancy angle \( \psi_D \):

\[
Q_D = \tau_{\text{res}} - \sigma_n \cdot \tan \psi_D \cdot c_D. \tag{3.105}
\]

In the case of tensile failure perpendicular to the discontinuity \( Q_D = F_D \) applies.

Inserting (3.105) into (3.100) and (3.101) yields:

\[
\dot{\varepsilon}_{n,D}^{\text{vp}} = - \frac{1}{\eta_D} \cdot F_D^* \cdot \tan \psi_D \quad \text{if } F_D > 0, \tag{3.106}
\]

\[
\dot{\gamma}_{\text{res},D}^{\text{vp}} = \frac{1}{\eta_D} \cdot F_D^* \quad \text{if } F_D > 0. \tag{3.107}
\]

Integration of (3.106) and (3.107) with respect to time yields the following relationship between viscoplastic normal strain and shear strain:

\[
\varepsilon_{n,D}^{\text{vp}} = - \tan \psi_D \cdot \gamma_{\text{res},D}^{\text{vp}}. \tag{3.108}
\]

Thus, with increasing viscoplastic shear strain, linearly increasing dilatant viscoplastic normal strain occurs. This is represented in the \( \delta_{n,D}^{\text{vp}}-\delta_{s,D} \) diagram in Fig. 3.27 (lower right) for normal stress levels \( \sigma_{n1} \) and \( \sigma_{n2} \) with blue lines. However, at large shear deformations a constant angle of dilatancy, which is not equal to zero, leads to an unrealistically large volumetric strain. Therefore \( \psi_D \) should be set equal to zero when a certain shear displacement \( \delta_{s0} \) is exceeded (Fig. 3.27, lower right).

In a rock mass with one discontinuity set, the discontinuities are not necessarily oriented parallel to one of the axes of the global coordinate system \((x,y,z)\). Then \( \sigma_n \) and \( \tau_{\text{res}}^{\text{vp}} \) are functions of the stress components \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz} \) and \( \tau_{zx} \). The strain rates \( \{\varepsilon_D^{\text{vp}}\} \) resulting from viscoplastic deformations on the discontinuities are then calculated using the chain rule:

\[
\{\dot{\varepsilon}_D^{\text{vp}}\} = \frac{1}{\eta_D} \cdot F_D^* \cdot \frac{\partial Q_D}{\partial \sigma}\frac{\partial Q_D}{\partial \tau_{\text{res}}} = \frac{1}{\eta_D} \cdot F_D^* \left( \frac{\partial Q_D}{\partial \sigma_n} \frac{\partial \sigma_n}{\partial \sigma} + \frac{\partial Q_D}{\partial \tau_{\text{res}}} \frac{\partial \tau_{\text{res}}}{\partial \sigma} \right) \left( \tan \psi_D \cdot \frac{\partial \sigma_n}{\partial \sigma} - \frac{\partial \tau_{\text{res}}}{\partial \sigma} \right). \tag{3.109}
\]
The viscoplastic strain rates of a rock mass with \( n \) discontinuity sets \( D_1 \) to \( D_n \) and a schistosity \( S \) are calculated by superimposing the strain rates evaluated for the intact rock and the different discontinuity sets:

\[
\{\varepsilon^\text{vp}\} = \frac{1}{\eta_{\text{IR}}} \cdot F_{\text{IR}}^* \cdot \frac{\partial Q_{\text{IR}}}{\partial \{\sigma\}} + \frac{1}{\eta_{\text{S}}} \cdot F_{\text{S}}^* \cdot \frac{\partial Q_{\text{S}}}{\partial \{\sigma\}} + \sum_{i=1}^{n} \frac{1}{\eta_{\text{Di}}} \cdot F_{\text{Di}}^* \cdot \frac{\partial Q_{\text{Di}}}{\partial \{\sigma\}}.
\]  

Equation (3.110) presupposes that the strength of the intact rock, the strength of the schistosity and the strength of the discontinuities are all exceeded. Otherwise, the appropriate term needs to be omitted from the summation.

**Stress determination**

To determine the stresses, the total strains need to be subdivided into elastic and the viscoplastic components:

\[
\{\varepsilon\} = \{\varepsilon^\text{el}\} + \{\varepsilon^\text{vp}\}.
\]

Only the elastic strains lead to stresses for which the following applies according to (3.3):

\[
\{\sigma\} = [D] \cdot \{\varepsilon^\text{el}\}.
\]

The stresses are thus calculated from the total strains as follows:

\[
\{\sigma\} = [D] \cdot (\{\varepsilon\} - \{\varepsilon^\text{vp}\}).
\]

Equation (3.113) represents the stress-strain relationship for an elastic-viscoplastic material.

**Summary and conclusions**

In Table 3.1 the most important parameters are compiled that are required for the description of the stress-strain behavior according to the proposed elastic-viscoplastic rock mechanical model (see also Figs. 2.36 and 2.37). In Table 3.1 the methods for determination of these parameters, to be described in Chapters 13 to 18, are also specified.

Not in every case do all the parameters need to be evaluated. In each particular case, however, the parameters that are of considerable influence must be determined or at least reliably estimated.
Table 3.1 Most important parameters for describing the stress-strain behavior of jointed rock, and the methods of their determination

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameters</th>
<th>Methods of determination</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rock mass</strong></td>
<td>Elastic constants</td>
<td>Experience, Field tests, Back analyses</td>
</tr>
<tr>
<td></td>
<td>Isotropic: $E$, $v$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Transversely isotropic: $E_1$, $E_2$, $G_2$, $v_1$, $v_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Orthotropic: $E_1$, $E_2$, $E_3$, $G_1$, $G_2$, $G_3$, $v_1$, $v_2$, $v_3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Depends on structural model (Fig. 2.33)</td>
<td></td>
</tr>
<tr>
<td><strong>Intact rock</strong></td>
<td>Shear parameters</td>
<td>Laboratory tests</td>
</tr>
<tr>
<td></td>
<td>Peak strength: $\phi_{IR}$, $c_{IR}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\phi_S$, $c_S$ (planar grain structure)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Residual strength: $\phi_{IR}$, $c_{IR}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\phi^<em>_S$, $c^</em>_S$ (planar grain structure)</td>
<td></td>
</tr>
<tr>
<td><strong>Tensile strength</strong></td>
<td>Peak strength: $\sigma_{IR}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Residual strength: $\sigma^*_{IR} = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>Discontinuities</strong></td>
<td>Shear parameters</td>
<td>Mapping, Experience, Field tests, Back analyses</td>
</tr>
<tr>
<td></td>
<td>Peak strength: $\phi_D$, $c_D$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Residual strength: $\phi^<em>_D$, $c^</em>_D$</td>
<td></td>
</tr>
</tbody>
</table>
3.4.3 Combination of the Homogeneous Model with Discrete Models of Individual Discontinuities

The homogeneous model allows us to consider sets of discontinuities in stability analyses with relatively little effort, taking into account the anisotropy of deformability and strength caused by the discontinuities. The application of the homogeneous model leads to reliable results if the dimensions of the engineering structure are large in comparison to the rock blocks that are bounded by the individual discontinuities; that is, the mean spacing $s$ of discontinuities must be small compared with the characteristic dimension $D$ of the tunnel, slope or dam. This condition can be considered as fulfilled if the spacing of the discontinuities is smaller than $1/8$ to $1/10$ of the characteristic dimension of the considered structure. Master joints and faults of large spacing must therefore be modeled discretely (Fig. 3.28).

Discrete models of individual discontinuities can be coupled with the homogeneous model by modeling large joints and faults discretely, while the influence of smaller joints and discontinuities on strength and deformability is taken into account using the homogeneous model.

As an example, Fig. 3.29 shows a three-dimensional finite element mesh for a tunnel. This mesh was used to simulate the driving of a tunnel through a steeply dipping fault zone with a thickness of 8 m, which intersects the tunnel axis at an angle.
of $15^\circ$. The fault zone is modeled by elements that have a higher deformability and a lower strength than the elements simulating the surrounding rock mass according to the homogeneous model.

Figure 3.29 Combination of a homogeneous rock mass with the discrete model of a fault zone (Wittke et al. 2006)