Solving vertical and horizontal well hydraulics problems analytically in Cartesian coordinates with vertical and horizontal anisotropies

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1. Introduction

Well hydraulics solutions have started to appear in the literature using cylindrical coordinates after Darcy's historical paper was published in 1856. This tradition has continued since then and numerous well hydraulics solutions have been developed since the 1860s and many of them are being used in various hydrologic and petroleum investigations as well as engineering design studies (e.g., Hantush, 1964;Bear, 1979;Batu, 1998). Since the 1960s, some well hydraulics solutions were derived in Cartesian coordinates (Papadopoulos, 1965; Neuman et al., 1984) without solving the corresponding differential equation in Cartesian coordinates, but using only the Theis solution (Thies, 1935). The Theis solution as well as other cylindrical coordinate-based solutions are all based on the assumption that the horizontal hydraulic conductivity is the same in all radial directions from a pumped well.

In this paper, a new generalized three-dimensional analytical solution is developed for a partially-penetrating vertical rectangular parallelepiped well screen in a confined aquifer by solving the three-dimensional transient ground water flow differential equation in x-y-z Cartesian coordinates system for drawdown by taking into account the three principal hydraulic conductivities \((K_x, K_y, K_z)\) along the x-y-z coordinate directions. The fully penetrating screen case becomes equivalent to the single vertical fracture case of Gringarten and Ramey (1973). It is shown that the new solution and Gringarten and Ramey solution (1973) match very well. Similarly, it is shown that this new solution for a horizontally tiny fully penetrating parallelepiped rectangular well screen case of this new solution match very well with Hantush (1964) solution. This new analytical solution can also cover a partially-penetrating horizontal well by representing its screen interval with vertically tiny rectangular parallelepipeds. Also, the solution takes into account both the vertical anisotropy \((a_y = K_y/K_z)\) as well as the horizontal anisotropy \((a_x = K_x/K_z)\) and has potential application areas to analyze pumping test drawdown data from partially-penetrating vertical and horizontal wells by representing them as tiny rectangular parallelepipeds as well as line sources. The solution has also potential application areas for a partially-penetrating parallelepiped rectangular vertical fracture. With this new solution, the horizontal anisotropy \((a_x = K_y/K_x)\) in addition to the vertical anisotropy \((a_y = K_y/K_z)\) can also be determined using observed drawdown data. Most importantly, with this solution, to the knowledge of the author, it has been shown the first time in the literature that some well-known well hydraulics problems can also be solved in Cartesian coordinates with some additional advantages other than the conventional cylindrical coordinates method.
length of the aquifer along the y coordinate direction is assumed to be infinite. The partially-penetrating parallelepiped well is located on the \( y = 0 \) plane. Therefore, the aquifer length is from \( y = -\infty \) to \( y = +\infty \). The confined aquifer is overlain by a confining layer whose bottom is the xOy plane. The origin of the x-y-z Cartesian coordinate system is at the upper right corner of the \( y = 0 \) plane. Because of symmetry with respect to the xOz plane, only one-half of the aquifer is shown in Fig. 1. Likewise, the planes at \( z = 0 \) and \( z = b \) are assumed to be impermeable as well. Physically, the plane at \( z = 0 \) can be the bottom of a confining layer and the plane at \( z = b \) can be the top of a bedrock. As shown in Fig. 1, the sizes of the rectangular parallelepiped screen interval of the extraction well are \( 2x_1, 2z_2 \) and 2d. \( x_1 \) and \( x_2 \) are the distances from the vertical edges of the screen interval to the impermeable boundaries at \( x = 0 \) and \( x = L \), respectively. Likewise, \( z_1 \) and \( z_2 \) are the distances from the horizontal edges of the screen interval to the impermeable boundaries at \( z = 0 \) and \( z = b \), respectively. Apart from the well, an observation is also shown in Fig. 1. As can be seen from Fig. 1, the screen interval of the observation well is located between the elevations \( z_{e1} \) and \( z_{e2} \).

The purpose of this work is to determine an analytical solution for the problem described above for the temporal drawdown variation at an observation well located horizontally at any point having x and y coordinates. Because of symmetry, the solution will be derived for only one-half of the aquifer shown in Fig. 1. For this derivation, the thickness of the rectangular parallelepiped (2d) is assumed to be negligible. In other words, it is assumed that the screen interval interacts hydraulically with its faces perpendicular to the y coordinate direction. The impermeable vertical planes at \( x = 0 \) and \( x = L \) are included in the analytical solution in order to make its mathematics simpler. Of course, these planes increase the generality of the solution in such a way that the effects of vertical impermeable planes can be taken into account as well. By assigning large values to \( x_1 \) and \( x_2 \), the solution should correspond to an aquifer which has infinite extent in all horizontal directions. This solution will be called as “\( K_r-K_r-K_r \) solution”.

It is important to list some special cases of Fig. 1 for large values of \( x_1 \) and \( x_2 \): (a) For \( z_1 = z_2 = 0 \), and small \( 2x_t \) (line source) values, the geometry becomes equivalent to the geometry of Theis solution for a fully-penetrating vertical well in a confined aquifer (Theis, 1935); (b) for large \( x_1 \), \( x_2 \), and nonzero values of \( z_1 \) and \( z_2 \), and small \( 2x_t \) (line source) values, the geometry becomes equivalent to the geometry of Hantush’s solution geometry for a partially-penetrating well in a confined aquifer (Hantush, 1964); and (c) for \( z_1 = z_2 = 0 \), and finite \( 2x_t \) values, the geometry becomes equivalent to the geometry of Gringarten and Ramey solution for a fully-penetrating single vertical fracture in a confined aquifer (Gringarten and Ramey, 1973). The Theis (1935) and Hantush (1964) solutions will be called as “\( K_r \) solution” and “\( K_r-K_r \) solution”, respectively. The Gringarten and Ramey (1973) solution will be called as “\( K_r-K_r \) solution”. Also, for finite and large \( x_1 \) and \( x_2 \), nonzero values of \( z_1 \) and \( z_2 \), and small \( 2x_t \) (line source) values, the geometry becomes equivalent to the geometry of a partially-penetrating horizontal well in a confined aquifer. The vertical well and fracture solutions mentioned in (a), (b), and (c) are based on the assumption that the horizontal hydraulic conductivity is the same in all radial direction (\( K_r = K_r = K_r \)) from the line-source well. As a result, the other purpose of this paper is to compare the Cartesian coordinates – derived analytical solution (\( K_r-K_r-K_r \) solution) with the classical solutions in the literature (\( K_r \) and \( K_r-K_r \) solutions) and determine the appropriateness and accuracy of the \( K_r-K_r-K_r \) solution.

3. Governing equations

The governing differential equation of saturated groundwater flow is

\[
\frac{\partial h}{\partial t} = \eta_s \frac{\partial^2 h}{\partial x^2} + \eta_s \frac{\partial^2 h}{\partial y^2} + \eta_s \frac{\partial^2 h}{\partial z^2}
\]

(1)

where

\[
\eta_s = \frac{K_s}{S_s} \quad \eta_y = \frac{K_y}{S_s} \quad \eta_z = \frac{K_z}{S_s}
\]

(2)

In Eq. (2), \( h(L) \) is the hydraulic head; \( K_s \) (LT\(^{-1}\)), \( K_y \) (LT\(^{-1}\)), and \( K_z \) (LT\(^{-1}\)) are the principal hydraulic conductivities in the x, y, and z directions.

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Fig. 1. The geometry of a vertical as well as horizontal rectangular parallelepiped partially-penetrating extraction and observation wells in a confined aquifer in Cartesian coordinates.
4. Initial and boundary conditions

It is assumed that initially the hydraulic head surface is planar and horizontal. Therefore, the initial condition is that the drawdown \( s \) is zero everywhere:

\[
s(x, y, z, t = 0) = 0
\]

(5)

It is assumed that at large distances from the extraction well the drawdown approaches zero. Therefore, the boundary condition for drawdown at infinity along the \( y \) coordinate direction is

\[
s(x, y = \infty, z, t) = 0
\]

(6)

The boundary conditions at the impermeable planes \((x = 0, x = L, z = 0, \) and \( z = b)\) are

\[
\begin{align*}
\frac{\partial s}{\partial x}(x = 0, y, z) &= 0, \\
\frac{\partial s}{\partial x}(x = L, y, z) &= 0, \\
\frac{\partial s}{\partial x}(x, y, z = 0) &= 0, \\
\frac{\partial s}{\partial x}(x, y, z = b) &= 0
\end{align*}
\]

(7)-(10)

The vertical plane at \( y = 0 \) in Fig. 1 is the plane where the rectangular screen interval of the well is located. Since here a well is rate-controlled, the following equation can be written for the Darcy flux perpendicular to the \( 2x_q \) by \( 2z_q \) area using the second expression in Eq. (4)

\[
q_M e^{\gamma t} = -K_y \frac{\partial s}{\partial y} \quad x_1 < x < x_1 + 2x_q, \quad y = 0, \quad z_1 < z < z_1 + 2z_q
\]

(11)

in which \( q_M \) is the maximum Darcy flux and \( \gamma (T^{-1}) \) is a parameter. Eq. (11) performs such that exponentially decaying or increasing fluxes with time can be taken into account. For \( \gamma = 0 \), Eq. (11) represents constant Darcy flux conditions. For negative values of \( \gamma \), the flux decreases exponentially with time. Likewise, for positive values of \( \gamma \), the flux increases exponentially with time. The negative sign in front of the right-hand side of Eq. (11) represents withdrawal conditions. If the sign is positive, it represents injection conditions. All the rest of the area on the \( y = 0 \) plane, besides the \( 2x_q \) by \( 2z_q \) area, the Darcy fluxes are assumed to be zero.

Because of symmetry, the relationship between the one-half extraction rate \((Q/2)\) from one side of the parallelepiped well and Darcy flux can be expressed as

\[
Q = q_M(2x_q)(2z_q)
\]

(12)

And from Eq. (12), \( q_M \) can be expressed as

\[
q_M = \frac{Q}{(2x_q)(2z_q)}
\]

(13)

5. Drawdown solution at a point

The problem defined above first is solved in the Laplace domain and details are given in Appendix A. Details of the inverse Laplace transform are given in Appendix F. Solution in the Laplace domain is given by Eq. (F1) of Appendix F and its inverse Laplace transform

\[
s(x, y, z, t) = s_1(x, y, z, t) + s_2(x, y, z, t) + s_3(x, y, z, t) + s_4(x, y, z, t)
\]

(14)

is the drawdown solution at a point in the flow domain.

The first drawdown expression component, \( s_1(x, y, z, t) \), is determined by combination of Eqs. (G6), (G7), and (G9) of Appendix F and the final result is

\[
s_1(x, y, z, t) = \frac{4q_Mx_2}{[z_1 + 2z_q + z_2][x_1 + 2x_q + x_2]} \frac{1}{K_y} \int_0^t \left( \frac{\eta_y}{\pi t} \right)^{\frac{1}{2}} \times \exp \left( -\frac{y^2}{4\eta_y u} \right) \exp \left( -\frac{y^2}{4\eta_y u} \right) \, du
\]

(15)

The second drawdown expression component, \( s_2(x, y, z, t) \), is determined by combination of Eqs. (G15), (G16), (G17), and (G20) into Eq. (G22) of Appendix G and the final result is

\[
s_2(x, y, z, t) = q_M \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4(n + m)}{(n + m)!} \sin \left[ \frac{\eta_y y}{\pi} \right] \left[ \sin \left( \frac{\eta_z z_1}{\pi} \right) \cos \left( \frac{\eta_y y}{\pi} \right) \right]
\]

\[
\frac{1}{K_y} \int_0^t \left[ \left( \frac{\eta_z z_1}{\pi} \right) \sin \left( \frac{\eta_y y}{\pi} \right) \right] \left( \frac{\eta_y}{\pi t} \right)^{\frac{1}{2}} \exp \left( -\frac{y^2}{4\eta_y u} \right) \exp \left( -\frac{y^2}{4\eta_y u} \right) \, du
\]

(16)

Using the procedure as outlined in Appendix G, the third drawdown expression component, \( s_3(x, y, z, t) \), is

\[
s_3(x, y, z, t) = q_M \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4(2z_2)}{(n + m)!} \left( \frac{\eta_z z_1}{\pi} \right) \left( \frac{\eta_y}{\pi t} \right)^{\frac{1}{2}} \times \exp \left( -\frac{y^2}{4\eta_y u} \right) \exp \left( -\frac{y^2}{4\eta_y u} \right)
\]

(17)

Using the procedure as outlined in Appendix G, the fourth drawdown expression component, \( s_4(x, y, z, t) \), is

\[
s_4(x, y, z, t) = q_M \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4(2z_2)}{(n + m)!} \left( \frac{\eta_z z_1}{\pi} \right) \left( \frac{\eta_y}{\pi t} \right)^{\frac{1}{2}} \times \exp \left( -\frac{y^2}{4\eta_y u} \right) \exp \left( -\frac{y^2}{4\eta_y u} \right)
\]

(18)

6. Dimensionless drawdown solution at a point

The dimensionless arguments used for Eq. (12) are as follows:

\[
X_y = \frac{x_1}{x_q}, \quad Y_y = \frac{y}{x_q}, \quad Z_y = \frac{z_1}{x_q}, \quad Z_1 = \frac{z_2}{x_q}, \quad Z_y = \frac{z_2}{x_q}, \quad X_y = \frac{x}{x_q}, \quad Y_y = \frac{Y}{x_q}, \quad Z_y = \frac{Z}{x_q}, \quad a_x = \frac{K_y}{K_x}, \quad T = K_y b, \quad \Gamma = \frac{\gamma S x_q^2}{T}, \quad B_y = \frac{b}{x_q}, \quad B_y = \frac{b}{z_q}
\]

(19)

The dimensionless drawdown is defined as

\[
s_0 = \frac{4\pi(K_y K_x)^{\frac{1}{2}} b s}{Q} = \frac{4\pi(K_y K_x)^{\frac{1}{2}}}{Q} (s_1 + s_2 + s_3 + s_4)
\]

(20)
Similarly, the second dimensionless drawdown component can be drawn expression given by Eq. (15) takes the form

\[ s_{1D} = \frac{2\pi B_{yf}}{(Z_{yf} + 2Z_{ff} + Z_{ff})} \int_0^{t_D} \frac{1}{\tau^2} \exp \left( -\frac{1}{4\tau} y f^2 \frac{1}{\alpha_{ns}} \right) \exp \left( \frac{\Gamma(t_D - \tau)}{d\tau} \right) d\tau \]  

(23)

where

\[ t_D = \frac{\tau D}{s_{1D}} \quad \tau = \frac{\tau D}{s_{1D}} \]  

(24)

Similarly, the second dimensionless drawdown component can be expressed as

\[ s_{2D} = \frac{2\pi^2 B_{yf}}{(Z_{yf} + 2Z_{ff} + Z_{ff})} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} \frac{1}{m} \left\{ \sin \left( \frac{n\pi(X_{yf} + 2)}{X_{yf} + 2 + X_{yf}} \right) - \sin \left( \frac{m\pi X_{yf}}{X_{yf} + 2 + X_{yf}} \right) \right\} \cos \left( \frac{m\pi Z_{yf}}{Z_{yf} + 2Z_{ff} + Z_{ff}} \right) \cos \left( \frac{m\pi Z_{ff} + Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \cos \left( \frac{m\pi Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \exp \left( -\frac{1}{4\tau} y f^2 \frac{1}{\alpha_{ns}} \right) \exp \left( \frac{\Gamma(t_D - \tau)}{d\tau} \right) d\tau \]  

(25)

Similarly, the second dimensionless drawdown component can be expressed as

\[ s_{3D} = \frac{2\pi^2 B_{yf}}{(Z_{yf} + 2Z_{ff} + Z_{ff})} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} \frac{1}{m} \left\{ \sin \left( \frac{n\pi(X_{yf} + 2)}{X_{yf} + 2 + X_{yf}} \right) - \sin \left( \frac{m\pi X_{yf}}{X_{yf} + 2 + X_{yf}} \right) \right\} \cos \left( \frac{m\pi Z_{yf}}{Z_{yf} + 2Z_{ff} + Z_{ff}} \right) \cos \left( \frac{m\pi Z_{ff} + Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \cos \left( \frac{m\pi Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \exp \left( -\frac{1}{4\tau} y f^2 \frac{1}{\alpha_{ns}} \right) \exp \left( \frac{\Gamma(t_D - \tau)}{d\tau} \right) d\tau \]  

(26)

Finally, the fourth dimensionless drawdown component can be expressed as

\[ s_{4D} = \frac{2\pi^2 B_{yf}}{(Z_{yf} + 2Z_{ff} + Z_{ff})} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} \frac{1}{m} \left\{ \sin \left( \frac{m\pi Z_{yf}}{Z_{yf} + 2Z_{ff} + Z_{ff}} \right) - \sin \left( \frac{m\pi Z_{ff} + Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \right\} \cos \left( \frac{m\pi Z_{ff}}{X_{yf} + 2 + X_{yf}} \right) \exp \left( -\frac{1}{4\tau} y f^2 \frac{1}{\alpha_{ns}} \right) \exp \left( \frac{\Gamma(t_D - \tau)}{d\tau} \right) d\tau \]  

(27)

7. Conversion of \( t_D \) expression into radial coordinates and usage of Gauss numerical integration method

In order to make comparisons with some well known well hydraulics solutions (Theis, 1935; Hantush, 1964), the \( t_D \) expression given by Eq. (24) must be converted into radial coordinates. The radial coordinates system is shown in Fig. 2. Therefore,

\[ x' = x - x_1 - x_1 \quad \frac{x'}{x_f} = \frac{x - x_1 - x_1}{x_f} - 1 \]  

(28)

where \( x' \) is measured from the plane located at \( x=x_1+x_f \). From Fig. 2, by considering that \( r^2 = x'^2 + y'^2 \), the dimensionless radial coordinate \( r_D \) is defined as

\[ r_D = \frac{(x'^2 + y'^2)^{1/2}}{x_f} = \left[ \left( \frac{x'}{x_f} \right)^2 + \left( \frac{y'}{x_f} \right)^2 \right]^{1/2} \]  

(29)

And from this
where

\[ t_d = \frac{r_0^2}{4} \tau \]

which is the time parameter for the well hydraulics solutions (e.g., Theis, 1935; Hantush, 1964) in radial coordinates and \( r = (x^2 + y^2)^{1/2} \).

The four drawdown components given by Eqs. (24)–(27) have integrals and the Gauss numerical integration method is used for them. Since the integrals have limits 0 and \( t_d \), whereas Gauss method (e.g., Batu, 1993) requires limits of -1 and +1, the following change of variable is performed:

\[ \tau = \frac{1}{8} r_0^2 \tau _{1}(u' + 1) \]

Thus when \( u' = -1, \tau = 0 \); when \( u' = +1, \tau = (1/4)r_0^2 \tau _{1} \) and \( t_d \) becomes the variable in the four Gauss integrations. From Eq. (34) one can write

\[ d\tau = \frac{1}{8} r_0^2 \tau _{1} du' \]

### 8. Average drawdown in an observation well

The drawdown measured in an observation well that is screened between elevations \( z_{x1} \) and \( z_{x2} \) (Fig. 1) is simply the average over that vertical distance and is given by

\[ s_{x1z_{x2}}(x, y, z, t) = \frac{1}{z_{x2} - z_{x1}} \int_{z_{x1}}^{z_{x2}} s(x, y, z, t) dz \]

This average drawdown is always greater than the fall of the hydraulic head surface at the point of observation (Neuman, 1974). From Eqs. (23), (25), (26), and (27) it can be observed that

\[ s_{1D} = s_{1D}(\tau) \]
\[ s_{2D} = s_{2D}(X_f, Y_f, Z_f, \tau) \]
\[ s_{3D} = s_{3D}(X_f, Y_f, Z_f, \tau) \]
\[ s_{4D} = s_{4D}(X_f, Y_f, Z_f, \tau) \]

Only, \( s_{2D} \) and \( s_{4D} \) are the function of \( Z_f \). Therefore, Eq. (21) can be written as

\[ S_{D(x1,x2)} = s_{1D} + s_{2D} + s_{3D} + s_{4D} \]

where

\[ \tilde{s}_{2D} = \frac{1}{Z_{ref} - Z_{ef}} \int_{Z_{ref}}^{Z_{ef}} s_{2D}(Z_f) dZ_f \]

and

\[ \tilde{s}_{4D} = \frac{1}{Z_{ref} - Z_{ef}} \int_{Z_{ref}}^{Z_{ef}} s_{4D}(Z_f) dZ_f \]

where

\[ Z_{ref} = \frac{z_{x1}}{x_f} \quad Z_{ef} = \frac{z_{x2}}{x_f} \]

Using Eqs. (25) and (27) in Eqs. (39) and (40), respectively, and performing the integrations one can write

\[ \tilde{s}_{2D} = \frac{1}{(Z_{ref} - Z_{ef}) \pi^2} \frac{2}{B_{ef}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \sin \left[ \frac{n\pi (X_f + 2)}{X_f + 2 + X_f} \right] \]

\[ s_{2D} = \frac{1}{(Z_{ref} - Z_{ef}) \pi^2} \frac{2}{B_{ef}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \sin \left[ \frac{n\pi (X_f + 2)}{X_f + 2 + X_f} \right] \]

\[ s_{4D} = \frac{1}{(Z_{ref} - Z_{ef}) \pi^2} \frac{2}{B_{ef}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \sin \left[ \frac{m\pi (Z_f + 2Z_f)}{Z_f + 2Z_f + Z_f} \right] \]

### 9. Comparison with some vertical well solutions in the literature and dimensional examples

As mentioned previously, some well known vertical well case solutions published in the literature since the 1930s are equivalent special cases of the equations given by Eqs. (21) and (38). Of these solutions, the first one is the Theis (1935) solution for a fully penetrating well in confined aquifers. The second one is the Hantush (1964) solution for a partially penetrating well in confined aquifers. And the third one is the Gringarten and Ramey (1973) solution for a vertical fully penetrating fracture in confined aquifers. Comparisons with these solutions are presented below.

### 9.1. Comparison with Theis solution

Theis solution is based on the assumption that the aquifer tends to infinity in all horizontal directions. As mentioned previously, for \( z_1 = z_2 = 0 \) and small \( 2x_f \) (line source) values, the geometry shown in Fig. 1 becomes equivalent to the geometry of Theis solution for a fully-penetrating well in a confined aquifer (Theis, 1935) for large values of \( x_1 \) and \( x_2 \). For fully-penetrating well cases, all solution components become zero in Eq. (38) with the exception of \( s_{1D} \) and \( s_{2D} \) as given by Eqs. (23) and (26). In other words, the solution components \( s_{2D} \) and \( s_{4D} \) as given by Eqs. (25) and (27), respectively, become zero. Theis solution and the corresponding \( s_0 = s_{1D} + s_{2D} \) solution will be called the “\( K_e \) solution” and “\( K_p-K_e \) solution” respectively. For all comparisons, \( 2x_f = 0.01 \) m is used. This is a reasonable value as will be seen later during the comparison process. However, smaller values as well as somewhat higher values can also be used to make equivalent cases for the Theis line source condition.
For the first set of comparisons $x_1 = x_2 = 10\, \text{m}$ are used. All dimensional and dimensionless parameters are given in the box in Fig. 3. As can be seen from Fig. 3, the $K_r$ and $K_{x} - K_y$ solutions compare well until $t'_{0} = 5.0E + 02$. Fig. 3 also shows that the discrepancies between the $K_r$ and $K_{x} - K_y$ solutions start for $t'_{0}$ values greater than $5.0E + 02$ which is due to the effects of the impermeable vertical planes. It is important to point out that the drawdown values do not depend on the values of $x'$ and $y$. This is shown by examples, but they are not presented here. In other words, other combinations of $x'$ and $y$ values give the same results which confirm that the present solution generates one type curve as the case for the Theis solution.

The above results clearly show that the Theis solution results can be generated from the $K_{x} - K_y$ solution under the conditions that small $x_1$ and large $x_1$ and $x_2$ values be assigned in the $K_{x} - K_y$ solution. Numerical experiments showed that keeping the ratios of $x_1/x_0$ and $x_2/x_1$ around 20,000 or greater should produce results with enough accuracy with the $K_r$ or Theis solution. In Fig. 4, besides the Theis curve, the curves for $a_{x} = K_f/K_y = 0.1$ and 0.5 for the $K_{x} - K_y$ solution are shown for $x' = 1\, \text{m}$, $y = 1\, \text{m}$, and $z = 0$. In Fig. 5, all input data are the same as Fig. 4 with the exception that $x' = 1\, \text{m}$, $y = 3\, \text{m}$, and $z = 0$. As expected, the results of the $K_{x} - K_y$ solution for different $x'$ and $y$ values under horizontal anisotropy cases are the same as shown in Figs. 4 and 5. In the other words, this shows that the results are independent on the values of $x'$ and $y$ if $a_{x}$ is not equal to unity. Fig. 4 or Fig. 5 also indicate that the horizontal anisotropy potentially affect the drawdown variation in aquifers.

Numerical experiments indicated that values of the infinite series term $N = 100$ or higher with the Gauss points 20 or higher in the numerical integration process can generate results with sufficient accuracy. However, as the values of $x_1$ and $x_2$ increase higher $N$ values are required.

9.2. Comparison with Gringarten and Ramey solution

Gringarten and Ramey (1973) solution is based on the assumption that the aquifer tends to infinity in all horizontal directions. As mentioned previously, for $z_1 = z_2 = 0$ and finite $2x_0$ (vertical fracture) values, the geometry shown in Fig. 1 becomes equivalent to the geometry of Gringarten and Ramey solution for a fully-penetrating thin rectangular parallellepiped well screen in a confined aquifer (Gringarten and Ramey, 1973) for large values of $x_1$ and $x_2$. For fully-penetrating parallellepiped well cases, all solution components become zero in Eq. (38) with the exception of $s_{11D}$ and $s_{33D}$ as given by Eqs. (23) and (26). In other words, the solution components $s_{22D}$ and $s_{34D}$ as given by Eqs. (25) and (27), respectively, become zero. Both the Gringarten and Ramey and the corresponding $S_D = S_{11D} + S_{33D}$ solution are “$K_r-K_y$ solution”.

For all comparisons, $2x_0 = 5\, \text{m}$ is used. Like comparisons with the Theis solution case, assigning relatively large values to $x_1$ and $x_2$, equivalent cases can be generated to the Gringarten and Ramey solution. By assigning higher values to $x_1$ and $x_2$, these effects can be reduced or eliminated. This situation can be observed in Figs. 6 and 7 for which $x_1 = x_2 = 5000\, \text{m}$. Figs. 6 and 7 correspond to the horizontal anisotropy ratio ($a_{x} = K_f/K_y$) values 0.1 and 0.01, respectively. All dimensional and dimensionless parameters used for these comparisons are shown in the boxes of Figs. 6 and 7.

Numerical experiments indicated that values of the infinite series term $N = 400$ or higher with the Gauss points 20 or higher in the numerical integration process can generate results with sufficient accuracy. However, as the values of $x_1$ and $x_2$ increase higher $N$ values are required.

The above results clearly show that Gringarten and Ramey solution results can be generated from the present $K_{x} - K_y$ solution under the conditions that large $x_1$ and $x_2$ values should assigned in the solution. Numerical experiments showed that keeping the ratios of $x_1/x_0$ and $x_2/x_0$ around 1,000 or higher should produce results with enough accuracy with the Gringarten and Ramey $K_r - K_y$ solution.

9.3. Comparison with Hantush solution

Hantush solution is based on the assumption that the aquifer tends to infinity in all horizontal directions. As mentioned previously, for small $2x_0$ (line source) values, the geometry shown in Fig. 1 becomes equivalent to the geometry of Hantush solution for a partially-penetrating well in a confined aquifer (Hantush, 1964) for large values of $x_1$ and $x_2$. For partially-penetrating well cases, all solution components in Eq. (38) are required. Hantush solution and the corresponding to $S_{p(2x_1, 2x_2)}$ solution will be called “$K_r-K_y$ solution” and “$K_{x} - K_y$ solution” respectively. For all comparisons, $2x_0 = 0.01\, \text{m}$ is used. This is a reasonable value as will be seen later during the comparison process. However, smaller values as well as somewhat higher values can also be used to make equivalent cases for the Hantush line source condition. The aquifer thickness $(b)$ is 10 m and the length of screen interval $(2z_0)$ is 3 m. The rest of parameters are shown in the box of Fig. 8.

Like comparisons with the Theis solution case, assigning relatively large values to $x_1$ and $x_2$, equivalent cases can be generated to the Hantush solution. Numerical experiments showed that $x_1 = x_2 = 1000\, \text{m}$ or higher values generate drawdown values at the observation well without the effects of the impermeable boundaries until $t'_{0} = 1.0E + 04$. As the length of the screen interval $(2z_0)$ increases higher values for $x_1$ and $x_2$ are required in order to determine drawdown values for infinite horizontal aquifer cases. This situation is shown with examples for the cases of Theis solution and Gringarten and Ramey solution as presented previously.

Numerical experiments indicated that values of Gauss points equal to 20 or higher in the numerical integration process can generate results with sufficient accuracy. However, under partially-penetrating wells condition, the results showed that the number of series parameter $(N)$ plays a critical role in order to generate converged results. Numerical experiments show that the present analytical solution rapidly approaches to the Hantush solution (for $K_f = K_y = K_x$) and for $N = 800$ the comparison becomes excellent and this is shown in Fig. 8.

Besides the Hantush (1964) solution for which $K_f = K_y = K_x$, two different horizontal anisotropy cases $K_f/K_y = 0.01$ and 0.05 are also presented in Fig. 8. As can be seen from Fig. 8, the horizontal anisotropy can potentially affect the drawdown variation in the aquifer depending on its value.

9.4. Dimensional examples

Figs. 9 and 10 present drawdown versus time curves in dimensional forms for $K_f = 1.0E - 02\, \text{cm/s}$ and $K_y = 1.0E - 03\, \text{cm/s}$, respectively. These figures correspond to temporal drawdown variation for $x' = 1\, \text{m}$, $y = y' = 1\, \text{m}$ ($r = 1.41\, \text{m}$), $z_1 = 2\, \text{m}$, $z_2 = 5\, \text{m}$. The rest of parameters are the same as in Fig. 8. As can be seen from Figs. 9 and 10, the horizontal anisotropy $K_f/K_y$ affects significantly the drawdown variation depending on its value. As the values of $K_f/K_y$ approach to 1, the drawdown curves coalesce to each other. The impermeable boundaries are 1000 m away from the well ($x_1 = x_2 = 1000\, \text{m}$) and their effects are visible in the figures depending on the values of hydraulic conductivities. Fig. 9 shows that for $K_f = 1.0E - 02\, \text{cm/s}$ the effects start to appear after approximately 50 days of elapsed time. As the hydraulic conductivity values become smaller, the effects of the impermeable boundaries appear later. This situation can be observed from Fig. 10 for which $K_f = 1.0E - 03\, \text{cm/s}$.
Fig. 3. Drawdown comparison with the Theis (1935) fully-penetrating vertical well solution for $K_y/K_x = 1$, $K_z/K_x = 1$; $x_1 = 10$ m, $x_2 = 10$ m, $x_f = 0.005$ m; $z_1 = 0$, $z_2 = 0$, and $z_f = 0.5$ m; and $x' = 1$ m, $y = 1$ m, and $z = 0$.

Fig. 4. Drawdown comparison with the Theis (1935) fully-penetrating vertical well solution for $K_y/K_x = 0.1, 0.5$, and 1; $K_z/K_x = 1$; $x_1 = 50$ m, $x_2 = 50$ m, $x_f = 0.005$ m; $z_1 = 0$, $z_2 = 0$, and $z_f = 0.5$ m; and $x' = 1$ m, $y = 1$ m, and $z = 0$.

Fig. 5. Drawdown comparison with the Theis (1935) fully-penetrating vertical well solution for $K_y/K_x = 0.1, 0.5$, and 1; $K_z/K_x = 1$; $x_1 = 50$ m, $x_2 = 50$ m, $x_f = 0.005$ m; $z_1 = 0$, $z_2 = 0$, and $z_f = 0.5$ m; and $x' = 1$ m, $y = 3$ m, and $z = 0$. 
Fig. 6. Drawdown comparison with the Gringarten and Ramey (1973) fully-penetrating rectangular vertical fracture solution for $K_y/K_x = 0.1$; $K_z/K_x = 1$; $x_1 = 5000$ m, $x_2 = 5000$ m, $x_f = 5$ m; $z_1 = 0$, $z_2 = 0$, and $z_f = 0.5$ m; and $x' = 10$ m, $y' = 10$ m, and $z = 0$.

Fig. 7. Drawdown comparison with the Gringarten and Ramey (1973) fully-penetrating rectangular vertical fracture solution for $K_y/K_x = 0.01$; $K_z/K_x = 1$; $x_1 = 5000$ m, $x_2 = 5000$ m, $x_f = 0.005$ m; $z_1 = 2$ m, $z_2 = 5$ m, and $z_f = 1.5$ m; and $x' = 10$ m, $y' = 10$ m, and $z = 0$.

Fig. 8. Drawdown comparison with the Hantush (1964) partially-penetrating vertical well solution for $K_y/K_x = 0.01$, $0.05$, and $1$; $K_z/K_x = 0.1$; $b = 10$ m; $x_1 = 1000$ m, $x_2 = 1000$ m, $x_f = 0.005$ m; $z_1 = 2$ m, $z_2 = 5$ m, and $z_f = 1.5$ m; $x' = 1$ m and $y' = 1$ m; and $z_1 = 6$ m and $z_2 = 8$ m.
10. Application to horizontal wells

As mentioned previously, for finite and large $x_1$ and $x_2$, nonzero values of $z_1$ and $z_2$, and small $2x_f$ (line source) values, the geometry in Fig. 1 becomes equivalent to the geometry of a partially-penetrating horizontal well in a confined aquifer. Like the vertical line source wells, for a line-source horizontal well $2x_f = 0.01$ m is used. The horizontal well length ($2x_f$) is 5 m. The rest of parameters are $x_1 = x_2 = 1000$ m, $z_1 = 2$ m, $z_2 = 5$ m, $x' = 1$ m and $y' = 1$ m; and $z_{e1} = 6$ m and $z_{e2} = 8$ m.

11. Conclusions

The main conclusions drawn from this work can be summarized as follows:

1. The $K_x$–$K_y$–$K_z$ solution covers vertical fully and partially penetrating line source wells in confined aquifers.

2. The $K_x$–$K_y$–$K_z$ solution covers horizontal fully and partially penetrating line source wells in confined aquifers.

3. The $K_x$–$K_y$–$K_z$ solution covers vertical fully and partially penetrating rectangular parallelepiped fractures in confined aquifers.

4. The analytical solutions of Theis (1935), Hantush (1964), and others are based on the infinite areal extent aquifer assumption. With the present $K_x$–$K_y$–$K_z$ solution, in addition of Theis and Hantush solutions, the effects of two vertical impermeable boundaries can be taken into account. With large $x_1$ and $x_2$ values; infinite areal aquifer cases, which are equivalent to Theis and Hantush cases, can also be handled.

5. The $K_x$–$K_y$–$K_z$ solution covers vertical fully and partially penetrating line source wells in confined aquifers.

6. Under special conditions (by assigning large values to $x_1$ and $x_2$ and very small values to $2x_f$), the drawdown values of the $K_x$–$K_y$–$K_z$ solution (Hantush, 1964) for a partially penetrating vertical well match very well with the present $K_x$–$K_y$–$K_z$ solution.
8. Under special conditions (by assigning large values to \( x_1 \) and \( x_2 \) and finite values to \( 2x_1 \)), the drawdown values of the \( K_x-K_y \) solution (Gringarten and Ramey, 1973) for a fully penetrating vertical fracture match very well with the present \( K_x-K_y-K_z \) solution.

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Appendix A

A.1. General solution in the Laplace domain

Taking the Laplace transform of Eq. (1) and using Eq. (5) one obtains

\[
p\tilde{s} = \frac{\partial^2 \tilde{s}}{\partial x^2} + \frac{\partial^2 \tilde{s}}{\partial y^2} + \frac{\partial^2 \tilde{s}}{\partial z^2} \tag{A1}
\]

in which

\[
\tilde{s}(x, y, z, s) = \mathcal{L}[s(x, y, z, t)] = \int_0^\infty e^{-st}s(x, y, z, t)dt \tag{A2}
\]

is the Laplace transform of \( s(x, y, z, t) \). The method of separation of variables (e.g., Spiegel, 1965) is applied to (Eq. A1). Therefore, with

\[
\tilde{s}(x, y, z, s) = X(x)Y(y)Z(z) \tag{A3}
\]

Eq. (A1) takes the form

\[
p\eta_x \frac{X''}{X} + \eta_y \frac{Y''}{Y} + \eta_z \frac{Z''}{Z} = 0 \tag{A4}
\]

Dividing both sides by \( \eta_x \) it becomes

\[
\frac{X''}{X} = \frac{\eta_y}{\eta_x} \frac{Y''}{Y} + \frac{\eta_z}{\eta_x} \frac{Z''}{Z} = \lambda^2 \tag{A5}
\]

From Eq. (A5) one can write

\[
X'' + \lambda^2 X = 0 \tag{A6}
\]

and

\[
\eta_y \frac{Y''}{Y} + \eta_z \frac{Z''}{Z} - \frac{p}{\eta_x} = \lambda^2 \tag{A7}
\]

or

\[
\frac{\eta_z}{\eta_x} \frac{Z''}{Z} + \frac{\eta_y}{\eta_x} \frac{Y''}{Y} - \lambda^2 \frac{p}{\eta_x} = K^2 \tag{A8}
\]

Multiplying both sides by \( \eta_x/\eta_z \), Eq. (A8) becomes

\[
\frac{Z''}{Z} \frac{\eta_y}{\eta_x} \frac{Y''}{Y} - \lambda^2 \frac{p}{\eta_x} = K^2 \tag{A9}
\]

From Eq. (A9) one can write

\[
Z'' + k^2 Z = 0 \tag{A10}
\]

and

\[
Y'' - \left( \frac{\eta_y}{\eta_x} \frac{\eta_z}{\eta_x} \frac{\eta_y}{\eta_x} + \frac{p}{\eta_y} \right) Y = 0 \tag{A11}
\]

Solutions of Eqs. (A6), (A10), and (A11), respectively, can be determined as

\[
X(x) = A_3 \sin(kx) + B_1 \cos(kx) \tag{A12}
\]

\[
Y(y) = A_3 \exp(Ky) + B_1 \exp(-Ky) \tag{A13}
\]

and

\[
Z(z) = A_2 \sin(kz) + B_2 \cos(kz) \tag{A14}
\]

where

\[
K = \sqrt{\frac{\eta_y}{\eta_x} \frac{\eta_z}{\eta_x} \frac{k^2}{2} + \frac{p}{\eta_y}} \tag{A15}
\]

Substitution of Eqs. A12, A13, and A14 into Eq. (A3), the general solution of Eq. (A1) in the Laplace domain can be written as

\[
\tilde{s}(x, y, z) = \left[ A_1 \sin(kx) + B_1 \cos(kx) \right] \left[ A_3 \exp(Ky) + B_3 \exp(-Ky) \right] \times \exp(-kz) \cdot \left[ A_2 \sin(kz) + B_2 \cos(kz) \right] \tag{A16}
\]

Appendix B

B.1. Usage of boundary conditions at infinity and at impermeable boundaries

The boundary condition at infinity given by Eq. (6), in the Laplace domain takes the form

\[
\eta_x \frac{X''}{X} \bigg|_{x=\infty} = 0
\]

\[
\eta_y \frac{Y''}{Y} \bigg|_{y=\infty} = 0
\]

\[
\eta_z \frac{Z''}{Z} \bigg|_{z=\infty} = 0
\]
\[
\dot{s}(x, y = \infty, z, p) = 0
\]  
(B1)

To satisfy this boundary condition, (A3) in Eq. (A16) must be zero. Then, Eq. (A16) becomes

\[
\dot{s}(x, y, z, p) = B_2[A_1 \sin(\lambda x) + B_2 \cos(\pi z)]B_2 \cos(\lambda x) + B_2 \sin(\lambda x) + B_2 \cos(\pi z)] \exp(-K y)
\]
(B2)

In the Laplace domain, the boundary conditions given by Eqs. (7)–(10), take respectively, the following forms:

\[
\frac{\partial \dot{s}(x, y, z, p)}{\partial x} = 0
\]  
(B3)

\[
\frac{\partial \dot{s}(x, y, z, p)}{\partial y} = 0
\]  
(B4)

\[
\frac{\partial \dot{s}(x, y, z, p)}{\partial z} = 0
\]  
(B5)

and

\[
\frac{\partial \dot{s}(x, y, z, p)}{\partial b} = 0
\]  
(B6)

The derivatives of Eq. (B2) with respect to \( x \) and \( z \) are

\[
\frac{\partial \dot{s}}{\partial x} = B_2[A_1 \sin(\lambda x) + B_2 \cos(\pi z)]B_2 \cos(\lambda x) + B_2 \sin(\lambda x) + B_2 \cos(\pi z)] \exp(-K y)
\]
and

\[
\frac{\partial \dot{s}}{\partial z} = B_2[A_2 \cos(\pi z) - B_2 \sin(\pi z)]B_2 \cos(\lambda x) + B_2 \sin(\lambda x) + B_2 \cos(\pi z)] \exp(-K y)
\]
(B7)

Eqs. (B3) and (B7) require that \( A_1 \) must be zero. Using \( A_1 = 0 \) and Eqs. (B4) and (B7), one can write

\[
\sin(\alpha L) = 0 \quad \lambda_m = \frac{m \pi}{L} \quad L = x_1 + 2x_y + x_2 \quad n = 0, 1, 2, \ldots
\]  
(B8)

Likewise, Eqs. (B5) and (B8) require that \( A_2 \) must be zero. Using \( A_2 = 0 \) and Eqs. (B6) and (B8), one can write

\[
\sin(\alpha b) = 0 \quad \kappa_m = \frac{m \pi}{L} \quad b = z_1 + 2z_y + z_2 \quad n = 0, 1, 2, \ldots
\]  
(B9)

Using \( A_1 = A_2 = 0 \), Eq. (B2) takes the form

\[
\dot{s}(x, y, z, p) = \dot{s}(x, y, z, p) = B \cos(\lambda x) \cos(\pi z) \exp(-K y)
\]
where \( B = B_1B_2B_2 \).

According to Eqs. (B9) and (B10), both \( \lambda \) and \( \kappa \) have infinite values. Since Eq. (A1) is a linear differential equation, the solution given by Eq. (B11) can be written as (e.g., Churchill, 1941; Carslaw and Jaeger, 1959; Spiegel, 1965):

\[
\dot{s}(x, y, z, p) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn} \cos(\lambda_m x) \cos(\kappa_n z) \exp(-K_{mn} y)
\]
(B12)

where

\[
K_{mn} = \left( \frac{\eta_x}{\eta_y} \right)^{1/2} \left( \frac{\eta_x}{\eta_y} \right)^{1/2} \left( \frac{\eta_x}{\eta_y} \right)^{1/2}
\]
(B13)

Appendix C

C.1. Usage of boundary conditions at \( y = 0 \)

From Eq. (B12)

\[
\frac{\partial \dot{s}(x, y, z, p)}{\partial y} = -\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} K_{mn} \exp(-K_{mn} y) \cos(\kappa_m z)
\]
\times \cos(\lambda_m x)
\]
(C1)

\[
\text{For } y = 0 \text{, Eq. (C1) becomes}
\]
\[
\frac{\partial \dot{s}(x, y = 0, z, p)}{\partial y} = -\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} K_{mn} \cos(\kappa_m z) \cos(\lambda_m x)
\]
\]
(C2)

The Laplace transform of Eq. (11) is

\[
q_{\alpha}(x, y = 0, z, p-y) = -K \frac{\partial \dot{s}}{\partial y} \quad x_1 < x < x_1 + 2x_y \quad y = 0 \quad z_1 < z < z_1 + 2z_y
\]
(C3)

Introducing Eq. (C1) into Eq. (C3) gives

\[
q_{\alpha}(x, y = 0, z, p-y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} E_{mn} \cos(\kappa_m z) \cos(\lambda_m x)
\]
(C4)

where

\[
E_{mn} = K_{mn} \cos(\lambda_m x)
\]
(C5)

Appendix D

D.1. Determination of the general form of \( E_{mn} \)

Substitution of Eqs. (B9) and (B10) into Eq. (C4) gives

\[
q_{\alpha}(x, y = 0, z, p-y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} E_{mn} \cos(\lambda_m x) \cos(\kappa_m z)
\]
\]
(D1)

The coefficients \( E_{mn} \) are the coefficients of the double Fourier cosine series expansion of \( q_{\alpha}(x, y = 0, z, p-y) \). \( E_{mn} \) can be obtained by the method given in applied mathematics (e.g., Hildebrand, 1976, p. 455).

If both sides of Eq. (D1) are multiplied by \( \cos(a_1 \pi x / b) \cos(a_2 \pi x / L) \) where \( a_1 \) and \( a_2 \) are arbitrary positive integers, and if the results are integrated over the rectangle, then follows:

\[
\int_0^b \int_0^{a_1 \pi x / b} \cos(\lambda_m x) \cos(\kappa_n z) \exp(-K_{mn} y) dx dz
\]
\]
\]
(D2)

The double integral on the right-hand side can be expressed as

\[
\int_0^b \int_0^{a_1 \pi x / b} \cos(\lambda_m x) \cos(\kappa_n z) \exp(-K_{mn} y) dx dz
\]
\]
\]
(D3)

Using the relation (e.g., Hildebrand, 1976, p. 217, Eq. (146))

\[
\int_0^l \cos(\lambda_m x) \cos(\kappa_n z) dx = 0 \quad m \neq n
\]
\]
(D4)

The product in Eq. (D3) vanishes unless \( a_1 = m \) and \( a_2 = n \), in which case it has the value

\[
\int_0^l \cos(\lambda_m x) \cos(a_2 \pi x / L) dx = \frac{1}{2} \quad \text{for } m = 1, 2, \ldots \quad n = 1, 2, 3, \ldots
\]
\]

Thus, the double series in the right-hand member of Eq. (D2) reduces to a single term, for which \( m = a_1 \) and \( n = a_2 \), and there follows:

\[
\int_0^l \cos(\lambda_m x) \cos(\kappa_n z) dx = 0 \quad m \neq n
\]
\]

(D5)
\[ E_{mm} = \frac{4}{bl} \int_0^b \int_0^b \frac{q_m(x,y = 0,z)}{p - y} \cos \left(\frac{m\pi x}{b}\right) \cos \left(\frac{n\pi x}{L}\right) dz \, dx \] (D6)

**Appendix E**

**E.1. Determination of the components of \( e_{mm} \) and \( b_{mm} \) for a single rectangular screen**

According to Eq. (11), the Darcy flux is valid over 2\(x_L \) by 2\(z_L \) rectangular area shown in Fig. 1. Using Eqs. (B9) and (B10), for the special case, Eq. (D6) takes the form

\[ E_{mm} = \frac{4}{bl} \frac{q_m}{(p - \gamma)} \int_{x_L}^{x_L + 2x_L} \int_{z_L}^{z_L + 2z_L} \cos \left(\frac{K_m z}{b}\right) \cos \left(\frac{\lambda_m x}{L}\right) dz \, dx \] (E1)

Eq. (E1) can also be expressed as

\[ E_{mm} = \frac{4}{bl} \frac{q_m}{(p - \gamma)} \int_{x_L}^{x_L + 2x_L} \int_{z_L}^{z_L + 2z_L} \cos(\lambda_m x) dz \cos(\lambda_m z) \, dx \] (E2)

After performing the integral in the right-hand side of Eq. (E2) with respect to \( x \), it becomes

\[ E_{mm} = \frac{4}{bl} \frac{q_m}{(p - \gamma)} \lambda_m \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] - \sin(\lambda_m z_1) \] (E3)

After performing the second integral and using Eqs. (B9) and (B10), Eq. (E3) takes the form

\[ E_{mm} = \frac{4}{bl} \frac{q_m}{(p - \gamma)} \frac{1}{b} \frac{4}{\pi} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \left[ \sin(\lambda_m z_1 + 2z_L) - \sin(\lambda_m z_1) \right] \] (E4)

Substitution of \( m = 0 \) in Eq. (E1) and performing the integration gives

\[ E_{00} = \frac{q_m}{b} \frac{4}{bl} \frac{4}{\pi} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \] (E5)

Similarly, for \( n = 0 \), Eq. (E1) gives

\[ E_{mm} = \frac{q_m}{(p - \gamma)} \frac{4}{bl} \frac{4}{\pi} \left[ \sin(\lambda_m z_1 + 2z_L) - \sin(\lambda_m z_1) \right] \] (E6)

And for \( m = n = 0 \), Eq. (E1) results

\[ E_{00} = \frac{4}{bl} \frac{q_m}{(p - \gamma)} (2x_L)(2z_L) \] (E7)

With the fact that \( B_{0m} \) and \( B_{nm} \) have coefficient of one-half and \( B_{00} \) has one-quarter for double Fourier series (e.g., Carslaw and Jaeger, 1959, p. 182), Eq. (B12) can be expressed as

\[ \frac{s(x,y,z,p) = \frac{1}{2} \left( \frac{1}{2} \right) \frac{1}{2} B_{00} \exp(-K_{00}y) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} \cos(K_m z) \right)}{\cos(\lambda_m z) \exp(-K_{mm}y)} \]

\[ \int \frac{\cos(\lambda_m x) \exp(-K_{mm}x)}{\cos(\lambda_m z) \exp(-K_{mm}y)} \int \frac{\cos(\lambda_m x)}{\cos(\lambda_m z) \exp(-K_{mm}y)} \int \] (E8)

The expression for \( K_m \) is given by Eq. (B13) in Appendix B. Therefore, from Eq. (B13) the following expressions can be written:

\[ K_{00} = \left( \frac{p}{\eta}\right)^2 \] (E9)

\[ K_{0n} = \left( \frac{\eta \pi}{b} \frac{\pi}{L} \right)^{\frac{1}{2}} \] (E10)

\[ K_{m0} = \left( \frac{\eta \pi}{b} \frac{\pi}{L} \right)^{\frac{1}{2}} \] (E11)

Also, from Eq. (C5) of Appendix C the following expressions can be written:

\[ E_{00} = K_B B_{00} K_{00} \] (E12)

\[ E_{0n} = K_B B_{0n} K_{0n} \] (E13)

\[ E_{m0} = K_B B_{m0} K_{m0} \] (E14)

From Eq. (C5) of Appendix C

\[ B_{nm} = \frac{E_{nm}}{K_B K_{nm}} \] (E15)

and using Eq. (E4) it becomes

\[ B_{00} = \frac{4}{b} \frac{q_m}{(p - \gamma)} \frac{1}{L} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \] (E16)

From Eq. (E12)

\[ B_{00} = \frac{E_{00}}{K_B K_{00}} \] (E17)

and using Eq. (E7) it becomes

\[ B_{00} = \frac{4}{b} \frac{q_m}{(p - \gamma)} \frac{1}{L} \left[ \sin(\lambda_m z_1 + 2z_L) - \sin(\lambda_m z_1) \right] \] (E18)

From Eq. (E13)

\[ B_{00} = \frac{E_{00}}{K_B K_{00}} \] (E19)

and using Eq. (E5) it becomes

\[ B_{00} = \frac{4}{b} \frac{q_m}{(p - \gamma)} \frac{1}{L} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \] (E20)

From Eq. (E14)

\[ B_{m0} = \frac{E_{m0}}{K_B K_{m0}} \] (E21)

and using Eq. (E6) it becomes

\[ B_{m0} = \frac{4}{b} \frac{q_m}{(p - \gamma)} \frac{1}{L} \left[ \sin(\lambda_m z_1 + 2z_L) - \sin(\lambda_m z_1) \right] \] (E22)

**Appendix F**

**F.1. Final solution in the Laplace domain**

According to Eq. (E8) of Appendix E, the solution in the Laplace domain has four components and Eq. (E8) can be written as

\[ \frac{s(x,y,z,p) = \frac{1}{2} \left( \frac{1}{2} \right) \frac{1}{2} \frac{q_m}{b} \frac{4}{bl} \left( \frac{4}{\pi} \right) \frac{4}{\pi} \frac{1}{\pi} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \exp(-K_{00}y) \] (F1)

Substitution of Eq. (E18) into the first term on the right-hand side of Eq. (E8) results

\[ \frac{s(x,y,z,p) = \frac{1}{2} \left( \frac{1}{2} \right) \frac{1}{2} \frac{q_m}{b} \frac{4}{bl} \left( \frac{4}{\pi} \right) \frac{4}{\pi} \frac{1}{\pi} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \exp(-K_{00}y) \] (F2)

Likewise, substitution of Eq. (E20) into the second term on the right-hand side of Eq. (E8) results

\[ \frac{s(x,y,z,p) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{4}{b} \frac{q_m}{(p - \gamma)} \frac{1}{L} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \exp(-K_{00}y) \] (F3)

Substitution of Eq. (E20) into the third term on Eq. (E8) gives

\[ \frac{s(x,y,z,p) = \sum_{m=1}^{\infty} \frac{1}{2} \frac{q_m}{b} \frac{4}{bl} \left( \frac{4}{\pi} \right) \frac{4}{\pi} \frac{1}{\pi} \left[ \sin(\lambda_m x_1 + 2x_L) - \sin(\lambda_m x_1) \right] \exp(-K_{00}y) \] (F4)
Finally, Substitution of Eq. (E22) into the fourth term of Eq. (E8) gives

\[ \tilde{s}_4(x, y, z, p) = \sum_{n=1}^\infty \frac{1}{L} \int_0^1 \frac{4q_m}{(p - \gamma)} \frac{1}{\pi} \sin(k_m(z_1 + 2z_\gamma)} - \sin(k_mz_1)) \frac{1}{K_yK_m} \cdot \cos(k_mx) \exp(-K_my) \] (F5)

### Appendix G

#### G.1. Inverse Laplace transform

The inverse Laplace transform of Eq. (F1) of Appendix F is Eq. (14). Therefore, from the inverse Laplace transforms of Eq. (F1) one can write

\[ L^{-1}[\mathfrak{S}(x, y, z, p)] = L^{-1}[\mathfrak{S}_1(x, y, z, p)] + L^{-1}[\mathfrak{S}_2(x, y, z, p)] + L^{-1}[\mathfrak{S}_3(x, y, z, p)] + L^{-1}[\mathfrak{S}_4(x, y, z, p)] \] (G1)

and from this Eq. (12) can be obtained. Now, the four components of the s(x,y,z,t) drawdown expression as given by Eq. (14) will be determined.

Determination of \( s_1(x, y, z, t) \)

Using Eq. (E9) of Appendix E in Eq. (F2) of Appendix F one can write

\[ \mathfrak{S}_1(x, y, z, p) = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \frac{4}{BL} \frac{q_m(2k_x)(2k_y)}{(p - \gamma)} e^{-\frac{t}{\gamma}} \frac{1}{K_y}\left(\frac{x}{\gamma}\right)^2 \] (G2)

Eq. (G2) can be written as

\[ \mathfrak{S}_1(x, y, z, p) = g_1(x, y, z, p)f_1(x, y, z, p) \] (G3)

where

\[ f_1(x, y, z, p) = e^{-\frac{t}{\gamma}} \frac{1}{K_y}\left(\frac{x}{\gamma}\right)^2 \] (G4)

and

\[ g_1(x, y, z, p) = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \frac{4}{BL} \frac{q_m(2k_x)(2k_y)}{(p - \gamma)} \] (G5)

From Carslaw and Jaeger (1959, p. 494, Eq. (12)), the inverse Laplace transform of Eq. (G4) is

\[ F_1(t) = L^{-1}[f_1(p)] = \left(\frac{\eta_x}{\pi t}\right)^{\frac{1}{4}} \frac{1}{K_y} \exp \left(-\frac{\eta_y}{4\eta_x t}\right) \] (G6)

The inverse Laplace transform of Eq. (G5) is

\[ G_1(t) = L^{-1}[g_1(p)] = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \frac{4}{BL} (2k_x)(2k_y) e^{\eta_x t} \] (G7)

According to the convolution or Faltung theorem (e.g., Spiegel, 1965, p. 45) if

\[ L^{-1}[f_1(p)] \cdot L^{-1}[g_1(p)] = G_1(t) \] (G8)

then

\[ L^{-1}[f_1(p)g_1(p)] = \int_0^t F_1(u)G_1(t - u)du \] (G9)

Substitution of Eqs. (G6) and (G7) into Eq. (G9) results Eq. (15).

Determination of \( s_2(x, y, z, t) \)

Using Eq. (B13) of Appendix B in Eq. (F3) of Appendix F, one can write

\[ \mathfrak{S}_2(x, y, z, p) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{4}{(m\pi)(n\pi)} q_m \sin(\lambda_m(x_1 + 2x_\gamma)} - \sin(\lambda_mz_1)) \cdot \sin(k_mz_1)) \cos(k_mz) \cos(\lambda_mx) \frac{1}{K_y} \exp \left[-\left(\frac{\eta_x}{\gamma} \right)^2 y \right] \] (G10)

Eq. (G10) can also be expressed as

\[ \mathfrak{S}_2(x, y, z, p) = f_2(x, y, z, p)g_2(x, y, z, p) \] (G11)

where

\[ f_2(x, y, z, p) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{4}{(m\pi)(n\pi)} q_m \sin(\lambda_m(x_1 + 2x_\gamma)} - \sin(\lambda_mz_1)) \cdot \sin(k_mz_1)) \cos(k_mz) \cos(\lambda_mx) \frac{1}{K_y} \exp \left[-\left(\frac{\eta_x}{\gamma} \right)^2 y \right] \] (G12)

\[ g_2(x, y, z, p) = \frac{1}{p - \gamma} \] (G13)

Eq. (G12) can also be expressed as

\[ f_2(x, y, z, p - a) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{4}{(m\pi)(n\pi)} q_m \sin(\lambda_m(x_1 + 2x_\gamma)} - \sin(\lambda_mz_1)) \cdot \sin(k_mz_1)) \cos(k_mz) \cos(\lambda_mx) \frac{1}{K_y} \exp \left[-\left(\frac{\eta_x}{\gamma} \right)^2 y \right] \] (G14)

where

\[ a = -\eta_x \lambda^2 - \eta_y \lambda^2 \] (G15)

The inverse Laplace transform of Eq. (G13) is

\[ G_2(t) = e^{\eta_x t} \] (G16)

According to the first translation or shifting property (e.g., Spiegel, 1965, Theorem 2-3, p. 43) if

\[ F_2(t) = L^{-1}[f_2(p - a)] = e^{\eta_x t}G_2(t) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{4}{(m\pi)(n\pi)} q_m \sin(\lambda_m(x_1 + 2x_\gamma)} - \sin(\lambda_mz_1)) \cdot \sin(k_mz_1)) \cos(k_mz) \cos(\lambda_mx) \frac{1}{K_y} \exp \left[-\left(\frac{\eta_x}{\gamma} \right)^2 y \right] \] (G17)

where

\[ L^{-1}[g_3(p)] = G_3(t) \] (G18)

where

\[ g_3(p) = \exp \left[-\left(\frac{\eta_x}{\gamma} \right)^2 y \right] \] (G19)
From Carslaw and Jaeger (1959, p. 494, Eq. (12)), the inverse Laplace transform of Eq. (G19) is

\[ G_3(t) = L^{-1}[g_3(p)] = \left(\frac{\eta_y}{\pi t}\right)^{\frac{1}{2}} \frac{1}{K_y} \exp\left(-\frac{y^2}{4\eta_y t}\right) \tag{G20} \]

According to the convolution or Faltung theorem (e.g., Spiegel, 1965, p. 45) if

\[ L^{-1}[f_2(p)] = F_2(t) \quad L^{-1}[g_2(p)] = G_2(t) \tag{G21} \]

then

\[ s_2(x,y,z,t) = L^{-1}[f_2(p)g_2(p)] = \int_0^t F_2(u)G_2(t-u)\,du \tag{G22} \]

Using of Eqs. (G15)–(G17), and (G20) into Eq. (G22), finally Eq. (16) can be obtained.

Determination of \( s_2(x,y,z,t) \).

The inverse Laplace transform of Eq. (F4) can similarly be taken and the final result is for \( s_3(x,y,z,t) \) is given by Eq. (17).

Determination of \( s_4(x,y,z,t) \).

The inverse Laplace transform of Eq. (F5) can similarly be taken and the final result is for \( s_4(x,y,z,t) \) is given by Eq. (18).

References


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