This paper follows the framework of P. Klein (1996) to price vulnerable options when the market is incomplete. Vulnerable options, which are usually traded in the over-the-counter market, may not only face the risk of default but also the risk of illiquidity. Thus, pricing such options under the assumption of market completeness, as was done by H. Johnson and R. Stulz (1987) and P. Klein (1996), seems to be a mistake. Accordingly, the proposed model uses the methodology proposed by J. H. Cochrane and J. Saá-Requejo (2000) to price vulnerable options under both deterministic and stochastic interest rates in an incomplete market. The model is found to perform well when the interest rate is stochastic. © 2005 Wiley Periodicals, Inc. Jrl Fut Mark 25:135–170, 2005
default risk of the security issuers may be neglected. However, this ideal trading condition only exists in classical economic theory. In the real world, default risk may even be associated with large, listed companies, as evidenced by the bankruptcies of Enron and Worldcom, but such risk is more likely to exist in the over-the-counter (OTC) market, which has no organized exchange, such as a clearing corporation. Hence, holders of these contracts are vulnerable to default risk. Derivatives traded in the OTC market typically not only face the risk of default but also the risk of illiquidity. The source of illiquidity may be the derivatives themselves, the underlying assets of the derivatives, the firm value of the counterparty who issues the derivatives, or even the collateral assets of the counterparty. When pricing the derivatives in such an imperfect OTC market, the total risk of a contingent claim should be divided into the default risk and the liquidity risk\(^1\) (Lotz, 1998; Ericsson & Renault, 2001). Only considering the default risk overestimates the value of the derivative.

A vulnerable option is, as first proposed by Johnson and Stulz (1987), an option whose counterparty may default on the obligations it has in the contract. Since options have traditionally been assumed to be default-free derivatives, the incorporation of default risk into the traditional model prevents any option-pricing formulae from evaluating vulnerable options. The vulnerable option pricing method of Johnson and Stulz (1987) assumes that the option is the only liability of the firm and the default occurs when the value of the option outgrows the value of the option writer’s collateral assets. Following Johnson and Stulz (1987), Hull and White (1995) proposed a model to price vulnerable options. Their model expanded that of Johnson and Stulz (1987) to allow the counterparty to have other liabilities of equal priority. When default occurs, only a proportion of the original claims are paid to the option holders. Jarrow and Turnbull (1995) provided a new methodology for pricing and hedging derivative securities subject to default risk. Their model can be applied to securities other than vulnerable options. Rich (1996) proposed a model to price European options subject to an intertemporal default risk. His model puts into perspective the timing of the default and the uncertain recovery value. It can also be used to evaluate current margin requirements made for exchange-trade options. Klein (1996) criticized Johnson and Stulz’s (1987) assumption that the option could be treated as the only liability in the option writer’s capital

\(^1\)Lotz (1998) divides the total risk of a contingent claim into traded risk and totally nontradable risk when the market is incomplete. Ericsson and Renault (2001) divide total risk into liquidity and credit risk.
structure, improving their model to allow the option writer to have other liabilities of equal priority payment under the option. However the assets of the counterparty are usually nontradable. Therefore, treating them as tradable assets when pricing vulnerable options seems to be a major drawback. Hence, a model must be developed to accommodate the incompleteness of the real market when pricing vulnerable options.

This paper follows the idea of Cochrane and Saá-Requejo (2000) to develop a method to price vulnerable options when the market is incomplete. Cochrane and Saá-Requejo (2000) proposed the “discount factor” asset-pricing method to deal with the evaluation of uncertain payoffs. This type of valuation lies between the rigidity of model-based pricing and the looseness of no-arbitrage pricing. This method is appropriate for situations in which purely preference-free approaches break down, such as during nontrading or thin trading circumstances. Additionally restricting the discount factor by both a positivity constraint and a volatility constraint causes the range of the price of the asset to be smaller than that obtained with general arbitrage; such constraints are called “good-deal bounds.” An exact solution for options for which the underlying asset is nontraded cannot be found, so, following Cochrane and Saá-Requejo (2000), “good-deal bounds” are found for the vulnerable option.

To see the impact of the interest rate on the good-deal bounds, the model is implemented under a deterministic interest rate as well as a stochastic interest rate. Amazingly, the performance of good-deal bounds is tighter given a stochastic interest than a deterministic interest rate. This condition is explained by the fact that when the interest rate is more volatile, the Sharpe ratio of the tradable assets increases if the interest rate is deterministic. The Sharpe ratio of the tradable assets then more closely approaches the volatility constraint. In this case, the good-deal bounds are much tighter. In addition to the general case that both the underlying asset as well as the assets of the counterparty are illiquid, we will also discuss the subcases when the assets of the counterparty are illiquid, and when the underlying asset of the vulnerable options is nontraded.

The article is structured as follows. The first section presents an overview of the research done on default risk models and the superiority of the proposed model in dealing with the problems of an incomplete market, which has been seldom discussed in previous articles. The second section introduces the proposed model. The third section shows that other pricing solutions in literature can be expressed as special cases of the proposed model. The fourth section numerically analyzes the model. Finally, the last section draws conclusions.
THE MAIN MODEL: BOTH THE ASSET UNDERLYING THE OPTION AND THE ASSETS OF COUNTERPARTY ARE NONTRADED

This section establishes the model of vulnerable options in an incomplete market. The expected actual payout $B^*$ is first defined as follows:

$$B^* = E^*[B \mid V_T \geq D^*] + E^*[B(1 - \alpha)V_T/D \mid V_T < D^*]$$  \hspace{1cm} (1)$$

where

1. $B^*$ represents the expected actual payout.
2. $B$ is the nominal claim on the counterparty.
3. $V_T$ is the total value of assets of the counterparty at time $T$.
4. $\alpha$ is the value of the deadweight costs associated with bankruptcy and is expressed as a percentage of the value of the assets of the counterparty.
5. $D$ is the total amount associated with claims.
6. $D^*$ is the critical value of claims; default occurs when $V_T < D^*$.

The expected actual payout in Equation (1) is the same as that in Klein (1996) and can be divided into two parts. The first part of Eq. (1) means that if $V_T \geq D^*$, then the terminal value of the counterparty’s total assets is enough to pay the claims on it, and no credit loss occurs. In contrast, if $V_T < D^*$, default occurs, and the counterparty pays out only the proportion $(1 - \alpha)V_T/D$ of the normal claim. After the actual payment is defined under conditions of default or otherwise, this payout is incorporated into the vulnerable Black-Scholes option formula, pricing such an option under incomplete market conditions.

Pricing vulnerable options in an incomplete market can be accomplished in many ways. As we have discussed before, the source of illiquidity may exist in various types. Let us first discuss the most general case “when both the underlying asset as well as the assets of the counterparty are illiquid.”

Suppose $S$ is a nontraded asset underlying the option with $Y$ as its tradable twin security. These two processes can be expressed as follows:

$$\frac{dS}{S} = \mu_S dt + \sigma_{Sz} dz^p_S + \sigma_{Su} dw^p_Y \quad \text{and} \quad \frac{dY}{Y} = \mu_Y dt + \sigma_Y dz^p_Y$$  \hspace{1cm} (2)$$

Since $V$, the assets of the counterparty, is also a nontraded asset, we must find its twin security $\xi$ in the market in order to observe its market
price through $\xi$. The processes of the nontraded asset of the counterparty $V$ and its tradable twin security $j$ can be expressed as follows:

\[
\frac{dV}{V} = \mu_V \, dt + \sigma_{Vz} \, dz^p_V + \sigma_{Vx} \, dx^p \quad \text{and} \quad \frac{d\xi}{\xi} = \mu_\xi \, dt + \sigma_\xi \, dz^p_\xi \quad (3)
\]

The relationship between $dz^p_V$, $dz^p_\xi$ and $dw^p$ are as follows:

\[
E((dz^p_V)^2) = E((dz^p_\xi)^2) = E((dw^p)^2) = 1,
\]

\[
\rho_{wz_V} = E(dw^p \, dz^p_V) = 0, \quad \rho_{wz_\xi} = E(dw^p \, dz^p_\xi) = 0,
\]

\[
\rho_{sz_V} = E(dx^p \, dz^p_V) = 0, \quad \rho_{sz_\xi} = E(dx^p \, dz^p_\xi) = 0
\]

Under the complete market assumption, an option is usually priced as follows:

\[
C_t = E_t^p \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \right]
\]

where $\frac{dS_U}{S_U} = \mu_S \, dt + \sigma_S \, dz^p$ is a tradable asset underlying the option, $\frac{d\Lambda_T}{\Lambda_T} = -r \, dt - h_V \, dz^p$ is the discount factor of the option payoff, and $h_V = \frac{\mu_V - r}{\sigma_V}$ is the Sharpe ratio.\(^2\) The exercise price $K$ is subtracted from the terminal value of the stock price $S_T$ and discounted back by $\frac{\Lambda_T}{\Lambda_t}$ to derive the price of the call option.

However, when the asset underlying the option and the assets of the counterparty are nontraded, perfect hedging is impossible. Hence, the usual “no arbitrage” methods employed in the Black-Scholes formula are not applicable. However, the concept behind “approximate hedging” can be used to price options whose underlying assets and the assets of the counterparty are non-tradable. Following Cochrane and Saá-Requejo (2000), weak restrictions are imposed on the discount factor to exclude arbitrage opportunities and high Sharpe ratios. Imposing the volatility and positivity constraints on the stochastic discount factor and trying all kinds of combinations of binding and nonbinding constraints, yields a lower bound for the call option given by the following Equation:

\[
C_t = \min_{\Lambda} E_t^p \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \right],
\]

\[
s.t. \quad S_t = E_t^p \left( \frac{\Lambda_u}{\Lambda_t} S_u \right); \quad \Lambda_u > 0; \quad \frac{1}{dt} \, E_t^p \frac{d\Lambda^2}{\Lambda^2} \leq \Lambda^2; \quad t \leq u \leq T \quad (4)
\]

\(^2\)Please see, for example, Cochrane (2001) for more details.
where $A$ is the volatility constraint. Similarly, the upper bound on the call option can be derived from the corresponding maximum. These bounds are called the “good-deal” bounds for the call option.\(^3\)

The differences between Cochrane and Saá-Requejo (2000) and our model are that there exists two nontraded securities in our model instead of theirs. When there are two nontraded securities, in addition to imposing the volatility and positivity constraints on the stochastic discount factor, one additional constraint has to be imposed in order to have a unique stochastic discount factor. Here, we assume that

\[ m \times \text{Cov}(\frac{d\Lambda}{\Lambda}, dw') = \text{Cov}(\frac{d\Lambda}{\Lambda}, dx'), \]

that is, the influence of white noise $dx$ is $m$ times greater on discount factor $\frac{d\Lambda}{\Lambda}$ than white noise $dw$. Hence, the lower bound for the call option is given by the following equation:

\[
C_t = \min_{\Lambda} E_t^p \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \right],
\]

s.t. \( S_t = E_t^p \left[ \frac{\Lambda_u}{\Lambda_t} S_u \right]; \quad \Lambda_u > 0; \quad \frac{1}{dt} E_t^p \frac{d\Lambda^2}{\Lambda^2} \leq A^2; \)

\[
m \times \text{Cov}(\frac{d\Lambda}{\Lambda_t}, dw') = \text{Cov}(\frac{d\Lambda}{\Lambda_t}, dx'); \quad t \leq u \leq T \quad (5)
\]

In this case, the discount factor becomes

\[
\frac{d\Lambda_T}{\Lambda_t} = -r dt - [\tilde{\mu}_Y \quad \tilde{\mu}_\xi] \begin{pmatrix} \frac{1}{\sigma^2_Y} & 0 \\ 0 & \frac{1}{\sigma^2_\xi} \end{pmatrix} \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_\xi \end{pmatrix} \begin{pmatrix} dz_Y \\ dz_\xi \end{pmatrix} \\
- \sqrt{\frac{1}{m + 1} \left( A^2 - [\tilde{\mu}_Y \quad \tilde{\mu}_\xi] \begin{pmatrix} \frac{1}{\sigma^2_Y} & 0 \\ 0 & \frac{1}{\sigma^2_\xi} \end{pmatrix} \begin{pmatrix} \tilde{\mu}_Y \\ \tilde{\mu}_\xi \end{pmatrix} \right)^2} dw^p \\
- \sqrt{\frac{m}{m + 1} \left( A^2 - [\tilde{\mu}_Y \quad \tilde{\mu}_\xi] \begin{pmatrix} \frac{1}{\sigma^2_Y} & 0 \\ 0 & \frac{1}{\sigma^2_\xi} \end{pmatrix} \begin{pmatrix} \tilde{\mu}_Y \\ \tilde{\mu}_\xi \end{pmatrix} \right)^2} dx^p \quad (6)
\]

where $\tilde{\mu}_Y = \mu_Y - r$, $\tilde{\mu}_\xi = \mu_\xi - r$, $m$ is a constant and $0 < m < \infty$.

\(^3\)For more details about this asset pricing method, please refer to Cochrane and Saá-Requejo (2000). We substitute for $\max(S_T - K, 0)$ and set the dividend ($x_\xi'$) to be zero in Equation (26) in Cochrane and Saá-Requejo (2000).
According to Equations (2), (3), and (6), the vulnerable option when both the asset underlying the option and the assets of the counterparty are nontraded can be expressed as follows:

\[
C_t = E_t^p \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \left[ 1 \mid V_T \geq D^* \right] \right. \\
+ \left. \left[ (1 - \alpha)V_T/D \mid V_T < D^* \right] \right] 
\]  
(7)

It is shown in Appendix A, the formula of the vulnerable option when both the asset underlying the option and the asset of the counterparty are nontraded can be derived as follows:

\[
C_t = S_t e^{\eta_0(T-t)}N(a_1^*, a_2^*) - Ke^{-r(T-t)}N(b_1^*, b_2^*) \\
+ S_t V_t e^{(r+\eta_1+\eta_2)(T-t)} \frac{(1 - \alpha)}{D} N(c_1^*, c_2^*) - KV_t e^{\eta_3(T-t)}N(d_1^*, d_2^*) 
\]  
(8)

where

\[
\sigma_S^2 = \sigma_S^2 + \sigma_S^2; \quad \sigma_V^2 = \sigma_V^2 + \sigma_V^2 \\
h_V = \frac{\mu_V - r}{\sigma_V} ; \quad h_Y = \frac{\mu_Y - r}{\sigma_Y} \\
h_S = \frac{\mu_S - r}{\sigma_S} ; \quad h_\xi = \frac{\mu_\xi - r}{\sigma_\xi} \\
a_1^* = \ln \frac{S_t}{K} + (r + \eta_1)(T - t) + \frac{1}{2} \sigma_S \sqrt{T - t} \\
a_2^* = \ln \frac{V_t}{D^*} (r + \eta_2)(T - t) + \frac{1}{2} \sigma_V \sqrt{T - t} \\
b_1^* = \ln \frac{S_t}{K} + (r + \eta_1)(T - t) - \frac{1}{2} \sigma_S \sqrt{T - t} \\
b_2^* = \ln \frac{V_t}{D^*} (r + \eta_2)(T - t) - \frac{1}{2} \sigma_V \sqrt{T - t} \\
c_1^* = \ln \frac{S_t}{K} + (r + \eta_1)(T - t) + \frac{1}{2} \sigma_S \sqrt{T - t} \\
c_2^* = -\ln \frac{V_t}{D^*} (r + \eta_2)(T - t) - \frac{1}{2} \sigma_V \sqrt{T - t} \\
d_1^* = \ln \frac{S_t}{K} + (r + \eta_1)(T - t) - \frac{1}{2} \sigma_S \sqrt{T - t} \\
d_2^* = -\ln \frac{V_t}{D^*} (r + \eta_2)(T - t) - \frac{1}{2} \sigma_V \sqrt{T - t} 
\]
SPECIAL CASES

This section shows that some other pricing methods are just special cases of the proposed formula.

The Assets of the Counterparty Are Nontraded

In this subsection, we will discuss the case when the underlying asset of the vulnerable option is tradable, but the assets of the counterparty are not. Let $S$ be the underlying asset and $V$ be the assets of the counterparty. Their processes can be expressed as follows:

\[ \frac{dS}{S} = \mu_S \, dt + \sigma_S \, dz^p \]  \hspace{1cm} (9)

\[ \frac{dV}{V} = \mu_V \, dt + \sigma_{Vz} \, dz^p + \sigma_{Vv} \, dw^p \]  \hspace{1cm} (10)

Since $S$ and $V$ are correlated and $S$ is a tradable asset, we use $S$ as the twin security of $V$. The discount factor that we use to price the vulnerable option is set up as below:

\[ \frac{d\Lambda}{\Lambda} = -r \, dt - h_S \, dz^p - \sqrt{A^2 - h_S^2} \, dw^p \]  \hspace{1cm} (11)

According to Equations (9), (10), and (11), the vulnerable option when the asset underlying the option is tradable, and the assets of the counterparty are not, can be expressed as follows:

\[ C_t = E^p_i \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \right( [1|V_T \geq D^*] + [(1 - \alpha)V_T/D|V_T < D^*] \) \right] \]  \hspace{1cm} (12)
The formula of a vulnerable option, when the underlying asset is tradable but the assets of the counterparty are not can be derived as follows:

\[
C_t = S_t N_2(a'_1, a'_2, \rho) - e^{-r(T-t)}K N_2(b'_1, b'_2, \rho) \\
+ e^{(\eta' r + \rho \sigma_V \sigma_S)(T-t)} \frac{1 - \alpha}{D} V_t S_t N_2(c'_1, c'_2, -\rho) \\
- \frac{V_t K}{D}(1 - \alpha)e^{\eta'(T-t)}N(d'_1, d'_2, -\rho) 
\]

(13)

where

\[
\begin{align*}
\sigma_v^2 &= \sigma_{Vz}^2 + \sigma_{wV}^2 \\
a'_1 &= \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2} \sigma_S^2)(T - t)}{\sigma_S \sqrt{T - t}} \\
a'_2 &= \frac{\ln \frac{V_t}{D} + (r + \eta' - \frac{1}{2} \sigma_V^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_V \sqrt{T - t}} \\
b'_1 &= \frac{\ln \frac{S_t}{K} + (r - \frac{1}{2} \sigma_S^2)(T - t)}{\sigma_S \sqrt{T - t}} \\
b'_2 &= \frac{\ln \frac{V_t}{D} + (r + \eta' - \frac{1}{2} \sigma_V^2)(T - t)}{\sigma_V \sqrt{T - t}} \\
c'_1 &= \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2} \sigma_S^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_S \sqrt{T - t}} \\
c'_2 &= -\frac{\ln \frac{V_t}{D} + (r + \eta' + \frac{1}{2} \sigma_V^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_V \sqrt{T - t}} \\
d'_1 &= \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2} \sigma_S^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_S \sqrt{T - t}} \\
d'_2 &= -\frac{\ln \frac{V_t}{D} + (r + \eta' + \frac{1}{2} \sigma_V^2)(T - t)}{\sigma_V \sqrt{T - t}} \\
h_V &= \frac{\mu_V - r}{\sigma_V} \\
h_S &= \frac{\mu_S - r}{\sigma_S} \\
\eta' &= [h_V - h_S \left( \rho - a \sqrt{\frac{A^2}{h_S^2} - 1} \sqrt{1 - \rho^2} \right)] \sigma_V \\
\rho &= \text{corr} \left( \frac{dV}{V}, \frac{dS}{S} \right) = \frac{\sigma_{Vz}}{\sigma_V} \\
a &= \begin{cases} +1 \text{ upper bound} \\ -1 \text{ lower bound} \end{cases}
\end{align*}
\]
It should be noted that the solution degenerates into the univariate normal when \( \rho = 1 \):\(^4\)

\[
C_t = \begin{cases} 
S_t \cdot N(a_3') - K \cdot e^{-r(T-t)} \cdot N(a_4') & \text{when } S_T > K > D^* \\
S_t \cdot N(b_3') - K \cdot e^{-r(T-t)} \cdot N(b_4') & \text{when } S_T > D^* > K \\
\frac{1-e^{(r+\sigma_v^2)(T-t)}}{D^*}S_t \cdot (N(c_3') - N(c_4')) & \text{when } D^* > S_T > K \\
-K \cdot \frac{1-e^{a^\prime}}{D}S_t \cdot (N(d_3') - N(d_4')) & \text{when } D^* > S_T > K
\end{cases}
\]

where

\[
a_3' = \frac{\ln \frac{S_t}{K} + (r + 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
a_4' = \frac{\ln \frac{S_t}{K} + (r - 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
b_3' = \frac{\ln \frac{S_t}{D^*} + (r + 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
b_4' = \frac{\ln \frac{S_t}{D^*} + (r - 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
c_3' = \frac{\ln \frac{S_t}{D^*} + (r + 3/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
c_4' = \frac{\ln \frac{S_t}{D^*} + (r + 3/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
d_3' = \frac{\ln \frac{S_t}{D^*} + (r + 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}} \\
d_4' = \frac{\ln \frac{S_t}{D^*} + (r + 1/2\sigma_v^2) (T - t)}{\sigma_v \sqrt{T - t}}
\]

As it has been discussed in Equation (1). If \( B \) is a fixed nominal payoff, the same as the one discussed by Klein (1996), then the discounted expected value of the amount recovered is

\[
e^{-r(T-t)}B^* = B \left( E^* \left[ \frac{\Lambda_T}{\Lambda_1} \mid V_T \geq D^* \right] + E^* \left[ \frac{\Lambda_T}{\Lambda_1} (1 - \alpha) V_T / D \mid V_T < D^* \right] \right) \tag{14}
\]

A series of calculations yields

\[
e^{-r(T-t)}B^* = B(e^{-r(T-t)}N_1(b_2') + e^{r(T-t)}N_1(d_2')(1 - \alpha) V_t / D) \tag{15}
\]

\(^4\)The derivation is available from the authors upon request.
where

\[
\begin{align*}
  b'_2 &= \ln \frac{V_t}{V_0} + (r + \eta' - \frac{1}{2} \sigma_V^2) (T - t) \\
  d'_2 &= -[b'_2 + \sigma_V \sqrt{T - t}]
\end{align*}
\]

The term \( r^* \) is the yield on the nontraded zero coupon bond; \( r \) is the yield on the traded riskless zero coupon bond; and \( N_1 \) represents the standard univariate normal cumulative distribution function. Setting \( B \) equal to 1, the credit spread on a nontraded bond can be expressed as follows:

\[
e^{-\left(r^*-r\right)(T-t)} = N_1\left(b'_2\right) + e^{(\eta'+r)(T-t)}N_1\left(d'_2\right) (1 - \alpha) V_t / D
\]

Equation (16) is compared to the corresponding equation in Klein (1996), and the credit spread of the model is found to include the non-tradable risk, which is expressed as \( \eta' \) in Equation (16). equation (16) implies that pricing the vulnerable option while ignoring its nontradability is somewhat imprecise.

**The Assets Underlying the Option Are Nontraded**

Another application of our model is to deal with the condition when the assets underlying the option are nontraded, whereas the assets of the counterparty are not. Let \( P \) be the natural measure; \( S_t \) be the value of the asset underlying the option at time \( t \) and \( V_t \) be the total value of the assets of the counterparty at time \( t \); both these two processes follow geometric Brownian motions:

\[
\begin{align*}
  \frac{dS}{S} &= \mu_S \, dt + \sigma_{Sz} \, dz^p + \sigma_{Sw} \, dw^p \\
  \frac{dV}{V} &= \mu_V \, dt + \sigma_V \, dz^p
\end{align*}
\]

Since \( S_t \) is not a tradable asset, it must be priced by finding a twin asset that is tradable in the financial market. Let \( V \) be the twin asset of \( S \), whose value also follows a geometric Brownian motion, as described in Equation (18) above. The precise value of \( S_T \) depends on the correlation between \( S_T \) and \( V_T \). When the correlation between \( S_T \) and \( V_T \) equals one, the proposed model reduces to the complete market case. Equations (17) and (18) indicate that \( S \) includes more added noise (\( dw^p \)) than \( V \). Process \( S_T \) is decomposed into two parts, \( dz^p \) and \( dw^p \). The part \( dz^p \) can be hedged by the twin asset \( V \), but \( dw^p \) cannot.
The relationship between \(dz^p\) and \(dw^p\) is as follows:

\[ E((dz^p)^2) = E((dw^p)^2) = 1, \quad \rho_{wz} = E(dw^p dz^p) = 0 \]

As shown in Cochrane and Saá-Requejo (2000), the “discount factor” is

\[
\frac{d\Lambda_T}{\Lambda_t} = -r \, dt - h^v \, dz^p \pm \sqrt{A^2 - h^v} \, dw^p
\]

This stochastic discount factor is used to price nontradable assets. Here, the nominal claim on the counterparty \(B\) in the proposed model is substituted for \(\max(ST/H_1, 0)\), and \(K\) represents the exercise price of the vulnerable option. The vulnerable option can be priced using the discount factor and is expressed as

\[
C_t = E^P \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \left( \left[ 1 \mid V_T = D \right] + \left[ \left( 1 - \alpha \right) V_T / D \mid V_T < D \right] \right) \right] (F_t) \quad (19)
\]

A series of detailed computations yields the value of a vulnerable option constructed in an incomplete market as follows:

\[
C_t = S e^{\eta(T-t)N_2(a_1, a_2, \rho)} - e^{-\eta(T-t)K}N_2(b_1, b_2, \rho) + e^{(\eta + \rho \sigma_S \sigma_Y)(T-t)} \frac{1 - \alpha}{D} V_t S t N_2(c_1, c_2, -\rho) - V_t \frac{K}{D} (1 - \alpha)N(d_1, d_2, -\rho) \quad (20)
\]

where

\[
\sigma^2_S = \sigma^2_{S_t} + \sigma^2_{S_w} \\
\sigma^2_Y = \sigma^2_{Y_t} + \rho \sigma_{Y_t} \sigma_S \\
a_1 = \frac{\ln \frac{S_t}{K} + (r + \eta + \frac{1}{2} \sigma^2_Y)(T - t)}{\sigma_S \sqrt{T - t}} \\
a_2 = \frac{\ln \frac{V_t}{D^*} + (r - \frac{1}{2} \sigma^2_Y + \rho \sigma_Y \sigma_S)(T - t)}{\sigma_Y \sqrt{T - t}} \\
b_1 = \frac{\ln \frac{S_t}{K} + (r + \eta - \frac{1}{2} \sigma^2_Y)(T - t)}{\sigma_S \sqrt{T - t}} \\
b_2 = \frac{\ln \frac{V_t}{D^*} + (r - \frac{1}{2} \sigma^2_Y)(T - t)}{\sigma_Y \sqrt{T - t}} \\
c_1 = \frac{\ln \frac{S_t}{K} + (r + \eta + \frac{1}{2} \sigma^2_S + \rho \sigma_Y \sigma_S)(T - t)}{\sigma_S \sqrt{T - t}}
\]
Similarly, the value of a vulnerable put option under an incomplete market can be expressed as follows:

$$c_2 = -\frac{\ln \frac{V_t}{K} + (r + \frac{1}{2}\sigma_V^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_V \sqrt{T - t}}$$

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma_S^2 + \rho \sigma_V \sigma_S)(T - t)}{\sigma_S \sqrt{T - t}}$$

$$d_2 = -\frac{\ln \frac{V_t}{D} + (r + \frac{1}{2}\sigma_V^2)(T - t)}{\sigma_V \sqrt{T - t}}$$

$$h_V = \frac{\mu_V - r}{\sigma_V}; \quad h_s = \frac{\mu_S - r}{\sigma_S}$$

$$\eta = \left[ h_s - h_V \left( \rho - a \sqrt{\frac{A^2}{h_V^2} - 1} \sqrt{1 - \rho^2} \right) \right] \sigma_S$$

$$\rho = \text{corr} \left( \frac{dV}{V}, \frac{dS}{S} \right) = \frac{\sigma_{sz}}{\sigma_S}$$

$$a = \begin{cases} +1 \text{ upper bound} \\ -1 \text{ lower bound} \end{cases}$$

Similarly, the value of a vulnerable put option under an incomplete market can be expressed as follows:

$$P_t = e^{-r(T-t)}KN_2(-b_1, b_2, \rho) + V_t \frac{K}{D} (1 - \alpha)N(-d_1, d_2, -\rho)$$

$$- S_t e^{\eta(T-t)}N_2(-a_1, a_2, \rho) - e^{(\eta + r + \rho \eta)(T-t)} \frac{1 - \alpha}{D} V_t S_t N_2(-c_1, c_2, -\rho) \quad (21)$$

**Formula in Klein (1996)**

When $\eta' = 0$ (or $\eta = 0$), Equation (13) (Equation (20)) becomes the formula in Klein (1996).

$$C_t = S_t N_2(a_1^*, a_2^*, \rho) - e^{-r(T-t)}KN_2(b_1^*, b_2^*, \rho)$$

$$+ e^{(r + \rho \eta)(T-t)} \frac{1 - \alpha}{D} V_t S_t N_2(c_1^*, c_2^*, -\rho) - V_t \frac{K}{D} (1 - \alpha)N(d_1^*, d_2^*, -\rho) \quad (22)$$

where

$$\sigma_S^2 = \sigma_{sz}^2 + \sigma_{sw}^2$$

$$a_1^* = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma_S^2)(T - t)}{\sigma_S \sqrt{T - t}}$$
Formula in Cochrane and Saá-Requejo (2000)

When $D^* \rightarrow 0$, $a_2, b_2 \rightarrow \infty$ and $c_2, d_2 \rightarrow -\infty$, Equation (20) becomes the model in Cochrane and Saá-Requejo (2000).

$$
C_t = S_t e^{\eta(T-t)} N_2(a_1, \infty, \rho) - e^{-r(T-t)} K N_2(b_1, \infty, \rho) \\
+ e^{(\eta + r + \rho \sigma_S)(T-t)} \frac{1-\alpha}{D} V_s S_t N_2(c_1, -\infty, -\rho) - V_s \frac{K}{D} (1-\alpha) N(d_1, -\infty, -\rho) \quad (23)
$$

Since $N_2(x, \infty, \rho) = N_1(x)$ and $N_2(x, -\infty, \rho) = 0$,

$$
C_t = S_t e^{\eta(T-t)} N(a_1) - e^{-r(T-t)} K N(b_1)\quad (24)
$$

and the proposed formula becomes Cochrane and Saá-Requejo's real option pricing formula.

Formula in Black and Scholes (1974)

It can be shown that if there is no risk of bankruptcy, i.e., $D^*$ equals zero and no risk of nontradeability, that is, $\eta^* = 0(\eta = 0)$, then Equation (13) (Equation (20) becomes the standard Black-Scholes formula.

$$
C_t = S_t N(a_1^*) - e^{-r(T-t)} K N(b_1^*) \quad (25)
$$
Under this circumstance, \( V \) and \( S \) are perfectly replicated. The diffusion term \( (dV) \) in \( dS \), which represents the part that cannot be hedged by \( dV \), becomes zero, implying that \( S \) also becomes a tradable asset and can be hedged by any other tradable asset perfectly. Comparing the proposed model to that of Cochrane and Saá-Requejo (2000) yields the effect of default risk. Comparing the proposed model with that of Klein (1996), or comparing the model of Cochrane and Saá-Requejo (2000) with that of Black and Scholes (1974) yields the difference between market completeness and incompleteness. The relationship among the models can be expressed as shown in Figure 1.

### Numerical Analysis

This section uses formulae derived in the previous section to perform some numerical analysis. Some examples of numerical results are considered to understand how to price vulnerable options when the market is incomplete. To see the impact of the interest rate on the good-deal bounds, the proposed numerical analysis is implemented under the assumption of different interest-rate term structures. We know the assumption of a deterministic interest rate is reasonable only for a short period of time. The maturity of an underlying asset of a vulnerable option, such as oil investment, is usually a long-term investment. Using
the stochastic interest rate under this condition would be more reasonable. Hence, numerical results are obtained for both the assumption of deterministic interest rates and stochastic interest rates.

Even if we have derived a more general model in Equation (8) than those in Klein (1996) and Cochrane and Saá-Requejo (2000). For easy comparison with Klein (1996) and Cochrane and Saá-Requejo (2000), we only use Equation (20) to do the numerical analysis.

Deterministic Interest Rates

This section compares the proposed model to those in Black and Scholes (1973), Cochrane and Saá-Requejo (2000), Klein (1996), and Johnson and Stulz (1987). Table I shows the numerical results obtained using the

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.0$</th>
<th>C-S</th>
<th>Klein ($\alpha = 0$)</th>
<th>B-S</th>
<th>J-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case (+)</td>
<td>3.935</td>
<td>3.693</td>
<td>3.451</td>
<td>2.967</td>
<td>4.030</td>
<td>3.005</td>
<td>3.070</td>
<td>1.960</td>
</tr>
<tr>
<td>Base case (-)</td>
<td>2.643</td>
<td>2.500</td>
<td>2.357</td>
<td>2.070</td>
<td>2.697</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_v = 0.2(\pm)$</td>
<td>4.205</td>
<td>3.961</td>
<td>3.718</td>
<td>3.232</td>
<td>4.267</td>
<td>3.031</td>
<td>3.070</td>
<td>1.920</td>
</tr>
<tr>
<td>$\sigma_v = 0.2(-)$</td>
<td>2.290</td>
<td>2.182</td>
<td>2.074</td>
<td>1.858</td>
<td>2.316</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_v = 0.4(\pm)$</td>
<td>3.785</td>
<td>3.543</td>
<td>3.302</td>
<td>2.818</td>
<td>3.915</td>
<td>2.977</td>
<td>3.070</td>
<td>2.000</td>
</tr>
<tr>
<td>$\sigma_v = 0.4(-)$</td>
<td>2.817</td>
<td>2.653</td>
<td>2.489</td>
<td>2.160</td>
<td>2.903</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_s = 0.2(\pm)$</td>
<td>2.927</td>
<td>2.737</td>
<td>2.546</td>
<td>2.164</td>
<td>3.004</td>
<td>2.116</td>
<td>2.164</td>
<td>1.770</td>
</tr>
<tr>
<td>$\sigma_s = 0.2(-)$</td>
<td>2.022</td>
<td>1.905</td>
<td>1.788</td>
<td>1.554</td>
<td>2.067</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_s = 0.4(\pm)$</td>
<td>4.961</td>
<td>4.668</td>
<td>4.375</td>
<td>3.789</td>
<td>5.075</td>
<td>3.895</td>
<td>3.976</td>
<td>2.040</td>
</tr>
<tr>
<td>$\sigma_s = 0.4(-)$</td>
<td>3.269</td>
<td>3.100</td>
<td>2.931</td>
<td>2.592</td>
<td>3.332</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.0833(\pm)$</td>
<td>1.655</td>
<td>1.552</td>
<td>1.450</td>
<td>1.244</td>
<td>1.675</td>
<td>1.444</td>
<td>1.460</td>
<td>1.330</td>
</tr>
<tr>
<td>$T = 0.999933(-)$</td>
<td>1.356</td>
<td>1.277</td>
<td>1.198</td>
<td>1.040</td>
<td>1.371</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.6433(\pm)$</td>
<td>5.798</td>
<td>5.444</td>
<td>5.090</td>
<td>4.383</td>
<td>5.985</td>
<td>4.068</td>
<td>4.181</td>
<td>2.180</td>
</tr>
<tr>
<td>$T = 0.083333(\pm)$</td>
<td>3.425</td>
<td>3.248</td>
<td>3.071</td>
<td>2.716</td>
<td>3.513</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$K = 50(\pm)$</td>
<td>0.677</td>
<td>0.656</td>
<td>0.634</td>
<td>0.591</td>
<td>0.684</td>
<td>0.430</td>
<td>0.434</td>
<td>0.340</td>
</tr>
<tr>
<td>$K = 50(-)$</td>
<td>0.348</td>
<td>0.339</td>
<td>0.330</td>
<td>0.313</td>
<td>0.351</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$S = 30(\pm)$</td>
<td>0.248</td>
<td>0.241</td>
<td>0.235</td>
<td>0.223</td>
<td>0.249</td>
<td>0.148</td>
<td>0.149</td>
<td>0.130</td>
</tr>
<tr>
<td>$S = 30(-)$</td>
<td>0.116</td>
<td>0.114</td>
<td>0.112</td>
<td>0.107</td>
<td>0.117</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$S = 50(\pm)$</td>
<td>0.2294</td>
<td>0.1125</td>
<td>0.1027</td>
<td>0.1277</td>
<td>0.1054</td>
<td>0.1032</td>
<td>0.1092</td>
<td>4.240</td>
</tr>
<tr>
<td>$S = 50(-)$</td>
<td>0.902</td>
<td>0.9018</td>
<td>0.8233</td>
<td>0.6665</td>
<td>10.153</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$V = 3(\pm)$</td>
<td>2.713</td>
<td>2.041</td>
<td>1.369</td>
<td>0.025</td>
<td>4.030</td>
<td>2.087</td>
<td>3.070</td>
<td>1.340</td>
</tr>
<tr>
<td>$V = 3(-)$</td>
<td>1.842</td>
<td>1.387</td>
<td>0.931</td>
<td>0.020</td>
<td>2.697</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$V = 10(-)$</td>
<td>2.697</td>
<td>2.697</td>
<td>2.697</td>
<td>2.697</td>
<td>2.697</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = 0.1(\pm)$</td>
<td>4.280</td>
<td>3.829</td>
<td>3.377</td>
<td>2.475</td>
<td>4.522</td>
<td>2.909</td>
<td>3.070</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = 0.1(-)$</td>
<td>2.550</td>
<td>2.287</td>
<td>2.023</td>
<td>1.496</td>
<td>2.690</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = 0.9(\pm)$</td>
<td>3.276</td>
<td>3.235</td>
<td>3.194</td>
<td>3.112</td>
<td>3.283</td>
<td>3.064</td>
<td>3.070</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = 0.9(-)$</td>
<td>2.954</td>
<td>2.924</td>
<td>2.893</td>
<td>2.832</td>
<td>2.959</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table I: Values of Vulnerable and Call Options

Note: This table reports the values of vulnerable call options. The parameter values are $T = 0.333, D = 5, D' = 5, S_T = 40, K = 40, V = 5, \sigma_v = 0.3, \sigma_s = 0.3, \rho = 0.5, \text{ and } \sigma_r = 0.048.$
new model. The values for the models of Cochrane and Saá-Requejo (2000), Klein (1996), Black and Scholes (1973), and Johnson and Stulz (1987) are reported in the last four columns of Table I for comparison.

The numerically obtained value of the vulnerable option in Table I is not a single value, as in Black and Scholes (1973), Klein (1996), or Johnson and Stulz (1987). Instead, it is “a range of option values,” because when markets are incomplete, the no-arbitrage method and the law of one price cannot be applied and a portfolio cannot be formed to hedge this asset perfectly (See Figure 2.) Since a perfect hedge is impossible, only a range of asset price values can be found, and the best effort is to make the bound on the price range as tight as possible. The price bounds, called “good-deal” asset bounds in Cochrane and Saá-Requejo (2000) have proven to outperform general arbitrage bounds. These good-deal asset bounds are tighter than the arbitrage bounds since they add an additional volatility constraint by eliminating unrealistic Sharpe ratios5 (larger than 2). The good-deal asset bounds of the numerical results are expressed by two values. The one indicated by (+) is the upper bound, and the one indicated by (−) is the lower bound. The true price of the vulnerable option must lie between these two values. The value in the base case is considered first. The Black and Scholes value of the nonvulnerable

---

5Ross (1976) showed that no portfolio can have more than twice the market Sharpe ratio and used it to bound the residuals of asset pricing theory.
option is 3.070. Setting \( \alpha = 0 \) in the formula in Klein (1996) yields the vulnerable option value of 3.005. Johnson and Stulz's vulnerable option value is 1.96. When the market fails, Cochrane and Saá-Requejo (C-S) find that the true value is between 4.030 and 2.697 for the nonvulnerable option, which can be compared with the nonvulnerable Black and Scholes option value of 3.070. The bounds for the vulnerable option according to the proposed model and on the nonvulnerable option in C-S seem a little wider in the base case when \( \rho = 0.5 \) than the case when \( \rho = 0.9 \). Increasing the correlation value, \( \rho \), from 0.5 to 0.9, cause the good-deal bounds according to the proposed model and C-S's model to become (3.276, 2.954) and (3.283, 2.959), respectively. (Also compare Figures 3 and 6.) When the market is incomplete, the new model proposes a precise bound range for the vulnerable option. Comparing the C-S model with the proposed model shows that the default risk changes the option value from (4.030, 2.697) to (3.935, 2.643). The difference between them are (0.095, 0.054), which can be compared with the change due to the default risk under the complete market assumption (B-S value minus Klein’s value) of 0.065. The value of the vulnerable option when the deadweight cost \( \alpha \) is changed from 0 to 1 is considered. When \( \alpha \) increases, the bounds fall below the C-S bounds, implying that default risk affects vulnerable option value by pulling down the range of nonvulnerable option values. This effect can also be seen by comparing Figures 3 and 4 or Figures 3 and 5. In Figure 3, when \( \alpha = 0 \), the vulnerable

![FIGURE 3](image)

The effect of variable \( S \) on the value of the vulnerable option. The parameter values are \( \alpha = 0, D = 5, D^* = 5 \), and \( \rho = 0.5 \).
option bounds almost coincide with the nonvulnerable option bounds. In Figure 4, when $\alpha$ is increased to 0.25, the vulnerable option bounds deviate greatly from the nonvulnerable option bounds, which is consistent with Klein (1996). The $\alpha$ effect in Figure 5 is not significant since $D^* = 3$ is far from the critical value of claims $D^* = 5$. 

FIGURE 4
The effect of variable $S$ on the value of the vulnerable option.
The parameter values are $\alpha = 0.25$, $D = 5$, and $D^* = 5$.

FIGURE 5
The effect of variable $S$ on the value of the vulnerable option.
The parameter values are $\alpha = 1$, $D = 5$, and $D^* = 3$. 

Figures 2 and 3 show the effect of the critical value of debt $D^*$. In Figure 2, when $D^* = 3$ is far lower than the value of the assets of the counterparty $V_T = 5$, the bound according to the model is almost equal to the bounds according to the C-S model. Increasing $D^* = 3$ to $D^* = 5$ causes the good deal bounds of the proposed model to move down from that of the C-S model, showing that the default risk reduces the bound of the option value. (See Figure 6.) Increasing the volatility of the value of the assets of the counterparty causes the bounds of the vulnerable option to move from the base case $(3.935, 2.643)$ to $(3.785, 2.817)$ in the $a = 0$, $V^0 = 0.4$ case. The upper bound moves down but the lower bound moves up, implying that when the volatility of the assets of the counterparty is increased, the possibility of default also increases. Consequently, the value of the vulnerable option decreases as $\sigma_V$ increases and the good-deal bounds move closer together as the volatility of the assets of the counterparty increase.

Increasing (decreasing) the exercise price of the vulnerable option reduces (increases) the option value according to the proposed model and Klein’s model. This result describes an expected property of an ordinary option. Figures 7 and 8 present these results with the effect of default risk $\alpha$ and $D^*$.

When time to maturity decreases from $T = 0.333$ to $T = 0.083$, the bounds of the vulnerable option move down from the base case $(3.935, 2.643)$ to $(1.655, 1.356)$, and when the time to maturity increases from...
0.333 to 0.583, the bounds of the vulnerable option increase to (5.798, 3.425). This result is the same as for an ordinary option. When the time to maturity increases, the time value of the option increases, and the possibility of this option’s moving far above the exercise price and deep...
into the money increases. Figures 9 and 10 present the effects of the default risk $\alpha$, $D^*$, and the time to maturity $T$ on the value of the option.

The volatility of the basis asset $S$ increases with the option value. A more volatile basis asset of an option is associated with a higher possibility

![FIGURE 9](image1.png)

**FIGURE 9**
The effect of variable $T$ on the value of the vulnerable option.
The parameter values are $\alpha = 0$, $D = 5$, and $D^* = 5$.

![FIGURE 10](image2.png)

**FIGURE 10**
The effect of variable $T$ on the value of the vulnerable option.
The parameter values are $\alpha = 1$, $D = 5$, and $D^* = 5$. 
of obtaining a higher option value. For example, as $\sigma_S$ increases from 0.3 to 0.4, the value of Klein’s model increases from 3.005 to 3.895, and the good-deal bounds of the proposed model increases from (3.935, 2.643) to (4.961, 3.269). Figure 11 shows the effects of default risk of $\alpha$, $D^*$, and the volatility of the basis asset $\sigma_S$.

As the value of the basis asset $S$ increases from 40 to 50, the value of the vulnerable option increases from (3.935, 2.643) to (12.294, 9.802). A more valuable basis asset is associated with a more valuable option. The value of the assets of the counterparty represents “how much protection the holder of the vulnerable option has against default risk,” as shown by the results in Table I for the $V = 10$ and $V = 3$ cases. When $V = 10$, the value of the vulnerable option seems not to differ among the four cases from $\alpha = 0$ to $\alpha = 1$. Their upper bounds are all equal to 4.030, and their lower bounds are all equal to 2.697. The same is true for Klein’s vulnerable option value and Black and Scholes’ nonvulnerable option value. When the value of the assets of the counterparty is sufficiently large, even if the counterparty defaults, the option holders can still recover their losses from the assets of the counterparty and the default risk does not change the value of the option. In the case of $V = 3$, the good-deal bounds on the vulnerable option value decrease dramatically from (2.713, 1.842) for $\alpha = 0$ to (0.025, 0.020) for $\alpha = 1$. This difference is the same when the market is complete; that is, Klein’s value (2.087) is much smaller than Black and Scholes’s value (3.070). The
correlation in the proposed model has two meanings. It represents the correlations either between the basis asset and the assets of the counterparty or between the basis asset and the twin asset of the nontraded basis asset. As Klein has shown in his 1996 paper, the default risk on Black-Scholes option values is much less when the assets of the counterparty and the assets underlying the options are positively correlated than when they are not. The proposed model supports Klein’s result, as seen in the case of $\rho = 0.1$. As $\alpha$ increases from 0 to 1, the bounds on the value of the vulnerable option change from (4.280, 2.550) to (2.475, 1.496), and the bound is much looser when the underlying asset is more weakly correlated with the twin asset. In the case of $\rho = 0.9$, the bound values of the vulnerable option change from (3.276, 2.954) to (3.112, 2.832), and the bounds are much tighter when the twin asset is more strongly correlated with the underlying asset. In conclusion, a stronger correlation between the assets of the counterparty and the assets that underlie the option corresponds to a smaller effect of default risk on the vulnerable option and a greater precision of the proposed model when used to price the nontraded basis asset underlying a vulnerable option. Treating $\rho$ as measuring the correlation between the basis asset and twin asset yields another interesting result. That is, as $\rho$ approaches 1, the nontradable basis asset becomes more liquid, and the effect of default risk on the vulnerable call option declines quickly. Also, default risk and liquidity risk are positively correlated. The Figures 12 and 13 show the effect of

![Image of a graph showing the effect of $\rho$ on the value of the vulnerable option. The parameter values are $\alpha = 0$, $D = 5$, and $D^* = 5$.]

FIGURE 12
The effect of variable $\rho$ on the value of the vulnerable option.
The parameter values are $\alpha = 0$, $D = 5$, and $D^* = 5$. 
default risk on call option value when the correlation between $V$ and $S$ changes from $-1$ to $1$. The effects for vulnerable put options are the opposite of vulnerable call options.

Table II is the same as Table I, except in that the numerical results are obtained by setting the risk-free rate to $0.098$ for comparison with the numerical results obtained under the stochastic interest rate. Table III shows the effects of all variables on the vulnerable option. The results in Table III can be obtained by performing a sensitivity analysis.

**Model Under Stochastic Interest Rates**

This section relaxes the constant risk-free interest rate assumption to a stochastic one, which exhibits mean-reverting as follows:

$$dr = \phi(\bar{r} - r(t))dt + \sigma_r dz^r_P$$  \hspace{1cm} (26)

where $\phi$, $\bar{r}$, and $\sigma_r$ are all constants, and $dz^r_P$ follows a standard Brownian motion under $P$ measure. The short-term interest rate considered above follows the well-known Vasicek (1977) term structure model. Other Gaussian interest rate models, such as that of Ho and Lee (1986), Hull and White (1990), and Heath, and Jarrow, and Morton (1992) can also be used as the interest-rate term structure for the simulation.
## TABLE II

Values of Vulnerable and Call Options

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.0$</th>
<th>C-S</th>
<th>Klein ($\alpha = 0$)</th>
<th>B-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case (+)</td>
<td>1.929</td>
<td>1.913</td>
<td>1.897</td>
<td>1.865</td>
<td>1.930</td>
<td>1.880</td>
<td>1.880</td>
</tr>
<tr>
<td>Base case (-)</td>
<td>1.854</td>
<td>1.840</td>
<td>1.826</td>
<td>1.845</td>
<td>1.855</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_V = 0.341(+)$</td>
<td>1.960</td>
<td>1.920</td>
<td>1.880</td>
<td>1.801</td>
<td>1.969</td>
<td>1.873</td>
<td>1.881</td>
</tr>
<tr>
<td>$\sigma_V = 0.341(-)$</td>
<td>1.929</td>
<td>1.891</td>
<td>1.852</td>
<td>1.776</td>
<td>1.937</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_V = 0.041(+)$</td>
<td>1.773</td>
<td>1.773</td>
<td>1.773</td>
<td>1.773</td>
<td>1.881</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_V = 0.041(-)$</td>
<td>1.535</td>
<td>1.535</td>
<td>1.535</td>
<td>1.535</td>
<td>1.535</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_S = 0.341(+)$</td>
<td>3.509</td>
<td>3.497</td>
<td>3.485</td>
<td>3.461</td>
<td>3.510</td>
<td>3.548</td>
<td>3.549</td>
</tr>
<tr>
<td>$\sigma_S = 0.341(-)$</td>
<td>3.349</td>
<td>3.339</td>
<td>3.329</td>
<td>3.309</td>
<td>3.350</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\sigma_S = 0.041(+)$</td>
<td>1.324</td>
<td>1.275</td>
<td>1.227</td>
<td>1.130</td>
<td>1.330</td>
<td>1.194</td>
<td>1.198</td>
</tr>
<tr>
<td>$\sigma_S = 0.041(-)$</td>
<td>1.294</td>
<td>1.248</td>
<td>1.201</td>
<td>1.109</td>
<td>1.330</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.1(+)</td>
<td>0.934</td>
<td>0.925</td>
<td>0.916</td>
<td>0.897</td>
<td>0.935</td>
<td>0.920</td>
<td>0.920</td>
</tr>
<tr>
<td>$T = 0.1(-)</td>
<td>0.912</td>
<td>0.903</td>
<td>0.895</td>
<td>0.877</td>
<td>0.912</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.5(+)</td>
<td>2.778</td>
<td>2.758</td>
<td>2.738</td>
<td>2.697</td>
<td>2.780</td>
<td>2.691</td>
<td>2.692</td>
</tr>
<tr>
<td>$T = 0.5(-)</td>
<td>2.645</td>
<td>2.628</td>
<td>2.610</td>
<td>2.576</td>
<td>2.646</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$K = 35(+)</td>
<td>6.061</td>
<td>5.769</td>
<td>5.477</td>
<td>4.894</td>
<td>6.104</td>
<td>5.990</td>
<td>6.032</td>
</tr>
<tr>
<td>$K = 35(-)</td>
<td>5.951</td>
<td>5.668</td>
<td>5.385</td>
<td>4.819</td>
<td>5.993</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$K = 45(+)</td>
<td>0.213</td>
<td>0.213</td>
<td>0.213</td>
<td>0.213</td>
<td>0.213</td>
<td>0.203</td>
<td>0.203</td>
</tr>
<tr>
<td>$K = 45(-)</td>
<td>0.197</td>
<td>0.197</td>
<td>0.197</td>
<td>0.197</td>
<td>0.197</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$S = 35(+)</td>
<td>0.123</td>
<td>0.123</td>
<td>0.123</td>
<td>0.123</td>
<td>0.123</td>
<td>0.117</td>
<td>0.116</td>
</tr>
<tr>
<td>$S = 35(-)</td>
<td>0.113</td>
<td>0.113</td>
<td>0.113</td>
<td>0.113</td>
<td>0.113</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$S = 45(+)</td>
<td>6.235</td>
<td>5.957</td>
<td>5.678</td>
<td>5.121</td>
<td>6.275</td>
<td>6.156</td>
<td>6.194</td>
</tr>
<tr>
<td>$S = 45(-)</td>
<td>6.113</td>
<td>5.844</td>
<td>5.575</td>
<td>5.037</td>
<td>6.151</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = 0.5(+)$</td>
<td>2.077</td>
<td>1.987</td>
<td>1.897</td>
<td>1.716</td>
<td>2.091</td>
<td>1.869</td>
<td>1.881</td>
</tr>
<tr>
<td>$\rho = 0.5(-)$</td>
<td>1.780</td>
<td>1.707</td>
<td>1.635</td>
<td>1.489</td>
<td>1.791</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\rho = -0.5(+)</td>
<td>2.139</td>
<td>1.843</td>
<td>1.548</td>
<td>0.957</td>
<td>2.214</td>
<td>1.814</td>
<td>1.881</td>
</tr>
<tr>
<td>$\rho = -0.5(-)$</td>
<td>1.836</td>
<td>1.576</td>
<td>1.315</td>
<td>0.795</td>
<td>1.903</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

**Note.** This table reports the values of vulnerable call options. The parameter values are $T = 0.3$, $D = 20$, $D^* = 20$, $S_T = 40$, $K = 40$, $S = 20$, $\sigma_V = 0.141$, $\sigma_S = 0.141$, $\rho = 0.9$, and $r_f = 0.098$.

## TABLE III

Sensitivity Analysis of Vulnerable Options

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Effect of parameters on vulnerable option value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_S$</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>-</td>
</tr>
<tr>
<td>$r_f$</td>
<td>+</td>
</tr>
<tr>
<td>$T$</td>
<td>+</td>
</tr>
<tr>
<td>$S$</td>
<td>+</td>
</tr>
<tr>
<td>$K$</td>
<td>-</td>
</tr>
<tr>
<td>$V$</td>
<td>+</td>
</tr>
<tr>
<td>$\rho$</td>
<td>+</td>
</tr>
<tr>
<td>$D^*$</td>
<td>-</td>
</tr>
<tr>
<td>$D$</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-</td>
</tr>
</tbody>
</table>

**Note.** This table reports the effects of parameters on the vulnerable options.
Under the stochastic interest rate assumption, the proposed model can be written as follows:

\[
C_t = E^P \left[ \frac{\Lambda_T}{\Lambda_t} \max(S_T - K, 0) \left( [1 | V_T \geq D^*] + [(1 - \alpha)V_T/D | V_T < D^*] \right) \right] F_t
\]  

(27)

where

\[
\frac{\Lambda_T}{\Lambda_t} = e^{-(\int_0^T (r(u) + \frac{1}{2}\sigma_s^2) du + \int_0^T \sigma_s dW - \frac{1}{2}\sigma_s^2 du + \int_0^T h_V dW - 1/2 h_V^2 du)}
\]

The discount factor changes with the stochastic interest rate. Table IV presents the notation used for the parameters. The parameters used to describe the one factor Vasicek interest rate process are taken from Bakshi, Madan, and Zhang (2001). These are estimated using U.S. treasury STRIPS data from the Lehman Brothers Fixed Income Database. The correlation coefficient between the interest rate and the observable stock price of \( V \), \( \rho_{rV} \), is set to -0.3 because stock price and the interest rate may be negatively correlated. Black (1976) called this effect the “leverage effect” of the market.

A Monte Carlo simulation is employed to find the numerical results given a stochastic interest rate. The proposed model involves four concurrent stochastic processes. In the simulation, they are \( dS, dV, dr, \) and \( d\Lambda \).

### Table IV
Parameter Values Used in Monte Carlo Simulation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definition of parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>long-term interest rate</td>
<td>0.098</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>instantaneous volatility of interest rate</td>
<td>0.077</td>
</tr>
<tr>
<td>( \phi )</td>
<td>mean-reversion speed of interest rate</td>
<td>0.379</td>
</tr>
<tr>
<td>( \sigma_s )</td>
<td>volatility of the non-traded asset underlying the option</td>
<td>0.141</td>
</tr>
<tr>
<td>( \sigma_V )</td>
<td>volatility of the assets of the counterparty</td>
<td>0.141</td>
</tr>
<tr>
<td>( \rho_{rV} )</td>
<td>correlation coefficient between interest rate and observable stock price of twin asset</td>
<td>-0.300</td>
</tr>
<tr>
<td>( T )</td>
<td>termination day of maturity</td>
<td>0.300</td>
</tr>
<tr>
<td>( S )</td>
<td>nontradable asset underlying the option</td>
<td>40</td>
</tr>
<tr>
<td>( K )</td>
<td>exercise price</td>
<td>40</td>
</tr>
<tr>
<td>( V )</td>
<td>twin asset and the assets of the counterparty</td>
<td>20</td>
</tr>
<tr>
<td>( \rho )</td>
<td>correlation between ( V ) and ( S )</td>
<td>0.900</td>
</tr>
<tr>
<td>( D^* )</td>
<td>critical value of default</td>
<td>20</td>
</tr>
<tr>
<td>( D )</td>
<td>total value of debt</td>
<td>20</td>
</tr>
</tbody>
</table>

*Note.* This table reports parameter values used in Monte Carlo Simulation. The definition and values of the parameters which are used to do the Monte Carlo simulation of the default vulnerable option pricing formula in an incomplete market with a stochastic interest rate are summarized in this table. Interest rate parameters are from Bakshi et al. (2001).
All processes can be easily simulated except for the process of the discount factor under stochastic interest rates. Its mean and variance are obtained in a complex calculation, and they are as follows:

$$E\left[ \ln \frac{\Lambda_T}{\Lambda_0} \right] = -\bar{r}T - \frac{1 - e^{-\phi T}}{\phi} (r_0 - \bar{r}) - \frac{1}{2} A^2 T + \frac{\sigma_r V}{\sigma_V} (1 - e^{-\phi t})$$

$$Var\left( \ln \frac{\Lambda_T}{\Lambda_0} \right) = \left( \frac{\mu_V - \bar{r}}{\sigma_V} \right)^2 T - 2 \left( \frac{\mu_V - \bar{r}}{\sigma_V^2} \right) (r_0 - \bar{r}) \left( 1 - e^{-\phi T} \right)$$

$$+ \frac{(r_0 - \bar{r})^2}{(2\phi\sigma_V^2)} (1 - e^{-2\phi T}) + \left( A^2 - \frac{\mu_V^2}{\sigma_V^2} + 2 \frac{\mu_V \bar{r}}{\sigma_V} - \frac{\bar{r}}{\sigma_V^2} - \frac{\sigma_r^2}{2\phi\sigma_V^2} \right) T$$

$$+ \frac{2(r_0 - \bar{r})}{\phi\sigma_V} (1 - e^{-\phi T}) \left( \frac{\mu_V - \bar{r}}{\sigma_V} \right) + \frac{\sigma_r}{4\phi^2\sigma_V^2} \left( 1 - e^{-2\phi T} \right)$$

Table V shows the numerical results of Equation (13). These numerical results were obtained by running the simulation 1,000,000 times and taking the average of 1,000,000 option values as the analytic result of the proposed pricing model, given a stochastic interest rate.

Table V includes two remarkable results. First, given a stochastic interest rate, the good-deal bounds of the proposed model become

**TABLE V**

Values of Vulnerable Call Options Given a Stochastic Interest Rate

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1.0$</th>
<th>C-S</th>
<th>B-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case (+)</td>
<td>1.929</td>
<td>1.913</td>
<td>1.897</td>
<td>1.865</td>
<td>1.930</td>
<td>1.880</td>
</tr>
<tr>
<td>Base case (-)</td>
<td>1.854</td>
<td>1.840</td>
<td>1.826</td>
<td>1.845</td>
<td>1.855</td>
<td>—</td>
</tr>
<tr>
<td>Stochastic interest rate (+)</td>
<td>1.924</td>
<td>1.766</td>
<td>1.609</td>
<td>1.285</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Stochastic interest rate (-)</td>
<td>1.923</td>
<td>1.765</td>
<td>1.607</td>
<td>1.280</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.1$ (+)</td>
<td>0.934</td>
<td>0.844</td>
<td>0.747</td>
<td>0.562</td>
<td>0.935</td>
<td>0.920</td>
</tr>
<tr>
<td>$T = 0.1$ (-)</td>
<td>0.934</td>
<td>0.839</td>
<td>0.747</td>
<td>0.559</td>
<td>0.912</td>
<td>—</td>
</tr>
<tr>
<td>$T = 0.5$ (+)</td>
<td>2.766</td>
<td>2.563</td>
<td>2.364</td>
<td>1.963</td>
<td>2.780</td>
<td>2.692</td>
</tr>
<tr>
<td>$T = 0.5$ (-)</td>
<td>2.759</td>
<td>2.562</td>
<td>2.360</td>
<td>1.956</td>
<td>2.646</td>
<td>—</td>
</tr>
<tr>
<td>$K = 35$ (+)</td>
<td>5.985</td>
<td>5.483</td>
<td>4.990</td>
<td>3.987</td>
<td>6.104</td>
<td>6.032</td>
</tr>
<tr>
<td>$K = 35$ (-)</td>
<td>5.983</td>
<td>5.481</td>
<td>4.986</td>
<td>3.985</td>
<td>5.993</td>
<td>—</td>
</tr>
<tr>
<td>$K = 45$ (+)</td>
<td>0.218</td>
<td>0.200</td>
<td>0.182</td>
<td>0.146</td>
<td>0.213</td>
<td>0.203</td>
</tr>
<tr>
<td>$K = 45$ (-)</td>
<td>0.218</td>
<td>0.199</td>
<td>0.181</td>
<td>0.146</td>
<td>0.197</td>
<td>—</td>
</tr>
<tr>
<td>$S = 35$ (+)</td>
<td>0.127</td>
<td>0.116</td>
<td>0.106</td>
<td>0.085</td>
<td>0.123</td>
<td>0.116</td>
</tr>
<tr>
<td>$S = 35$ (-)</td>
<td>0.126</td>
<td>0.116</td>
<td>0.105</td>
<td>0.084</td>
<td>0.113</td>
<td>—</td>
</tr>
<tr>
<td>$S = 45$ (+)</td>
<td>6.157</td>
<td>5.649</td>
<td>5.135</td>
<td>4.104</td>
<td>6.275</td>
<td>6.194</td>
</tr>
<tr>
<td>$S = 45$ (-)</td>
<td>6.155</td>
<td>5.647</td>
<td>5.133</td>
<td>4.102</td>
<td>6.151</td>
<td>—</td>
</tr>
</tbody>
</table>

*Note.* This table reports values of vulnerable call options given a stochastic interest rate. The parameter values are $T = 0.3$, $D = 20$, $D^* = 20$, $S_r = 40$, $K = 40$, $V = 20$, $\sigma_V = 0.141$, $\sigma_S = 0.141$, $\rho = 0.9$, and $r_T = 0.098$. 
tighter than when a deterministic interest rate is used. Given a stochastic interest rate, the base case bounds on the vulnerable option value are (1.924, 1.923) for \((\alpha = 0)\). These bounds should be compared with the vulnerable option value in the base case, presented in Table II. The first two rows of Table V presents the base case. The base case given a deterministic interest rate has bound values for the value of the vulnerable option of (1.929, 1.854), representing a wider good-deal bound than in the case under a stochastic interest rate. This result may follow from the fact that when the interest rate is more volatile, the Sharpe ratio of tradable assets \((h_v)\) becomes larger than when the interest rate is deterministic; then, \(\frac{\lambda^2}{h^2}\) in the expression for \(\eta\) more closely approaches 1. In such a case, the good-deal bounds move closer together. Second, the value of the vulnerable option decreases faster given stochastic interest rates than given deterministic interest rates, as the dead weight cost \(\alpha\) increases from 0 to 1. In the example of the base case, the bounds of the vulnerable option are (1.865, 1.845) when \(\alpha = 1\) under the deterministic interest rate, but they decrease to (1.285, 1.280) when \(\alpha = 1\) under a stochastic interest rate. Figure 14 depicts the random walk of Vasicek.

![Figure 14](image)

**FIGURE 14**
A simulated path of the Vasicek term structure model. The parameter values are \(\bar{r} = 0.098, \phi = 0.379, \) and \(\sigma_r = 0.077\).
term-structure interest rate when the parameters of this process are shown in Table IV. The distribution of interest rates is plotted to prove that, in the long term, the interest rate is 0.098 in the presented numerical analysis, and a negative interest rate is never obtained when a Vasicek term structure model is used.

CONCLUSION

This study extended the model of Klein (1996) and Cochrane and Saá-Requejo (2000) to obtain a vulnerable option pricing formula in an incomplete market. The proposed formula overcomes the drawbacks in other models, such as that of Klein (1996), which do not consider market incompleteness in pricing vulnerable options. The proposed formula also overcomes the shortcomings of models, such as that of Cochrane and Saá-Requejo (2000), which price real options underlying nontraded or thinly traded assets without considering default risk. The proposed formula incorporates the advantages of both models and is better able to evaluate nontraded assets underlying vulnerable options. The performance of the proposed model under different interest rate assumptions, including deterministic and stochastic interest rates, is investigated. The numerical results reveal that “good-deal” bounds are tighter when the interest rate is stochastic.

APPENDIX A

Proof of the Model When Both Asset Underlying the Option and the Assets of the Counterparty are Nontraded

Let \( m \) be a constant, \( 0 < m < \infty \).

\[
\frac{d \Lambda_T}{\Lambda_t} = -r dt - \tilde{\mu} \Sigma^{-1} \sigma dz - \sqrt{\frac{1}{m + 1} (A^2 - \tilde{\mu} \Sigma^{-1} \tilde{\mu})} dw \\
- \sqrt{\frac{m}{m + 1} (A^2 - \tilde{\mu} \Sigma^{-1} \tilde{\mu})} dx
\]
\[
\frac{d\Lambda_t}{\Lambda_t} = -rdt - \left[\tilde{\mu}_Y \ - \tilde{\mu}_\xi\right]\begin{pmatrix}
\frac{1}{\sigma_Y^2} & 0 \\
0 & \frac{1}{\sigma_\xi^2}
\end{pmatrix}\begin{pmatrix}
\sigma_Y & 0 \\
0 & \sigma_\xi
\end{pmatrix}
\left[\begin{array}{c}
dz_Y \\
dz_\xi
\end{array}\right]
\]
\[
-\sqrt{\frac{1}{m+1}\left(A^2 - \left[\tilde{\mu}_Y \ - \tilde{\mu}_\xi\right]\begin{pmatrix}
\frac{1}{\sigma_Y^2} & 0 \\
0 & \frac{1}{\sigma_\xi^2}
\end{pmatrix}\begin{pmatrix}
\tilde{\mu}_Y \\
\tilde{\mu}_\xi
\end{pmatrix}\right)}\ dw^p
\]
\[
-\sqrt{\frac{m}{m+1}\left(A^2 - \left[\tilde{\mu}_Y \ - \tilde{\mu}_\xi\right]\begin{pmatrix}
\frac{1}{\sigma_Y^2} & 0 \\
0 & \frac{1}{\sigma_\xi^2}
\end{pmatrix}\begin{pmatrix}
\tilde{\mu}_Y \\
\tilde{\mu}_\xi
\end{pmatrix}\right)}\ dx^p
\]

where \(\tilde{\mu}_Y = \mu_Y - r, \tilde{\mu}_\xi = \mu_\xi - r\).

\[
E(dz^2) = E(dw^2) = E(dx^2) = 1, \quad \rho_{zw} = E(dz \ dw) = 0,
\]
\[
\rho_{xz} = E(dz \ dx) = 0, \quad \rho_{wx} = E(dw \ dx) = 0
\]
\[
\frac{dS}{S} = \mu_sd \ t + \sigma_{sz}dz_Y^p + \sigma_{su}dw^p \Leftrightarrow \frac{dY}{Y} = \mu_Yd \ t + \sigma_Ydz_Y^p
\]
\[
\frac{dV}{V} = \mu_vd \ t + \sigma_{vz}dz_\xi^p + \sigma_{v_\xi}dx^p \Leftrightarrow \frac{dx}{x} = \mu_\xi d \ t + \sigma_\xi dz_\xi^p
\]

\[
C_t = E_t^p\left[\frac{\Lambda_T}{\Lambda_t}\max(S_T - K, 0)\left[1 \ | \ V_T \geq D^*\right] + [(1 - \alpha)V_T/\text{D} \ | \ V_T < D^*]\right]
\]

\[
C_t = E_t^p[e^{-r(T-t)}(T-t)^{-\frac{3}{2}}\pi_t]\left[\frac{\Lambda_T}{\Lambda_t}\max(S_T - K, 0)\left[1 \ | \ V_T \geq D^*\right] + [(1 - \alpha)V_T/\text{D} \ | \ V_T < D^*]\right]
\]

\[
dZ_Y^p = dZ_Y^R - \frac{\tilde{\mu}_Y}{\sigma_Y}d \ t
\]
\[
dZ_\xi^p = dZ_\xi^R - \frac{\tilde{\mu}_\xi}{\sigma_\xi}d \ t
\]
\[
dW^p = dW^R - \sqrt{\frac{1}{m+1}\left(A^2 - \left(\frac{\tilde{\mu}_Y^2}{\sigma_Y^2} + \frac{\tilde{\mu}_\xi^2}{\sigma_\xi^2}\right)\right)}d \ t
\]
\[
dX^p = dX^R - \sqrt{\frac{m}{m+1}\left(A^2 - \left(\frac{\tilde{\mu}_Y^2}{\sigma_Y^2} + \frac{\tilde{\mu}_\xi^2}{\sigma_\xi^2}\right)\right)}d \ t
\]

\[
C_t = e^{-r(T-t)}E_t^R[\max(S_T - K, 0)\left[1 \ | \ V_T \geq D^*\right] + [(1 - \alpha)V_T/\text{D} \ | \ V_T < D^*]\]
\]
\[
\frac{dS}{S} = \mu_S dt + \sigma_S (dZ_S^R - h_Y dt) + \sigma_{Sv} \left( dW^R - \sqrt{\frac{1}{m+1}(A^2 - (h_Y^2 + h_\xi^2))} dt \right) \\
= \left( \mu_S - \sigma_S h_Y - \sigma_{Sv} \sqrt{\frac{1}{m+1}(A^2 - (h_Y^2 + h_\xi^2))} \right) dt + \sigma_S dZ_S^R + \sigma_{Sv} dW^R \\
\frac{dV}{V} = \mu_V dt + \sigma_V (dZ_V^R - h_\xi dt) + \sigma_{Vx} \left( dX^R - \sqrt{\frac{m}{m+1}(A^2 - (h_\xi^2 + h_Y^2))} dt \right) \\
= \left( \mu_V \sigma_V - h_\xi \sigma_{Vx} \sqrt{\frac{m}{m+1}(A^2 - (h_\xi^2 + h_Y^2))} \right) dt + \sigma_V dZ_\xi^R + \sigma_{Vx} dX^R \\
C_t = e^{-r(T-t)} E_t^R \left[ \max(S_T - K, 0) \left( [1 \mid V_T \geq D^*] + [(1 - \alpha) V_T / D \mid V_T < D^*] \right) \right] \\
= e^{-r(T-t)} E_t^R \left[ (S_T - K) 1_{\{S_T > K\}} \left( [1 \mid V_T \geq D^*] + [(1 - \alpha) V_T / D \mid V_T < D^*] \right) \right] \\
= e^{-r(T-t)} E_t^R \left[ S_T V_T 1_{\{S_T > K, V_T \geq D^*\}} \right] - K e^{-r(T-t)} \frac{1 - \alpha}{D} E_t^R \left[ V_T 1_{\{S_T < K, V_T < D^*\}} \right] \\
+ \frac{(1 - \alpha)}{D} e^{-r(T-t)} E_t^R \left[ S_T V_T 1_{\{S_T > K, V_T < D^*\}} \right] - K e^{-r(T-t)} \frac{1 - \alpha}{D} E_t^R \left[ V_T 1_{\{S_T < K, V_T < D^*\}} \right] \\
\text{Part I:} \\
\text{Part II:} \\
\text{Part III:} \\
\text{Part IV:} \\
\\n\eta_1 = \left[ h_S - a * h_Y \left( \rho_1 - \sqrt{1 - \rho_1^2} \right) \sqrt{\frac{1}{m+1}(A^2 - (h_Y^2 + h_\xi^2))} \right] * \sigma_S \\
dZ_Y = dZ_Y^R + \sigma_{S2} dt \\
dZ_\xi = dZ_\xi^R \\
dW^R = dW^R + \sigma_{Sw} dt \\
\frac{dX^R}{\sqrt{m+1}} = S e^{\eta_1} E_t^R [1_{\{S_T > K, V_T > D^*\}}] \\
= S e^{\eta_1} P_t^{R1} \{ S_T > K, V_T > D^* \} \\
= S e^{\eta_1} P_t^{R1} \{ \ln S_T > \ln K, \ln V_T > \ln D^* \}
\[ S_t e^{\eta_t P^R_1} \left\{ \ln S_t + \left( \mu_S - h_V \sigma_S - \sigma_Sh_{\tilde{\xi}} \sqrt{\frac{1}{m+1} \left( A^2 - (h_V^2 + h_{\tilde{\xi}}^2) \right)} \right) + \frac{1}{2} \sigma_S^2 (T - t) + \sigma_S Z_{S_{T-t}}^R + \sigma_S W_{T-t}^R > \ln K \right\} \]
\[ \times \sqrt{\frac{m}{m+1} \left( A^2 - (h_V^2 + h_{\tilde{\xi}}^2) - \frac{1}{2} \sigma_V^2 \right) (T - t)} + \sigma_V Z_{V_{T-t}}^R + \sigma_V X_{V_{T-t}}^R > \ln D^* \}

Let \[ J_1 = -\frac{\sigma_S Z_{S_{T-t}}^R + \sigma_S W_{T-t}^R}{\sigma_S \sqrt{T - t}}, \quad J_2 = -\frac{\sigma_V Z_{V_{T-t}}^R + \sigma_V X_{V_{T-t}}^R}{\sigma_V \sqrt{T - t}} \]
\[ = S_t e^{\eta_t P^R_1} \left\{ J_1 < \frac{\ln S_t + (r + \eta_1 + \frac{1}{2} \sigma_S^2) (T - t)}{\sigma_S \sqrt{T - t}}, J_2 < \frac{\ln V_t + (r + \eta_2) (T - t)}{\sigma_V \sqrt{T - t}} \right\} \]
\[ - \frac{1}{2} \sigma_V \sqrt{T - t} \}

where
\[ \eta_2 = \left[ h_V - a * h_{\tilde{\xi}} (\rho_2 - \sqrt{1 - \rho_2^2}) \sqrt{\frac{m}{m+1} \left( A^2 - \frac{(h_V^2 + h_{\tilde{\xi}}^2)}{h_{\tilde{\xi}}^2} \right)} \right] * \sigma_V \]

As with the proofs for Part I–III, Part IV can be derived as:

**Part II:** Let \[ J_3 = -\frac{\sigma_S Z_{S_{T-t}}^R + \sigma_S W_{T-t}^R}{\sigma_S \sqrt{T - t}}, \quad J_4 = -\frac{\sigma_V Z_{V_{T-t}}^R + \sigma_V X_{V_{T-t}}^R}{\sigma_V \sqrt{T - t}} \]

\[ Part II = -K e^{-r(T-t)} E_t [1_{\{S_t < K, V_t < D^*\}}] \]
\[ = -K e^{-r(T-t)} P^R_1 \left\{ J_3 < \frac{\ln S_t + (r + \eta_1) (T - t)}{\sigma_S \sqrt{T - t}} - \frac{1}{2} \sigma_S \sqrt{T - t}, \right. \]
\[ J_4 < \frac{\ln V_t + (r + \eta_2) (T - t)}{\sigma_V \sqrt{T - t}} - \frac{1}{2} \sigma_V \sqrt{T - t} \}

**Part III:** Let \[ J_5 = -\frac{\sigma_S Z_{S_{T-t}}^R + \sigma_S W_{T-t}^R}{\sigma_S \sqrt{T - t}}, \quad J_6 = -\frac{\sigma_V Z_{V_{T-t}}^R + \sigma_V X_{V_{T-t}}^R}{\sigma_V \sqrt{T - t}} \]

\[ Part III = \frac{1 - \alpha}{D} e^{-r(T-t)} E_t [1_{\{S_T > K, V_t < D^*\}}] \]
\[ dZ_t^V = dZ^R + \sigma_S dt \]
Part IV: Let

\[ dZ^R = dZ^{R2} + \sigma_{Vz} \, dt \]
\[ dW^R = dW^{R2} + \sigma_{Sw} \, dt \]
\[ dX^R = dX^{R2} + \sigma_{Vx} \, dt \]

\[
= \frac{1 - \alpha}{D} e^{(r + \eta_1 + \nu_2)(T-t)} S_t V_t E_t^{R2} [1_{\{S_t > K, V_t < D^*\}}]
\]

\[
= \frac{1 - \alpha}{D} e^{(r + \eta_1 + \nu_2)(T-t)} S_t V_t P_t^{R2} \left\{ J_5 \leq \frac{\ln S_t - (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} + \frac{1}{2} \sigma_S \sqrt{T - t}, \right. \]

\[
J_6 < - \frac{\ln V_t - (r + \eta_2)(T - t)}{\sigma_V \sqrt{T - t}} - \frac{1}{2} \sigma_V \sqrt{T - t} \right\}
\]

Part IV: Let \( J_7 = -\frac{\sigma_S Z_x^{R3}}{\sigma_V \sqrt{T - t}} + \frac{\sigma_S W_x^{R3}}{\sqrt{T - t}}, \quad J_8 = -\frac{\sigma_V Z_x^{R3}}{\sigma_V \sqrt{T - t}} + \frac{\sigma_V W_x^{R3}}{\sqrt{T - t}} \)

Part IV = \(-K e^{-r(T-t)} \frac{(1 - \alpha)}{D} E_t^{R} [V_t] 1_{\{S_t < K, V_t < D^*\}}]

\[
= e^{\eta_2(T-t)} V_t E_t^{R3} [1_{\{S_t > K, V_t < D^*\}}]
\]

\[
= e^{\eta_2(T-t)} V_t P_t^{R3} \left\{ J_7 \leq \frac{\ln S_t - (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} + \frac{1}{2} \sigma_S \sqrt{T - t}, \right. \]

\[
J_8 < - \frac{\ln V_t - (r + \eta_2)(T - t)}{\sigma_V \sqrt{T - t}} - \frac{1}{2} \sigma_V \sqrt{T - t} \right\}
\]

We can recombine the four parts in the following formula:

\[
C_t = S_t e^{\eta_1(T-t)} N(a_t, a_t, \rho) - K e^{-r(T-t)} N(b_t^*, b_t^*, -\rho) + S_t V_t e^{(r + \eta_1 + \eta_2)(T-t)} \frac{(1 - \alpha)}{D} N(c_t^*, c_t^*, \rho) - KV_t e^{\eta_2(T-t)} N(d_t^*, d_t^*, -\rho)
\]

where

\[
\sigma_S^2 = \sigma_{S_z}^2 + \sigma_{S_w}^2 \quad ; \quad \sigma_V^2 = \sigma_{V_z}^2 + \sigma_{V_w}^2
\]
\[
\begin{align*}
    h_V &= \frac{\mu_V - r}{\sigma_V} \quad ; \quad h_Y = \frac{\mu_Y - r}{\sigma_Y} \\
    h_S &= \frac{\mu_S - r}{\sigma_S} \quad ; \quad h_\xi = \frac{\mu_\xi - r}{\sigma_\xi} \\
    a_1'' &= \frac{\ln \frac{S}{K} + (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} + \frac{1}{2} \sigma_S \sqrt{T - t} \\
    a_2'' &= \frac{\ln \frac{V_i}{D^2} (r + \eta_2)(T - t)}{\sigma_Y \sqrt{T - t}} - \frac{1}{2} \sigma_Y \sqrt{T - t} \\
    b_1'' &= \frac{\ln \frac{S}{K} + (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} - \frac{1}{2} \sigma_S \sqrt{T - t} \\
    b_2'' &= \frac{\ln \frac{V_i}{D^2} (r + \eta_2)(T - t)}{\sigma_Y \sqrt{T - t}} - \frac{1}{2} \sigma_Y \sqrt{T - t} \\
    c_1'' &= \frac{\ln \frac{S}{K} + (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} + \frac{1}{2} \sigma_S \sqrt{T - t} \\
    c_2'' &= -\frac{\ln \frac{V_i}{D^2} (r + \eta_2)(T - t)}{\sigma_Y \sqrt{T - t}} - \frac{1}{2} \sigma_Y \sqrt{T - t} \\
    d_1'' &= \frac{\ln \frac{S}{K} + (r + \eta_1)(T - t)}{\sigma_S \sqrt{T - t}} - \frac{1}{2} \sigma_S \sqrt{T - t} \\
    d_2'' &= -\frac{\ln \frac{V_i}{D^2} (r + \eta_2)(T - t)}{\sigma_Y \sqrt{T - t}} - \frac{1}{2} \sigma_Y \sqrt{T - t} \\
    \rho_1 &= \frac{\sigma_{S_0}}{\sigma_S} \quad \rho_2 &= \frac{\sigma_{V_1}}{\sigma_Y} \\
    \eta_1 &= \left[ h_S - a \ast h_Y \left( \rho_1 - \sqrt{1 - \rho_1^2} \sqrt{\frac{1}{m + 1} \left( \frac{A^2 h_Y^2}{h_Y} + \frac{h_\xi^2}{h_\xi^2} \right) } \right) \right] \ast \sigma_S \\
    \eta_2 &= \left[ h_V - a \ast h_\xi \left( \rho_2 - \sqrt{1 - \rho_2^2} \sqrt{\frac{m}{m + 1} \left( \frac{A^2 h_\xi^2}{h_\xi} - \frac{h_Y^2}{h_Y^2} \right) } \right) \right] \ast \sigma_Y \\
    a &= \begin{cases} 
        +1 \text{ upper bound} \\
        -1 \text{ lower bound} 
    \end{cases}
\end{align*}
\]

**BIBLIOGRAPHY**

