Chapter Eight

The BEM for Potential Problems in Inhomogeneous Anisotropic Bodies

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8.1 INTRODUCTION

In this chapter, the boundary element method (BEM) is developed for solving problems described by the general second order elliptic partial differential equation with variable coefficients. Many potential problems in physics and engineering science are described by this equation. As we know, the BEM can be applied only when the integral representation of the solution is available. This requires the establishment of a reciprocal identity for the governing operator and its fundamental solution. While for the problem at hand the reciprocal identity is known (see Section 2.5), unfortunately, it is not possible...
to establish the fundamental solution for the general second order elliptic partial differential equation, except only for some special forms of the equation governing particular problems, for example, Helmholtz equation, heat conduction equation in homogeneous bodies, potential problems in anisotropic bodies with constant material properties (see Sections 3.6 and 6.2.4). Even when the fundamental solution can be established, it is different for different problems and it is expressed by special mathematical functions, whose manipulation to derive the boundary integral equation is a difficult and tedious task. The whole procedure demands special care for the evaluation of the derivatives, the derivation of boundary integral equation, as well as their numerical solution.

These problems discouraged the researchers to develop and use the BEM as a computational method for solving boundary value problems described by such equations. Thus, very early on the investigators looked for BEM formulations that would solve the general potential problems using the known simple fundamental solution of the Laplace equation at the cost of introducing domain integrals, which evidently annul the pure boundary character of the BEM and may reduce to certain degree its advantages over the domain type methods, for example, finite different method (FDM) and finite element method (FEM). Several such methods are reported in the literature. Among them we distinguish the dual reciprocity method (DRM) and the analog equation method (AEM) as the most efficient methods. Both methods maintain the boundary-only character in the sense that the discretization is restricted to the boundary. The DRM introduced by Nardini and Brebbia [1] appeared as the most promising method to overcome the lack of a known fundamental solution. Many problems have been successfully solved by this method [2,3,4]. However, this method is subject to a major limitation. It works only if for a given nonstandard differential operator a dominant operator with known fundamental solution can be extracted. In general this is not feasible, for example, in differential equations with variable coefficients, coupled differential equations, or nonlinear ones [5,6]. On the other hand the AEM not only reduces a given problem to one with a simple known fundamental solution but it is alleviated from any restriction inherent in DRM [7]. Thus, an integral representation of the solution can always be established and the standard BEM can be applied. It can solve efficiently not only linear but also nonlinear problems, static as well as dynamic, and it is problem independent for systems described by a differential equation of the same order. In this chapter, both methods will be presented for the solution of the potential problems described by the complete second order elliptic partial differential equation.

8.2 THE GENERAL SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

The boundary value problem for the complete second order elliptic partial differential equation in two dimensions is stated as

\[ A u_{xx} + 2Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu = f(x), \quad x (x, y) \in \Omega \quad (8.1) \]
subject to the boundary conditions

\[ u = \alpha(x), \quad x \in \Gamma_1 \quad (8.2a) \]

\[ \nabla u \cdot \mathbf{m} = \gamma(x), \quad x \in \Gamma_2 \quad (8.2b) \]

where \( \Gamma_1 \cup \Gamma_2 = \Gamma; \ u = u(x) \) represents the unknown field function; \( A(x), B(x), \ldots, F(x) \) are position dependent coefficients satisfying the ellipticity condition \( B^2 - AC < 0 \) [8]. The quantity \( \nabla u \cdot \mathbf{m} \) represents the physical boundary quantity, that is, the flux in the direction of the conormal to the boundary \( \mathbf{m} = (An_x + Bn_y)i + (Bn_x + Cn_y)j \). Apparently, the direction of the conormal coincides with the normal to the boundary if \( A = C \) and \( B = 0 \). Finally, \( \alpha(x) \) and \( \gamma(x) \) are functions specified on boundary. The boundary \( \Gamma \) may be multiply-connected. The coefficients satisfy the self-adjointness conditions

\[ A_{,x} + B_{,y} = D, \quad B_{,x} + C_{,y} = E \quad (8.3a, b) \]

which ensure that the operator

\[ L = A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F \quad (8.4) \]

is self-adjoint, that is, \( L^* = L \) (see Section 2.5). Note that condition (8.2a) represents the Dirichlet boundary condition, while (8.2b) the Neumann boundary condition (see Fig. 8.1).

We consider the functional

\[ J(u) = \int_\Omega \left[ \frac{1}{2} \left( Au_{,x}^2 + 2Bu_{,x}u_{,y} + Cu_{,y}^2 - Fu^2 \right) + fu \right] d\Omega - \int_\Gamma \gamma u \, ds \quad (8.5) \]

The condition \( \delta J(u) = 0 \) produces the boundary value problem described by Eqs. (8.1) and (8.2a,b) as the Euler-Lagrange equation with the associated

\[ \text{FIGURE 8.1} \text{ Domain } \Omega \text{ with mixed boundary conditions.} \]
boundary conditions. The derivation is achieved via the calculus of variations (see Section 2.7.3). This derivation is proposed as an exercise.

The boundary value problem (8.1), (8.2a,b) under the conditions (8.3a,b) for suitable meaning of its coefficients occurs in many problems in engineering and mathematical physics such as heat transfer, electrostatic, seepage problems, membranes on elastic subgrade, etc., where the involved media may exhibit heterogeneous anisotropic properties.

**8.3 THE DUAL RECIPROCITY METHOD**

**8.3.1 The fundamentals of the DRM**

From the historical point of view, the DRM was introduced by Nardini and Brebbia [1] in their effort to avoid the use of the dynamic fundamental solution for the wave equation. The dynamic fundamental solution complicates the derivation of the boundary integral equations as it requires the dynamic reciprocal theorem, and special care for the subsequent numerical solution [9]. They developed a BEM formulation using the simple static fundamental solution, namely that of the Laplace equation. This formulation, however, introduced domain integrals of the unknown field function. Thus, the resulting integral equations are boundary-domain integral equations. In order to maintain the pure boundary character of their method, they developed a technique to convert the domain integrals to boundary line integrals, which allowed the problem to be solved using the standard BEM. The method was further developed and applied to solve linear problems for which the fundamental solution either could not be determined or, even available, it was too difficult to derive the boundary integral equations and solve them numerically. The method was also extended to nonlinear problems, where no reciprocal identity and fundamental solution exist.

The fundamental concept of the DRM is described as follows:

Let us consider the differential equation

\[ N(u) = f \]  \hspace{1cm} (8.6)

where \( N \) is a generic in general nonlinear operator with variable coefficients and \( f \) a generic source term. Let us suppose now that a linear dominant operator \( L \), that is an operator of the same order as \( N \), can be extracted from \( N \), and whose adjoint operator \( L^* \) has a known fundamental solution. We can then split the operator \( N \) as

\[ N = L + \hat{N} \]  \hspace{1cm} (8.7)

where \( \hat{N} \) represents the remaining part of \( N \) after extracting \( L \). This allows us to write Eq. (8.6) as

\[ L(u) = f - \hat{N}(u) \]  \hspace{1cm} (8.8)
or

\[ L(u) = b \] (8.9)

The source \( b = f - \hat{N}(u) \) is treated as unknown generalized source term. Now proceeding as the standard boundary element formulation (see Section 3.3) we obtain the integral representation formula

\[
u(x) = \int_{\Omega} v b \, d\Omega - \int_{\Gamma} [P^*(v) \, Q(u) - P(u) \, Q^*(v)] \, ds \] (8.10)

for the solution of the differential equation (8.9), in which \( v \) is the fundamental solution of the adjoint operator \( L^* \), that is, a singular particular solution of the equation

\[ L^*(u) = \delta(x, \xi) \] (8.11)

and \( P, Q \) are linear operators related to \( u \) and \( P^*, Q^* \) their adjoint operators related to \( v \). Their order is at least by one less than that of \( L \). Eq. (8.10) can be used to evaluate the solution if first the following two problems are solved: (1) The boundary quantities \( P(u), Q(u) \) are expressed in terms of the specified boundary conditions; (2) The domain integral in Eq. (8.10) is evaluated.

To make concrete these concepts we illustrate the DRM starting with the Poisson equation. The procedure adheres to the steps presented in [2] with a slight difference of the notation to comply with that employed in this book.

### 8.3.2 The DRM for the Poisson equation

The DRM is explained with reference to the Poisson equation

\[ \nabla^2 u = b \quad \text{in} \ \Omega \] (8.12)

Here, the source term \( b = b(x, y) \), is a known function of position only.

The integral representation of the solution of Eq. (8.12) is given as (cf. Section 3.4)

\[
\varepsilon u(x) = \int_{\Omega} v b \, d\Omega - \int_{\Gamma} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, ds, \quad x (x, y) \in (\Omega \cup \Gamma) \] (8.13)

To maintain the pure boundary character of the BEM, hence its advantages over the domain type methods, the domain integral in Eq. (8.13), namely

\[
I(x) = \int_{\Omega} v(x, y) \ b(y) \, d\Omega_y, \quad x (x, y) \in (\Omega \cup \Gamma), \quad y(x, y) \in \Omega \] (8.14)

must be converted to boundary line integral.
The technique presented in Section 3.5 (ii) could be used. This technique, though effective, has an inherent drawback, which is the determination of a particular solution of the equation

\[ \nabla^2 \hat{u} = b \]  

for a given source \( b = b(x, y) \). Apparently, this procedure cannot be embedded in a computer code, since the user is responsible for providing the function \( \hat{u} \) and its normal derivative on the boundary according to Eq. (3.53). This drawback can be overcome by employing the DRM, which proposes the use of a series of particular solutions \( \hat{u}_j \) instead of a single function \( \hat{u} \). The number of \( \hat{u}_j \) used is taken equal to the total number of nodes in the problem, which may consist of \( N \) boundary nodes and \( L \) internal nodes, Fig. 8.2.

In the DRM the source \( b(x, y) \) is approximated by the series

\[ b \approx \sum_{j=1}^{N+L} a_j \phi_j \]  

where \( a_j \) are initially unknown coefficients and \( \phi_j = \phi_j(x, y) \) spatial approximation functions. The particular solutions \( \hat{u}_j \) are obtained from the solution of the equation

\[ \nabla^2 \hat{u}_j = \phi_j \]  

Evidently, Eq. (8.16) is exact at the nodal points. Next application of Green’s reciprocal identity (2.16) for \( u = \nabla^2 \hat{u}_j \) yields

\[ \int_{\Omega} \left( \nabla^2 \hat{u}_j - \hat{u}_j \nabla^2 v \right) \, d\Omega = \int_{\Gamma} \left( \nabla \hat{u}_j \cdot \hat{n} - \hat{u}_j \frac{\partial v}{\partial n} \right) \, ds \]  

**FIGURE 8.2** Boundary and internal nodes in DRM.
or taking into account Eq. (8.17) and \( v = \ln r / 2\pi \) we obtain

\[
\int_{\Omega} v(x, y) \phi_j(r_{jx}) \, d\Omega_y = \varepsilon \hat{u}_j(r_{jx}) + \int_{\Gamma} \left( v(x, \xi) \frac{\partial \hat{u}_j(r_{jx})}{\partial n_\xi} - \hat{u}_j(r_{jx}) \frac{\partial v(x, \xi)}{\partial n_\xi} \right) \, ds_\xi
\]  

(8.19)

Hence, on the bases of Eqs. (8.14) and (8.16) we can write

\[
\int_{\Omega} v_b \, d\Omega = \sum_{j=1}^{N+L} a_j \left( \varepsilon \hat{u}_j + \int_{\Gamma} v \frac{\partial \hat{u}_j}{\partial n} \, ds - \int_{\Gamma} \hat{u}_j \frac{\partial v}{\partial n} \, ds \right)
\]

(8.20)

Thus, Eq. (8.13) when written for the node \( i \) reads

\[
\varepsilon_i u_i - \int_{\Gamma} u \frac{\partial v}{\partial n} \, ds + \int_{\Gamma} v \frac{\partial u}{\partial n} \, ds = \\
\sum_{j=1}^{N+L} a_j \left( \varepsilon_i \hat{u}_{ij} + \int_{\Gamma} v \frac{\partial \hat{u}_j}{\partial n} \, ds - \int_{\Gamma} \hat{u}_j \frac{\partial v}{\partial n} \, ds \right)
\]

(8.21)

We observe that Eq. (8.21) was derived by applying Green’s reciprocal identity, i.e., the reciprocity principle, twice to take all terms to the boundary, hence the name DRM.

The next step is to write Eq. (8.21) in discretized form, where the integrals are replaced with summations over the elements.

\[
\varepsilon_i u_i + \sum_{k=1}^{N} v^*_k q \, ds - \sum_{k=1}^{N} u q^*_k \, ds = \\
\sum_{j=1}^{N+L} a_j \left( \varepsilon_i \hat{u}_{ij} + \sum_{k=1}^{N} v^*_j q \, ds - \sum_{k=1}^{N} \hat{u}_j q^*_k \, ds \right)
\]

(8.22)

where it has been set for convenience \( \partial u / \partial n = q, \ v = v^*, \ \partial v / \partial n = q^*, \ \partial \hat{u}_j / \partial n = \hat{q}_j; \ \hat{u}_{ij} \) is the value of the function \( \hat{u}_j \) at point \( i \). Since \( \hat{u}_j \) and \( \hat{q}_j \) are known functions, there is no need to approximate them within the elements. However, it is convenient to do so and use the matrices \( H \) and \( G \) defined in Section 4.2. This procedure introduces an approximation in the evaluation of the terms in the right-hand side of Eq. (8.22). However, it is shown that this error is small, while the efficiency of the method is increased. Thus, after
using the boundary element technique for the evaluation of the integrals over
the elements, the above equation is written in terms of the nodal values as

\[ \varepsilon_i u_i + \sum_{k=1}^{N} G_{ik} q_k - \sum_{k=1}^{N} \hat{H}_{ik} \hat{u}_k = \]

\[ \sum_{j=1}^{N+L} a_j \left( \varepsilon_i \hat{u}_{ij} + \sum_{k=1}^{N} G_{ik} \hat{q}_k - \sum_{k=1}^{N} \hat{H}_{ik} \hat{u}_k \right) \]

(8.23)

After application of the above equation to all boundary nodes, we can
write Eq. (8.23) in matrix form:

\[ H u - G q = \sum_{j=1}^{N+L} a_j (H u_j - G q_j) \]

(8.24)

The free term coefficient \( \varepsilon_i \) has been incorporated in matrix \( H \). For con-
stant elements the matrices \( H \) and \( G \) are given by Eqs. (4.3) and (4.5).
If each of the vectors \( u_j \), \( q_j \) is considered to represent a column of the
matrices \( \hat{U} \) and \( \hat{Q} \), respectively, we may define

\[ \hat{U} = [ \hat{u}_1 \ \hat{u}_2 \ \ldots \ \hat{u}_{N+L} ] \]

(8.25a)

\[ \hat{Q} = [ \hat{q}_1 \ \hat{q}_2 \ \ldots \ \hat{q}_{N+L} ] \]

(8.25b)

\[ \alpha = \{ \alpha_1 \ \alpha_2 \ \ldots \ \alpha_{N+L} \}^T \]

(8.25c)

Then the summations in Eq. (8.24) can be dropped and written

\[ H u - G q = (H \hat{U} - G \hat{Q}) \alpha \]

(8.26)

Equation (8.26) is the basis for the development of the DRM. Apparently, it is a boundary-only method in the sense that only boundary
discretization is performed. Domain nodes may be included, when it
is desirable to evaluate the solution at these points. The use of internal
nodes depends also on the type of the approximation functions \( \phi_j \). Anyhow, the use of domain nodes makes the method more robust. For
a better inspection, Eq. (8.26) can be represented schematically as shown
in Fig. 8.3.
THE VECTOR $\alpha$

The source term $b(x, y)$ is known for Poisson’s equation. This allows the evaluation of the coefficients $a_j$, which is achieved as follows:

By taking the value of $b$ at the $(N + L)$ nodal points, a set of equations of the form

$$b_i = \sum_{j=1}^{N+L} \alpha_j \phi_{ji}$$

(8.27)

is obtained; this may be expressed in matrix form as

$$b = \Phi \alpha$$

(8.28)

where each column of $\Phi$ consists of a vector $\phi_j$ containing the values of the function $\phi_j$ at the $(N + L)$ DRM collocation points, that is, $\phi_j(x_i, y_i)$, $i = 1, 2, \ldots, (N + L)$. Thus Eq. (8.28) may be inverted to give

$$\alpha = \Phi^{-1} b$$

(8.29)

The right-hand side of Eq. (8.26) is thus a known vector. Eq. (8.26) may be further written as

$$Hu - Gq = F$$

(8.30)

where

$$F = (H\hat{U} - G\hat{Q})\Phi^{-1} b$$

(8.31)

We see that the vector $F$ can be obtained directly by multiplying known matrices and vectors. Eq. (8.30) may be compared with Eq. (4.32), which gives the same result by integrating over internal cells.
Applying the boundary conditions (3.35) as explained in Chapter 3, this equation reduces to the form (4.9)

\[ \mathbf{Ax} = \mathbf{y} \quad (8.32) \]

where the vector \( \mathbf{x} \) contains \( N \) unknown boundary values of \( u \) or \( q \).

Noting that for the Laplace operator it is \( \mathbf{m} = \mathbf{n} \), hence \( \nabla u \cdot \mathbf{m} = u, n = q \), the boundary conditions (8.2) may be written in the more convenient form

\[ c_1 u + c_2 q = c_3 \quad (8.33) \]

where \( c_i(\mathbf{x}) \) \( (i = 1, 2, 3) \) are quantities specified on the boundary \( \Gamma \). Apparently, all types of boundary conditions are obtained from Eq. (8.33) for appropriate values of \( c_i \).

Application of the boundary conditions (8.33) to all boundary points yields the system of \( N \) equations

\[ c_1 \mathbf{u} + c_2 \mathbf{q} = c_3 \quad (8.34) \]

where \( c_1, c_2 \) are diagonal matrices and \( c_3 \) a vector containing the \( N \) values of \( c_i \).

Equations (8.30) and (8.34) can be solved simultaneously to give the boundary values of \( u \) and \( q \). This procedure, though it increases the number of unknowns by \( N \), it simplifies the programming of the solution procedure as it treats all types of boundary conditions in the same way.

**INTERIOR SOLUTION**

Once the nodal boundary values of \( u \) and \( q \) are known, the values at any internal node can be computed from Eq. (8.23) with \( \varepsilon_i = 1 \)

\[
\begin{align*}
  u_i = & - \sum_{k=1}^{N} G_{ik} q_k + \sum_{k=1}^{N} \hat{H}_{ik} u_k \\
  & + \sum_{j=1}^{N+L} \alpha_j \left( \hat{u}_{ij} + \sum_{k=1}^{N} G_{ik} \hat{q}_k - \sum_{k=1}^{N} \hat{H}_{ik} \hat{u}_k \right) 
\end{align*}
\quad (8.35)
\]

Note that the row matrices \([\hat{H}_{ik}]\) and \([G_{ik}]\) are evaluated for each interior point, where the solution is desired.

After application of Eq. (8.35) to all internal nodes we may write it in matrix form

\[
\mathbf{H}^* \mathbf{u} - \mathbf{I} \mathbf{u}^* - \mathbf{G}^* \mathbf{q} = (\mathbf{H}^* \hat{\mathbf{U}} - \mathbf{G}^* \hat{\mathbf{Q}}) \mathbf{\alpha} - \hat{\mathbf{U}}^* \mathbf{\alpha} 
\quad (8.36)
\]

The star designates quantities referred to the interior points. Thus, the matrices have the dimensions: \( \mathbf{H}^*, \mathbf{G}^* \ L \times N; \hat{\mathbf{U}}, \hat{\mathbf{Q}} \ N \times (N + L); \hat{\mathbf{U}}^* \ L \times (N + L); \mathbf{u}^* \ L \times 1; \mathbf{I} \) is the \( L \times L \) unit matrix.
Equations (8.26) and (8.36) can be combined as

$$\begin{bmatrix} H & 0 \\ H^* & -I \end{bmatrix} \begin{bmatrix} u \\ u^* \end{bmatrix} - \begin{bmatrix} G & 0 \\ G^* & 0 \end{bmatrix} \begin{bmatrix} q \\ 0 \end{bmatrix} = \left( \begin{bmatrix} H & 0 \\ H^* & -I \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{U}^* \end{bmatrix} - \begin{bmatrix} G & 0 \\ G^* & 0 \end{bmatrix} \begin{bmatrix} \hat{Q} \\ 0 \end{bmatrix} \right) \alpha$$

(8.37)

For better inspection the above equation is represented schematically in Fig. 8.4. We observe that Eq. (8.37) has the form of Eq. (8.26). Thus, we may write it as

$$Hu - Gq = (H\hat{U} - G\hat{Q})\alpha$$

(8.38)

provided that the matrices and vectors are redefined according to Eq. (8.37).

This representation with all matrices in \((N + L) \times (N + L)\) form simplifies the understanding of the method and will be used in the next sections for problems for which the source \(b\) is an unknown function.

**TYPE OF INTERPOLATION FUNCTIONS**

The success of DRM depends on the choice of the interpolation functions \(\phi_j(x, y)\) which are used to approximate the source \(b(x, y)\). To obtain accurate and reliable results with the DRM, the approximation functions must be appropriately chosen and satisfy the following requirements [10].

1. For \(N + L \to \infty\), the series (8.16) must converge to \(b\).
2. The rate of convergence must be high, so that the source term \(b\) is represented accurately even when few interpolation points are used.
3. The accuracy of the computations should not be influenced by the geometry of the domain \(\Omega\).
4. The particular solutions corresponding to the interpolation functions \(\phi_j(x, y)\) should be available in closed form. This ensures fast computations and simple implementation of the computer program.

The interpolation functions \(\phi_j(x, y)\) used in DRM analysis can be classified into *local* and *global* functions. The latter interpolate over the entire domain, while the former interpolate only in the neighborhood of a particular

**FIGURE 8.4** Schematic representation of the matrices in DRM (Eq. 8.37).
point. The most widely used global functions are the so-called radial basis functions (RBFs) \( \phi_j = \phi_j(r) \), which are functions of the Euclidean distance

\[
r = |\mathbf{x} - \mathbf{x}_j| = \sqrt{(x-x_j)^2 + (y-y_j)^2}
\]

between the field point \( \mathbf{x}(x, y) \) and the collocation point \( \mathbf{x}_j(x_j, y_j) \). Many types of RBFs have been reported in the literature. The most commonly used in DRM analysis are parts of the polynomial \( \phi_j = 1 + r + r^2 + \ldots + r^k \), multiquadrics (MQs) \( \phi_j = (r^2 + c^2)^{1/2} \) with \( c \) being an arbitrary constant, the so-called shape parameter, which must be appropriately chosen to obtain optimum accuracy, thin plate spines, compact support RBFs, etc. Best results are obtained by combining RBFs augmented with elements of a polynomial series. An overview of the developments on RBFs can be found in the articles [11–15] and more recently in [16]. However, the use of such special functions requires a lot of experience with the problem at hand and is therefore not recommended for a general analysis. Instead, the use of the very simple function

\[
\phi(r) = 1 + r
\]

has been shown to yield accurate results for a wide variety of problems [1,2,10,17]. Also, the corresponding particular solutions are available in closed form and are therefore easy to implement. Table 8.1 gives the particular solutions for some commonly used RBFs.

### Table 8.1 Particular Solutions for Various RBFs

<table>
<thead>
<tr>
<th>RBF ( \phi(r) )</th>
<th>Particular Solution ( \hat{u}(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi = 1 + r )</td>
<td>( \hat{u} = \frac{1}{4} r^2 + \frac{1}{7} r^3 )</td>
</tr>
<tr>
<td>( \phi = 1 + r + r^2 + \ldots + r^k )</td>
<td>( \hat{u} = \frac{1}{4} r^2 + \frac{1}{7} r^3 + \frac{1}{16} r^4 + \ldots + \frac{1}{(2+k)^{2+k}} r^{2+k} )</td>
</tr>
<tr>
<td>( \phi = \sqrt{c^2 + r^2} )</td>
<td>( \hat{u} = \frac{1}{7} \phi^3 + \frac{1}{7} \phi^2 c - \frac{1}{7} \ln(c + \phi) )</td>
</tr>
</tbody>
</table>

### 8.3.3 The DRM for equations of the type

\( \nabla^2 u = b(x, y, u, u_x, u_y) \)

Equations of this type result, if Eq. (8.1) can be written in its canonical form [8], that is,

\[
\nabla^2 u + D u_x + E u_y + F u = f(x, y)
\]

(8.41)

Keeping the dominant part of the operator in the left-hand side and shifting the remaining part in the right-hand side, Eq. (8.41) is written as

\[
\nabla^2 u = b(x, y, u, u_x, u_y)
\]

(8.42)
where
\[ b(x, y, u, u_x, u_y) = f - (Du_x + Eu_y + Fu) \] (8.43)

THE EQUATION \( \nabla^2 u + u = 0 \)

In order to derive the necessary basic relationships, the DRM will be applied first to the simplest equation of this type

\[ \nabla^2 u + u = 0 \] (8.44)

Apparently, in this case it is \( b(x, y, u, u_x, u_y) = -u \). Thus, from Eq. (8.29) we have

\[ \alpha = -\Phi^{-1}u \] (8.45)

and Eq. (8.38) becomes

\[ Hu - Gq = - (H\hat{U} - G\hat{Q})\Phi^{-1}u \] (8.46)

or

\[ Hu - Gq = -Su \] (8.47)

where

\[ S = (H\hat{U} - G\hat{Q})\Phi^{-1} \] (8.48)

is a known matrix.

After rearranging, Eq. (8.47) is written as

\[ (H + S)u = Gq \] (8.49)

Equation (8.49) represents a system of \( (N + L) \) equations for \( (2N + L) \) unknowns, that is, \( 2N \) boundary values of \( u \) and \( q \), and \( L \) internal values of \( u \). This equation can be combined with Eq. (8.34) and solved to yield all the unknown quantities.

Equations (8.46)–(8.49) form the basis of application of the DRM to equations of the form (8.42).

THE EQUATION \( \nabla^2 u + u_{xx} = 0 \)

We consider now the equation

\[ \nabla^2 u + u_{xx} = 0 \] (8.50)
In this case it is \( b(x, y, u, u_x, u_y) = -u_x \) and Eq. (8.29) becomes
\[
\alpha = -\Phi^{-1}u_x
\]  
which is inserted in Eq. (8.38) to give
\[
Hu - Gq = -Su_x
\]  
A mechanism must now be established to relate the nodal values of \( u \) to the nodal values of the derivative \( u_x \). This is achieved using the approximation
\[
u = \Phi\beta
\]
where \( \beta \neq \alpha \). Differentiating Eq. (8.53) yields
\[
u_x = \Phi_x \beta
\]
Equation (8.53) gives \( \beta = \Phi^{-1}u \), which is substituted in Eq. (8.54) to give
\[
u_x = \Phi_x \Phi^{-1}u
\]
Substituting now into Eq. (8.52) gives
\[
Hu - Gq = -S\Phi_{,x} \Phi^{-1}u
\]
where \( S \) is defined by Eq. (8.48).
Setting
\[
R = S\Phi_{,x} \Phi^{-1}
\]
and rearranging yields the system of equations
\[
(H + R)u = Gq
\]
which is treated exactly as Eq. (8.49) to obtain the solution.

**VARIABLE COEFFICIENTS**

The DRM can be extended to equations with variable coefficients. Consider the Helmholtz equation
\[
\nabla^2 u + F(x, y)u = 0
\]
where the DRM source is now \( b = -Fu \), which is applied to all nodes to give
\[
b = -Fu
\]
where \( F \) is an \((N + L) \times (N + L)\) diagonal matrix containing the nodal values of the function \( F(x, y) \), that is,

\[
F = \begin{bmatrix}
F(x_1, y_1) & 0 & \cdots & 0 \\
0 & F(x_2, y_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & F(x_{N+L}, y_{N+L})
\end{bmatrix}
\]  

(8.61)

Thus, Eq. (8.29) becomes

\[
\alpha = -\Phi^{-1} Fu
\]  

(8.62)

which is inserted in Eq. (8.38) to produce

\[
Hu - Gq = -SFu
\]  

(8.63)

where \( S \) is defined by Eq. (8.49). After rearranging, Eq. (8.63) becomes

\[
(H + SF)u = Gq
\]  

(8.64)

which is treated exactly as Eq. (8.49) to obtain the solution.

**THE DRM FOR THE COMPLETE ELLIPTIC DIFFERENTIAL EQUATION**

The cases studied in the previous Sections “The Equation \( \nabla^2 u + u = 0 \)”, “The Equation \( \nabla^2 u + u_{,x} = 0 \)”, “Variable Coefficients”, and Section 8.3.2 enable the application of the DRM to the complete elliptic differential equation Eq. (8.41). We readily derive the following DRM equation

\[
(H + R)u = Gq + Sf
\]  

(8.65)

where

\[
R = S\left[(D\Phi_{,x} + E\Phi_{,y})\Phi^{-1} + F\right]
\]  

(8.66)

in which \( D, E, F \) represent \((N + L) \times (N + L)\) diagonal matrices containing the nodal values of the position dependent coefficients \( D(x, y), E(x, y), F(x, y) \), respectively, and \( f \) is the vector containing the nodal values of the source \( f(x, y) \); the matrix \( S \) is defined by Eq. (8.48).

Equations (8.65) and (8.34) are combined to give the system of equations

\[
\begin{bmatrix}
H + R & -G \\
\text{c}_1 & \text{c}_2
\end{bmatrix}
\begin{bmatrix}
u \\
q
\end{bmatrix}
= 
\begin{bmatrix}
Sf \\
\text{c}_3
\end{bmatrix}
\]  

(8.67)

which is solved to yield the solution.
DERIVATIVES OF THE FIELD FUNCTION AT INTERNAL NODES
The derivatives at internal nodes are obtained by direct differentiation of Eq. (8.13) with \( \varepsilon = 1 \)

\[
\begin{align*}
    u_{,x}(x) &= \int_{\Omega} v_{,x} b \, d\Omega - \int_{\Gamma} \left( v_{,x} \frac{\partial u}{\partial n} - u \frac{\partial v_{,x}}{\partial n} \right) \, ds, \quad x (x, y) \in \Omega \quad (8.68a) \\
    u_{,y}(x) &= \int_{\Omega} v_{,y} b \, d\Omega - \int_{\Gamma} \left( v_{,y} \frac{\partial u}{\partial n} - u \frac{\partial v_{,y}}{\partial n} \right) \, ds, \quad x (x, y) \in \Omega \quad (8.68b)
\end{align*}
\]

This approach requires the evaluation of new influence matrices and transformation of the domain integrals to the boundary. The use of the DRM provides a simple alternative to Eqs. (8.68), using already computed matrices, independently of the governing equation and the type of the source. This is achieved using the relation (8.55), that is,

\[
\begin{align*}
    u_{,x} &= \Phi_{,x} \Phi^{-1} u \quad (8.69a) \\
    u_{,y} &= \Phi_{,y} \Phi^{-1} u \quad (8.69b)
\end{align*}
\]

SOLUTION PROCEDURE STEPS
A computer code can be written for the solution of the boundary value problem (8.1), (8.2) by adhering to the following steps:
1. Compute the \( N \times N \) matrices \( G \) and \( H \) for the boundary nodes using Eqs. (4.3) and (4.5).
2. Choose RBFs and compute the \( N \times (L + N) \) matrices \( \hat{U}, \hat{Q} \) for the boundary nodes using Eqs. (8.25a,b).
3. Compute the \( L \times N \) matrices \( G^*, H^* \) for the internal nodes using Eqs. (4.3).
4. Compute the \( L \times (L + N) \) matrices \( \hat{U}^*, \hat{Q}^* \) for the internal nodes using Eqs. (8.25a,b).
5. Formulate the matrices \( G, H, \hat{U}, \hat{Q} \) defined in Eq. (8.37).
6. Compute the matrices \( \Phi, S, \) and \( R \) using Eqs. (8.28), (8.48), and (8.66).
7. Solve the system of linear equations (8.67).

EXAMPLES
On the basis of the previous solution procedure steps a FORTRAN code has been written which implements the DRM and example problems are solved.

EXAMPLE 8.1
In this example the DRM is employed for the solution of the boundary value problem

\[
\begin{align*}
    \nabla^2 u + u_{,x} + u_{,y} - 2u &= \sin(x + y) - 4\sin x \sin y \quad \text{in} \ \Omega \quad (8.70a) \\
    u &= \sin x \sin y \quad \text{on} \ \Gamma \quad (8.70b)
\end{align*}
\]
The domain $\Omega$ is the ellipse with semi axes $a = 5, b = 3$. The problem admits an exact solution $u_{\text{exact}} = \sin x \sin y$. The employed RBFs are the MQs with shape parameter $c = 1$. The numerical results have been obtained using $N = 80$ constant boundary elements and $L = 101$ domain nodal points distributed as shown in Fig. 8.5. Table 8.2 shows the computed values of the solution and its first derivatives at certain points as compared with the exact ones. Fig. 8.6 shows the solution $u$ and its derivatives $u_x, u_y$ along the line $x = y$. Finally, Fig. 8.7 shows the normal derivative $q = u_n$ along the boundary.

**TABLE 8.2** Solution and its Derivatives in Example 8.1. Upper Value: Computed; Lower Value: Exact

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$u$</th>
<th>$u_x$</th>
<th>$u_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0300</td>
<td>1.9021</td>
<td>0.8102</td>
<td>0.5171</td>
<td>-0.2786</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.8107</td>
<td>0.4867</td>
<td>-0.2788</td>
</tr>
<tr>
<td>1.9593</td>
<td>1.6180</td>
<td>0.9226</td>
<td>-0.3876</td>
<td>-0.0455</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9244</td>
<td>-0.3783</td>
<td>-0.0437</td>
</tr>
<tr>
<td>3.1702</td>
<td>0.6180</td>
<td>-0.0175</td>
<td>-0.5843</td>
<td>-0.0327</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.0165</td>
<td>-0.5792</td>
<td>-0.0233</td>
</tr>
<tr>
<td>3.9627</td>
<td>0.7725</td>
<td>-0.5114</td>
<td>-0.4603</td>
<td>-0.5325</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.5185</td>
<td>-0.4756</td>
<td>-0.5242</td>
</tr>
<tr>
<td>-2.0225</td>
<td>-0.8816</td>
<td>0.6941</td>
<td>0.3337</td>
<td>-0.5752</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.6944</td>
<td>0.3369</td>
<td>-0.5720</td>
</tr>
<tr>
<td>-0.6742</td>
<td>0.2939</td>
<td>-0.1809</td>
<td>0.2254</td>
<td>-0.5970</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1808</td>
<td>0.2263</td>
<td>-0.5975</td>
</tr>
</tbody>
</table>
FIGURE 8.6 Solution $u$ and derivatives $u_x, u_y$ along the line $x = y$ in Example 8.1.
In Section 8.3 the DRM BEM was presented for the solution of the second order elliptic equation. It was shown that the application of the method was possible when the equation was in its canonical form, Eq. (8.41). This form allows the use of the Laplace fundamental solution to develop the DRM BEM. However, this is not possible for the general potential equation, Eq. (8.1). The inability to apply DRM may also appear in nonlinear equations. For example the equation of a surface with constant Gaussian curvature \( K \) [6]

\[
\begin{align*}
    u_{xx} - u_{yy} &= K(1 + u_x^2 + u_y^2) = 0 \\
    \frac{1}{1 + u_y^2} u_{yy} - 2u_{xy}u_{xy} + (1 + u_x^2)u_{xx} &= 0
\end{align*}
\]  

(8.71a, 8.71b)

or Plateau’s equation for the minimal surface cannot be treated by the DRM.

The AEM is alleviated from any restriction and can be employed to solve efficiently any differential equation not only linear but also nonlinear and systems of them as well. The AEM is based on the principle of the analog equation (PAE), which reduces all potential problems, linear or nonlinear, to the Poisson equation.
8.4.1 The principle of the analog equation

The PAE was introduced by Katsikadelis in 1994 [18] and was used in conjunction with the BEM to develop the analog equation method or AEM as it is known with its acronym. The PAE can be stated as:

*Any differential equation, be it linear or nonlinear, can be replaced by another differential equation of the same order under an unknown fictitious source. The substitute equation is termed the analog equation.*

To make this concept more concrete we illustrate the PAE as follows:

We consider the boundary value problem

\[
N(u) = g(x), \ x \in \Omega \tag{8.72a}
\]
\[
B(u) = \overline{g}(x), \ x \in \Gamma \tag{8.72b}
\]

where \( N \) and \( B \) are linear or nonlinear differential operators.

Let \( u(x) \) be the solution of the problem \((8.72a,b)\). If \( \tilde{N} \) is another differential operator (linear or nonlinear) of the same order with \( N \) and it is applied to \( u(x) \) yields

\[
\tilde{N}(u) = b(x), \ x \in \Omega \tag{8.73}
\]

where now \( b(x) \) is an unknown function (source, load). Eq. (8.73) together with the boundary condition (8.72b) can give the solution of the original problem, provided that the source \( b(x) \) is established. Eq. (8.73) is the analog equation of the original problem, which together with the boundary condition (8.72b) constitutes the equivalent or the substitute problem. The replacement of the actual problem by the substitute one expresses the PAE.

The PAE applies also to a system of coupled differential equations. In this case the analog equations may constitute a set of uncoupled equations. The PAE applies to elliptic, hyperbolic and parabolic differential equations of integer or fractional order derivative [19]. Obviously, the practical significance of the PAE for the BEM is very important as a linear equation with simple known fundamental solution can be chosen as the analog equation. Then the standard BEM is readily employed to solve the substitute problem and the solution of the original problem is obtained from the integral representation of the substitute problem. The fictitious source is first established using a BEM-based procedure. The PAE is depicted in Fig. 8.8. This figure shows the deflection curve \( u(x) \) of a beam with variable stiffness \( EI(x) \) under the load \( g(x) \). According to the PAE, the function \( u(x) \) can result as the deflection curve of a beam with constant unit stiffness, say \( EI(x) = 1 \), under an appropriate load \( b(x) \). Obviously, the solution of the substitute beam problem is simple if the fictitious load \( b(x) \) is first established. Some examples illustrating the PAE are presented in Table 8.3—8.5.
8.4.2 The AEM for the elliptic partial differential equation

Let \( u = u(x) \) be the sought solution to the boundary value problem (8.1), (8.2). This function is two times continuously differentiable in \( \Omega \). Since Eq. (8.1) is first of the second order, the operator \( \hat{N} = \nabla^2 \), that is, the Laplace operator, can be employed to produce the analog equation. This yields

\[
\nabla^2 u = b(x), \quad x(x, y) \in \Omega
\]  

(8.74)
TABLE 8.4 Example 2. The PAE Applied to a Nonlinear Problem

**Real Problem**

\[
\begin{align*}
  u_{xx} + u_{xy} - (1 + u_x^2 + u_y^2) + 5u &= 3(1 + xy) \quad \text{in } \Omega \\
  u &= x^2 - xy + y^2 \quad \text{on } \Gamma_u, \quad u_{x_n}^2 = 4x^2 - 4xy + y^2 \quad \text{on } \Gamma_n
\end{align*}
\]

The domain \( \Omega \) is the rectangle of Example 1

**Substitute Problem**

\[
\begin{align*}
  \nabla^2 u &= b(x, y) \quad \text{in } \Omega \\
  u &= x^2 - xy + y^2 \quad \text{on } \Gamma_u, \quad u_{x_n}^2 = 4x^2 - 4xy + y^2 \quad \text{on } \Gamma_n
\end{align*}
\]

If \( b(x, y) = 4 \), both problems assume the same solution \( u = x^2 - xy + y^2 \)

---

TABLE 8.5 Example 3. The PAE Applied to a System of Two Coupled Equations

**Real Problem**

\[
\begin{align*}
  \nabla^2 u + \frac{1 + \nu}{1 - \nu} (u_{xx} + v_{xx}) + \left( \frac{\lambda_x}{\mu \lambda_x} + 2 \frac{\mu_{xy}}{\mu} \right) u_x + \frac{\mu_{xy}}{\mu} u_y + \frac{\mu_y}{\mu} v_x + \frac{\lambda_x}{\mu} v_y &= -\frac{f_x}{\mu} \quad \text{in } \Omega \\
  \nabla^2 v + \frac{1 + \nu}{1 - \nu} (u_{xy} + v_{xy}) + \left( \frac{\lambda_y}{\mu \lambda_y} + 2 \frac{\mu_{xy}}{\mu} \right) u_y + \frac{\mu_{xy}}{\mu} u_x + \frac{\lambda_y}{\mu} v_x + \frac{\mu_y}{\mu} v_y &= -\frac{f_y}{\mu} \quad \text{in } \Omega
\end{align*}
\]

\[
\begin{align*}
  u &= xy + 0.2y^2, \quad v = xy + 0.2x^2 \quad \text{on } \Gamma_u \\
  t_x &= \lambda^*(x + y) + 2\mu y, \quad v = xy + 0.2x^2 \quad \text{on } \Gamma_t
\end{align*}
\]

where

\[
\begin{align*}
  \lambda^* &= \frac{\nu E}{1 - \nu^2}, \quad \mu = \frac{E}{2(1 + \nu)} \\
  f_x &= E_0(1 + x)[1 - 0.1(1 + x)(7 + 3\nu) - 2(\nu x + y)]/(1 - \nu^2) \\
  f_y &= E_0(1 + x)[1 - 0.1(1 + x)(7 + 3\nu) - 1.4(\nu x + y)]/(1 - \nu^2) \\
  E &= E_0(1 + x)^2, \quad E_0 = 2 \times 10^5 \text{ kN/m}^2, \quad \nu = 0.2
\end{align*}
\]

**Substitute Problem**

\[
\begin{align*}
  \nabla^2 u &= b_x(x, y) \quad \nabla^2 v &= b_y(x, y) \quad \text{in } \Omega \\
  u &= xy + 0.2y^2, \quad v = xy + 0.2x^2 \quad \text{on } \Gamma_u \\
  t_x &= \lambda^*(x + y) + 2\mu y, \quad v = xy + 0.2x^2 \quad \text{on } \Gamma_t
\end{align*}
\]

If \( b_x(x, y) = b_y(x, y) = 0.4 \), both problems assume the same solution

\[
\begin{align*}
  u &= xy + 0.2y^2, \quad v = xy + 0.2x^2
\end{align*}
\]
Equation (8.74) indicates that the solution of Eq. (8.1) could be established by solving Eq. (8.74) under the boundary conditions (8.2a,b), if the fictitious source $b(x)$ is first established. This is accomplished adhering to the following procedure:

We write the solution of Eq. (8.74) in integral form. Thus, we have (see Section 3.3)

$$
\varepsilon u(x) = \int_{\Omega} u^* b \, d\Omega - \int_{\Gamma} (u^* q - q^* u) \, ds \quad x \in \Omega \cup \Gamma
$$

(8.75)

We recall that $u^* = \ln r / 2\pi$ is the fundamental solution of Eq. (8.74) and $q^* = u^*_n$ its derivative normal to the boundary with $r = |\xi - x| = [(\xi - x)^2 + (y - \eta)^2]^{1/2}$ being the distance between any two points $x \in \Omega \cup \Gamma$ and $\xi(\xi, \eta) \in \Gamma$; $\varepsilon$ is the free term coefficient, which takes the values $\varepsilon = 1$ if $x \in \Omega$, $\varepsilon = \alpha / 2\pi$ if $x \in \Gamma$, and $\varepsilon = 0$ if $x \notin \Omega \cup \Gamma$; $\alpha$ is the interior angle between the tangents of boundary at point $x$. Note that it is $\varepsilon = 1/2$ for points where the boundary is smooth.

The next step is to approximate the source in Eq. (8.74) using local or global approximation functions. The first approach has been employed in [15]. Here, the AEM is developed using RBF approximation functions as in DRM. Since $b(x)$ is defined in $\Omega$, only internal collocation points are taken, that is,

$$
b \approx \sum_{j=1}^{L} a_j \phi_j
$$

(8.76)

where $L$ is the number of the internal nodes.

After converting the domain integral to boundary line integral and discretizing the boundary using $N$ elements, Eq. (8.75) becomes (see Eq. (8.26))

$$
Hu - Gq = T\alpha
$$

(8.77)

where

$$
T = (\hat{H}\hat{U} - \hat{G}\hat{Q})
$$

(8.78)

$$
\alpha = \Phi^{-1}b, \quad b = \{ b_1 \ b_2 \ \cdots \ b_L \}^T, \quad \alpha = \{ a_1 \ a_2 \ \cdots \ a_L \}^T
$$

(8.79)

The matrices $\hat{U}$ and $\hat{Q}$ have dimensions $N \times L$ and are defined as in Eqs. (8.25a,b), that is,

$$
\hat{U} = [\hat{u}_1 \ \hat{u}_2 \ \cdots \ \hat{u}_L]
$$

(8.80a)

$$
\hat{Q} = [\hat{q}_1 \ \hat{q}_2 \ \cdots \ \hat{q}_L]
$$

(8.80b)
Equation (8.77) represents a set of \( N \) equations for the boundary quantities \( u, q \). Another set of \( N \) equations for these quantities is obtained from the boundary conditions (8.2a,b), which may be combined as

\[
c_1 u + c_2 \nabla u \cdot \mathbf{m} = c_3
\]

(8.81)

where \( c_i(x) \ (i = 1, 2, 3) \) are quantities specified on the boundary \( \Gamma \). Apparently, all types of boundary conditions are obtained from Eq. (8.81) for appropriate values of \( c_i \).

Before using Eq. (8.81), the flux \( \nabla u \cdot \mathbf{m} \) must be first expressed in terms of \( q = u_{,n} \) and \( u \). This is achieved as follows.

It can be shown that (*)

\[
\nabla u \cdot \mathbf{m} = (\mathbf{m} \cdot \mathbf{n}) u_{,n} + (\mathbf{m} \cdot \mathbf{t}) u_{,t}
\]

(8.82)

The tangential derivative \( u_{,t} \) can be expressed in terms of \( u \). This can be accomplished either by establishing the integral representation of the tangential derivative (see Problem 3.4) or using finite differences as in Section 5.2.2. The latter technique is accurate and easy to implement. Backward and forward difference schemes are employed at points adjacent to corner points. Thus we obtain

\[
\mathbf{u}_{,t} = \mathbf{d} u
\]

(8.83)

where

\[
\mathbf{d} = \frac{1}{2\ell}
\]

\[
\begin{bmatrix}
-3 & 4 & -1 \\
-1 & 0 & 1 \\
-1 & 0 & 1 \\
1 & -4 & 3 \\
\end{bmatrix}
\]

\[
\begin{array}{cccc}
\text{j} & \text{j+1} & \\
\end{array}
\]

\[
\begin{array}{cccc}
\ldots & \ldots & \ldots & \ldots \\
-1 & 0 & 1 & \\
1 & -4 & 3 & \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{j-1} & \text{j} & \text{j+1} & \text{j+2} \\
\end{array}
\]

(*) \( \nabla u \cdot \mathbf{m} = [m_x \ m_y] \begin{bmatrix} u_{,x} \\ u_{,y} \end{bmatrix} = [m_x \ m_y] \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{bmatrix} u_{,n} \\ u_{,t} \end{bmatrix} = (\mathbf{m} \cdot \mathbf{n}) u_{,n} + (\mathbf{m} \cdot \mathbf{t}) u_{,t} \)
In the above matrix scheme, $j$ and $j + 1$ designate nodes adjacent to a corner and $l$ the common length of the boundary elements.

Applying the boundary condition (8.81) at the $N$ boundary nodes and using Eq. (8.82) we obtain
\[ c_1 u + c_2 [(m \cdot n)q + (m \cdot t)u,] = c_3 \]  
(8.84)
which after the replacement of the tangential derivative becomes
\[ C_1 u + C_2 q = c_3 \]  
(8.85)
where
\[ C_1 = c_1 + c_2 (m \cdot t) d, \quad C_2 = c_2 (m \cdot n) \]  
(8.86)
with $c_i (i = 1, 2)$, $(m \cdot t)$, and $(m \cdot n)$ being diagonal matrices and $c_3$ a vector containing the $N$ values of the respective quantities.

Equations (8.77) and (8.85) can be combined as
\[ \begin{bmatrix} H & -G \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix} = \begin{bmatrix} T \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} 0 \\ c_3 \end{bmatrix} \]  
(8.87)
which gives
\[ \begin{bmatrix} u \\ q \end{bmatrix} = Z \alpha + e \]  
(8.88)
where
\[ Z = \begin{bmatrix} H & -G \\ C_1 & C_2 \end{bmatrix}^{-1} \begin{bmatrix} T \\ 0 \end{bmatrix}, \quad e = \begin{bmatrix} H & -G \\ C_1 & C_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_3 \end{bmatrix} \]  
(8.89a, b)

The next step is to express the derivatives at the internal points in terms of the boundary quantities and the fictitious source. For this purpose, Eq. (8.75) with $\varepsilon = 1$ is differentiated to yield the derivatives involved in Eq. (8.1). Thus, we obtain
\[ u_{,x}(x) = \int_\Omega u_{,x}^* b d\Omega - \int_\Gamma (u_{,x}^* q - q_{,x}^* u) ds \quad x \in \Omega \]  
(8.90a)
\[ u_{,y}(x) = \int_\Omega u_{,y}^* b d\Omega - \int_\Gamma (u_{,y}^* q - q_{,y}^* u) ds \quad x \in \Omega \]  
(8.90b)
\[ u_{,xx}(x) = \int_\Omega u_{,xx}^* b d\Omega - \int_\Gamma (u_{,xx}^* q - q_{,xx}^* u) ds \quad x \in \Omega \]  
(8.90c)
\[ u_{,xy}(x) = \int_\Omega u_{,xy}^* b d\Omega - \int_\Gamma (u_{,xy}^* q - q_{,xy}^* u) ds \quad x \in \Omega \]  
(8.90d)
\[ u_{,yy}(x) = \int_\Omega u_{,yy}^* b d\Omega - \int_\Gamma (u_{,yy}^* q - q_{,yy}^* u) ds \quad x \in \Omega \]  
(8.90e)
Equations (8.90) can be written in the compact form

\[ u_{kl}(x) = \int_{\Omega} u_{kl}^* b \, d\Omega - \int_{\Gamma} (u_{kl}^* q - q_{kl}^* u) \, ds \quad x \in \Omega, \quad k, l = 0, x, y \quad (8.91) \]

This notation implies \( u_{x0} = u, u_{x0} = u_{xx}, u_{y0} = u_{xy} \), etc.

Applying the procedure described in Section 8.3.2 gives

\[
\int_{\Omega} u^*(x, y)_{xx} \phi_j(r_{jy}) d\Omega_y = \varepsilon \hat{u}_j(r_{jx})_{xx} + \int_{\Gamma} \left( u^*(x, \xi)_{xx} \frac{\partial \hat{u}_j(r_{j\xi})}{\partial n_{\xi}} - \hat{u}_j(r_{j\xi}) \frac{\partial u^*(x, \xi)_{xx}}{\partial n_{\xi}} \right) \, ds_{\xi}
\]

(8.92)

where \( x, y \in \Omega, \xi \in \Gamma, r_{jy} = |x_j - y|, r_{j\xi} = |x_j - \xi| \). The subscript in the differentials and normal derivatives indicates the point, with respect to which the integration or differentiation is performed. Fig. 8.9 shows the points and their distances involved in Eq. (8.92).

For convenience, the arguments and subscripts in Eq. (8.92) may be omitted. Thus, this equations reads

\[
\int_{\Omega} u^*_{xx} \phi_j \, d\Omega = \varepsilon \hat{u}_j(r_{jx})_{xx} + \int_{\Gamma} (u^*_{xx} \hat{q}_j - \hat{u}_j q^*_{xx}) \, ds
\]

(8.93)

where \( q^* = \partial u^*/\partial n \) and \( \hat{q}_j = \partial \hat{u}_j/\partial n \).

By virtue of Eqs. (8.93) and (8.76), Eq. (8.90a) becomes

\[
u_{x}(x) = \sum_{j=1}^{L} \left[ \hat{u}_{jx} + \int_{\Gamma} a_j(u_{x}^* q_j - \hat{u}_j q^*_{x}) \, ds \right] - \int_{\Gamma} (u_{x}^* q - q^*_{x} u) \, ds
\]

(8.94)

**FIGURE 8.9** Boundary and internal collocation points, and radial distances in AEM.
After discretizing the boundary using $N$ elements, Eq. (8.94) is written for the internal node $i$

$$u_{i,x} = - \sum_{k=1}^{N} G_{ik,x} q_k + \sum_{k=1}^{N} \hat{H}_{ik,x} u_k$$

$$+ \sum_{j=1}^{N+L} a_j \left( \hat{u}_{ij,x} + \sum_{k=1}^{N} G_{ik,x} \hat{q}_k - \sum_{k=1}^{N} \hat{H}_{ik,x} \hat{u}_k \right)$$  \hspace{1cm} (8.95)

Next applying Eq. (8.95) to the $L$ internal nodes and using matrix form yields

$$\mathbf{u}_x = - \mathbf{G}_x \mathbf{q} + \hat{\mathbf{H}}_x \mathbf{u} + (\hat{\mathbf{U}}_x + \mathbf{G}_x \hat{\mathbf{Q}} - \hat{\mathbf{H}}_x \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96a)

Similarly, we obtain

$$\mathbf{u}_y = - \mathbf{G}_y \mathbf{q} + \hat{\mathbf{H}}_y \mathbf{u} + (\hat{\mathbf{U}}_y + \mathbf{G}_y \hat{\mathbf{Q}} - \hat{\mathbf{H}}_y \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96b)

$$\mathbf{u}_{xx} = - \mathbf{G}_{xx} \mathbf{q} + \hat{\mathbf{H}}_{xx} \mathbf{u} + (\hat{\mathbf{U}}_{xx} + \mathbf{G}_{xx} \hat{\mathbf{Q}} - \hat{\mathbf{H}}_{xx} \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96c)

$$\mathbf{u}_{xy} = - \mathbf{G}_{xy} \mathbf{q} + \hat{\mathbf{H}}_{xy} \mathbf{u} + (\hat{\mathbf{U}}_{xy} + \mathbf{G}_{xy} \hat{\mathbf{Q}} - \hat{\mathbf{H}}_{xy} \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96d)

$$\mathbf{u}_{yy} = - \mathbf{G}_{yy} \mathbf{q} + \hat{\mathbf{H}}_{yy} \mathbf{u} + (\hat{\mathbf{U}}_{yy} + \mathbf{G}_{yy} \hat{\mathbf{Q}} - \hat{\mathbf{H}}_{yy} \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96e)

$$\mathbf{u}^* = - \mathbf{G} \mathbf{q} + \hat{\mathbf{H}} \mathbf{u} + (\hat{\mathbf{U}} + \mathbf{G} \hat{\mathbf{Q}} - \hat{\mathbf{H}} \hat{\mathbf{U}}) \alpha$$  \hspace{1cm} (8.96f)

The matrices $\hat{\mathbf{U}}_{k,l}$, $k, l = 0, x, y$ are derived from $\hat{\mathbf{U}}$ by performing the indicated differentiation. Here, $\mathbf{u}^*$ represents the vector of the values of $u$ at the internal nodes to distinguish it from the vector $\mathbf{u}$ designating the boundary values of $u$. The superscript star is dropped, if there is no chance of confusion.

Equations (8.96) by virtue of Eq. (8.88) become

$$\mathbf{u}_x = \mathbf{S}_x \alpha + \mathbf{z}_x$$  \hspace{1cm} (8.97a)

$$\mathbf{u}_y = \mathbf{S}_y \alpha + \mathbf{z}_y$$  \hspace{1cm} (8.97b)

$$\mathbf{u}_{xx} = \mathbf{S}_{xx} \alpha + \mathbf{z}_{xx}$$  \hspace{1cm} (8.97c)

$$\mathbf{u}_{xy} = \mathbf{S}_{xy} \alpha + \mathbf{z}_{xy}$$  \hspace{1cm} (8.97d)

$$\mathbf{u}_{yy} = \mathbf{S}_{yy} \alpha + \mathbf{z}_{yy}$$  \hspace{1cm} (8.97e)

$$\mathbf{u} = \mathbf{S} \alpha + \mathbf{z}$$  \hspace{1cm} (8.97f)
where
\[ S_{kl} = [\hat{H}_{,kl} - G_{,kl}]Z + (\hat{U}_{,kl} + G_{,kl} \hat{Q} - \hat{H}_{,kl} \hat{U}), \quad k, l = 0, x, y \] (8.98a)
are known matrices and
\[ z_{,kl} = [\hat{H}_{,kl} - G_{,kl}]e, \quad k, l = 0, x, y \] (8.98b)
are known vectors. Eqs. (8.97) give the vectors containing the values of field function and its derivatives at the \( L \) internal nodes.

The final step of the AEM is to collocate Eq. (8.1) at the \( L \) internal nodes. This yields
\[ A_u_{,xx} + 2Bu_{,xy} + Cu_{,yy} + Du_{,x} + Eu_{,y} + Fu = f \] (8.99)
where \( A, B, \ldots, F \) are \( L \times L \) diagonal matrices containing the values of the coefficients \( A, B, \ldots, F \) at the internal nodes and \( f \) is a vector containing the values of the actual source at the same points.

Substituting Eqs. (8.97) in Eq. (8.99) gives
\[ \bar{K}\alpha = \mathbf{p} \] (8.100)
where
\[ \bar{K} = A S_{,xx} + 2BS_{,xy} + CS_{,yy} + DS_{,x} + ES_{,y} + S \] (8.101a)
\[ \mathbf{p} = f - (A z_{,xx} + 2B z_{,xy} + C z_{,yy} + D z_{,x} + E z_{,y} + z) \] (8.101b)

Equation (8.100) constitute a set of \( L \) equations, which is solved for the vector \( \alpha \).

Finally, the solution and its derivatives at the internal nodes are obtained from Eqs. (8.97). Moreover, the solution and its derivatives at any point \( x \in \Omega \) can be evaluated using also Eq. (8.97), but in this case the involved matrices must be evaluated with respect to the point \( x \).

EVALUATION OF THE INFLUENCE MATRICES
The evaluation of the matrices \( G, H, G_{,x}, \hat{H}_{,x}, \ldots, G_{,yy}, \hat{H}_{,yy} \) depends on the type of element used for the boundary discretization. For constant elements they are evaluated as follows:

The matrices \( H, G \), in Eq. (8.77) are evaluated using the relations (cf. Eqs. (4.3) and (4.5))
\[ G_{ij} = \int_{\Gamma_j} v(x_i, y) ds_y, \quad H_{ij} = \int_{\Gamma_j} \frac{\partial v(x_i, y)}{\partial n} ds_y - \frac{1}{2} \delta_{ij}, \] (8.102)
\[ (i, j = 1, 2, \ldots, N), \quad x_i, y \in \Gamma, \]
Their numerical evaluation is presented in Section 4.2. The matrices \( G, \hat{H}, G_{x}, \hat{H}_{x}, \ldots, G_{yy}, \hat{H}_{yy} \) in Eqs. (8.96) are given by the relations (cf. Eq. (4.17))

\[
G_{ik} = \int_{\Gamma_k} v(x_i, y) dy, \quad \hat{H}_{ik} = \int_{\Gamma_k} q(x_i, y) dy \tag{8.103a}
\]

\[
G_{ik,x} = \int_{\Gamma_k} \frac{\partial v(x_i, y)}{\partial x} dy, \quad \hat{H}_{ik,x} = \int_{\Gamma_k} \frac{\partial q(x_i, y)}{\partial x} dy \tag{8.103b}
\]

\[
G_{ik,y} = \int_{\Gamma_k} \frac{\partial v(x_i, y)}{\partial y} dy, \quad \hat{H}_{ik,y} = \int_{\Gamma_k} \frac{\partial q(x_i, y)}{\partial y} dy \tag{8.103c}
\]

\[
G_{ik,xx} = \int_{\Gamma_k} \frac{\partial^2 v(x_i, y)}{\partial x^2} dy, \quad \hat{H}_{ik,xx} = \int_{\Gamma_k} \frac{\partial^2 q(x_i, y)}{\partial x^2} dy \tag{8.103d}
\]

\[
G_{ik,xy} = \int_{\Gamma_k} \frac{\partial^2 v(x_i, y)}{\partial x \partial y} dy, \quad \hat{H}_{ik,xy} = \int_{\Gamma_k} \frac{\partial^2 q(x_i, y)}{\partial x \partial y} dy \tag{8.103e}
\]

\[
G_{ik,yy} = \int_{\Gamma_k} \frac{\partial^2 v(x_i, y)}{\partial y^2} dy, \quad \hat{H}_{ik,yy} = \int_{\Gamma_k} \frac{\partial^2 q(x_i, y)}{\partial y^2} dy \tag{8.103f}
\]

\[(i = 1, 2, \ldots, L), \quad (k = 1, 2, \ldots, N), \quad x_i \in \Omega, \quad y \in \Gamma.\]

The boundary integrals in Eqs. (8.103) are regular, because \( r \neq 0 \) and thus they can be computed using Gauss integration for regular integrals. Attention should be paid to points \( x_i \) near the boundary. At those points, methods for near-singular integrals may be employed to obtain accurate results (see Section 5.6). The expressions for the derivatives of the fundamental solution involved in Eq. (8.103) can be established using the relevant relations given in Appendix A.

EVALUATION OF THE DERIVATIVES OF MATRIX \( \hat{U} \)

The matrix \( \hat{U} \) in Eqs. (8.96) contains the values of the particular solution \( \hat{u}_j = \hat{u}(r_{jx}), r_{jx} = |x - x_j| = \sqrt{(x - x_j)^2 + (y - y_j)^2}, x \in \Omega, x_j \) at the \( L \) internal collocation points. Thus, the elements of matrices \( \hat{U}_{1x}, \hat{U}_{1y}, \hat{U}_{2x}, \hat{U}_{2y}, \hat{U}_{3x}, \hat{U}_{3y} \) in Eqs. (8.96a)–(8.96e) result by differentiation of the function \( \hat{u}_j = \hat{u}(r_{jx}) \). Hence, we have

\[
\hat{u}_{j,x} = \hat{u}_j' r_{jx} \tag{8.104a}
\]

\[
\hat{u}_{j,y} = \hat{u}_j' r_{jy} \tag{8.104b}
\]

\[
\hat{u}_{j,xx} = \hat{u}_j'' r_{jx}^2 + \hat{u}_j' r_{jxx} \tag{8.104c}
\]

\[
\hat{u}_{j,xy} = \hat{u}_j'' r_{jx} r_{jy} + \hat{u}_j' r_{jxy} \tag{8.104d}
\]
\[ \hat{u}_{j,yy} = \hat{u}_{j,x}^2 r_{y}^2 + \hat{u}_{j,y} r_{yy} \]  
(8.104e)

where (see Appendix A)

\[
\begin{align*}
  r_{x} &= \frac{x - x_j}{r},
  r_{y} &= \frac{y - y_j}{r} \\
  r_{x,x} &= \frac{r_{x}^2}{r},
  r_{y,y} &= \frac{r_{y}^2}{r},
  r_{x,y} &= -\frac{r_{x} r_{y}}{r} 
\end{align*}
\]  
(8.105a, b)

For the off-diagonal elements \((i \neq j)\) of the matrices \(\hat{U}_{x}, \hat{U}_{y}, \hat{U}_{x,x}, \hat{U}_{x,y}, \hat{U}_{y,y}\), it is \(r \neq 0\). Hence, the expressions (8.104) can be used to compute the values of these elements. However, for the diagonal elements \((i = j)\) it is \(r = 0\) and their values are computed via a limiting process.

For the RBFs given in Table 8.1, it is shown that

a. \(\phi = \sqrt{r^2 + c^2}\)

\[
\lim_{r \to 0} \hat{u}_{j,x} = 0, \quad \lim_{r \to 0} \hat{u}_{j,y} = 0 
\]  
(8.106a, b)

\[
\lim_{r \to 0} \hat{u}_{j,x,x} = \frac{c}{2}, \quad \lim_{r \to 0} \hat{u}_{j,y,y} = \frac{c}{2}, \quad \lim_{r \to 0} \hat{u}_{j,x,y} = 0 
\]  
(8.106c, d, e)

b. \(\phi = 1 + r + r^2 + \ldots + r^k\)

\[
\lim_{r \to 0} \hat{u}_{j,x} = 0, \quad \lim_{r \to 0} \hat{u}_{j,y} = 0 \quad \text{for } k \geq 0 
\]  
(8.107a, b)

\[
\lim_{r \to 0} \hat{u}_{j,x,x} = \frac{1}{2}, \quad \lim_{r \to 0} \hat{u}_{j,y,y} = \frac{1}{2}, \quad \lim_{r \to 0} \hat{u}_{j,x,y} = 0 
\]  
(8.107c, d, e)

**OPTIMAL SOLUTION**

The accuracy of the solution depends on the number of internal nodes, their position, the type of the employed RBF, and the shape parameter of the RBF, if it includes such a parameter. A major drawback of the MQs is the uncertainty of the choice of the shape parameter [20]. Extended research has been carried out to obtain an optimum value of \(c\) and formulas have been proposed for its approximation [21]. The minimization of the functional \(J(u)\), Eq. (8.5), has been employed to obtain optimal values of the shape parameter and the position of the centers of the RBFs [22,23].

The use of the latter method requires the evaluation of the domain integral in Eq. (8.5). This is facilitated, if the domain integral is converted to a boundary line integral using the DRM. Thus denoting the integrand of the domain integral by

\[
R(x) = \frac{1}{2} (A u_x^2 + 2B u_x u_y + C u_y^2 - F u^2) + f u 
\]  
(8.108)
and approximating it with

\[ R(x) \simeq \sum_{j=1}^{L} \beta_j \phi_j(r) \]  

(8.109)

we obtain

\[ \int_{\Omega} R(x) d\Omega = (H \hat{U} - G \hat{Q}) \Phi^{-1} R \]  

(8.110)

where

\[ \Phi = [\phi(r_{ij})], \quad R = \{R(x_i)\}, \quad i, j = 1, 2, \ldots, L \]  

(8.111)

SOLUTION PROCEDURE STEPS

A computer code can be written for the solution of the boundary value problem (8.1), (8.2) using the AEM by adhering to the following steps:

1. Compute the \( N \times N \) matrices \( G \) and \( H \) for the boundary nodes using Eqs. (4.3) and (4.5).
2. Compute the \( N \times L \) matrices \( \hat{U}, \hat{Q} \) and \( T \) using Eqs. (8.78) and (8.80).
3. Compute the \( N \times N \) diagonal matrices \( C_1, C_2 \) and the vector \( c_3 \) using Eqs. (8.85) and (8.86).
4. Compute the \( 2N \times L \) matrix \( Z \) and the vector \( e \) using Eqs. (8.89).
5. Compute the \( L \times L \) matrices \( G, G_{xx}, G_{xy}, \ldots, G_{yy}, \hat{H}, \hat{H}_{,x}, \hat{H}_{,y}, \ldots, \hat{H}_{yy} \) for the internal nodes using Eqs. (8.103).
6. Compute the \( L \times L \) matrices \( \hat{U}, \hat{U}_{,x}, \hat{U}_{,y}, \ldots, \hat{U}_{yy} \) using Eqs. (8.80) and (8.104).
7. Compute the \( L \times L \) matrices \( S, S_{,kl} \) and the vectors \( z, z_{,kl} \) using Eqs. (8.98).
8. Compute the \( L \times L \) matrix \( K \) and the vector \( p \) using Eqs. (8.101) and solve Eq. (8.100) for the vector \( \alpha \).
10. If desired, optimize the solution by minimizing the functional (8.5).

Remark

In this section, the AEM was presented as a boundary-only method, in the sense that the discretization and integration are restricted only to the boundary as in DRM. Thus the AEM maintains the boundary character and, therefore, the advantages of the BEM. However, the AEM can be efficiently implemented as D/BEM. In this case, the domain integrals in Eqs. (8.75) and (8.91) are approximated using domain discretization. Then after elimination of the boundary quantities by virtue of the boundary conditions (8.85), the values of the field function and its derivatives at the domain nodal points are expressed in terms of the nodal values of the fictitious source \( b(x) \). Subsequent substitution in Eq. (8.99) yields the algebraic equation for the evaluation of the fictitious source. In this approach, the domain \( \Omega \) is divided into a finite number of elements over which the law of variation of \( b(x) \) is assumed known, for example, constant, linear, etc. Triangular elements with linear
variation of $b(x)$ are very convenient, because, on the one hand, they give good accuracy and, on the other hand, the discretization in triangles is readily performed using the Delaunay triangulation. There are ready to use subroutines for this purpose (see the relevant Matlab function). Though the domain discretization spoils the pure boundary character of the method, it may be preferable as it is alleviated from the drawbacks related to the RBFs. The implementation of the AEM with domain discretization is presented in detail in [19].

EXAMPLES
On the basis of the procedure presented in the Section “Solution Procedure Steps”, a FORTRAN code has been written for the solution of the boundary value problem (8.1), (8.2). The employed RBFs $\phi_j$ are multiquadrics (MQs). This computer program uses constant boundary elements and has been given the name AEMLABE. The electronic version of this program is given on this book’s companion website.

Certain examples problems are presented which demonstrate the efficiency and accuracy of the AEM.

Example 8.2
As a first example we consider a benchmark problem [24]. This problem is governed by the Poisson equation

$$\nabla^2 u = -\frac{10^6}{52} \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma$$

(8.12)

where $\Omega$ is the rectangular domain $-0.3 \leq x \leq 0.3$, $-0.2 \leq y \leq 0.2$. The exact value of $u$ at the center is $u(0,0) = 310.10$. The solution has been obtained using $N$ constant boundary elements and $L$ domain nodal points uniformly distributed on the rectangular domain. The obtained results for various values of $N$, $L$ and $c$ are shown in Table 8.6. The solution converges for $N = 160$.

<table>
<thead>
<tr>
<th>$c$ = 1</th>
<th>$c$ = 1.5</th>
<th>$c$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$ = 25</td>
<td>$L$ = 49</td>
<td>$L$ = 25</td>
</tr>
<tr>
<td>20</td>
<td>313.30</td>
<td>313.28</td>
</tr>
<tr>
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<td>310.78</td>
<td>310.79</td>
</tr>
<tr>
<td>80</td>
<td>310.24</td>
<td>310.23</td>
</tr>
<tr>
<td>160</td>
<td>310.10</td>
<td>310.10</td>
</tr>
</tbody>
</table>

TABLE 8.6 Dependence of the Solution $u(0,0)$ on $N$, $L$, and $c$ in Example 8.2
Example 8.3
In this example we obtain the solution for the following complete second
order elliptic partial differential equation
\[
(1 + y^2)u_{xx} + 2xyu_{xy} + (1 + 2x^2)u_{yy} + xu_x + yu_y + u = \frac{7x^2 - 5xy + 5y^2 + 4}{\sqrt{x^4 + y^2}} \quad \text{in } \Omega \tag{8.113}
\]
where the domain \( \Omega \) is the ellipse with semi axes \( a = 5 \) and \( b = 3 \).

Three types of boundary conditions are studied
\begin{enumerate}
  \item \( u = \alpha(x) \) on \( \Gamma \) (Dirichlet)
  \item \( \nabla u \cdot \mathbf{m} = \gamma(x) \) on \( \Gamma \) (Neumann)
  \item \( \nabla u \cdot \mathbf{m} = \gamma(x) \) on \( \Gamma_m, \ u = \alpha(x) \) on \( \Gamma_u \) (mixed)
\end{enumerate}
where \( \Gamma_m = \{ y = b\sqrt{1 - x^2/a^2}, \ 0 \leq x \leq a \}, \ \Gamma_u = \Gamma - \Gamma_m \) and
\[
\alpha(x) = x^2 - xy + y^2 \tag{8.114a}
\]
\[
\gamma(x) = \frac{(1 + y^2)xb^2 + xy^2a^2)(2x - y) + \left[ b^2x^2y + a^2(1 + 2x^2)y \right](-x + 2y)}{\sqrt{b^4x^2 + a^4y^2}} \tag{8.114b}
\]
In all three cases the problem admits an analytical solution
\[
u_{\text{exact}} = x^2 - xy + y^2 \tag{8.115}
\]
The solution has been obtained using \( N = 100 \) constant boundary
elements and \( L = 93 \) internal points, located as shown in Fig. 8.10.

![FIGURE 8.10 Elliptic domain and nodal points in Example 8.3.](image-url)
The computed nodal values of the solution and its derivatives are plotted in Fig. 8.11 as compared with the exact values. In all three cases the computed results are practically identical with the exact ones. Moreover, Figs. 8.12 and 8.13 show the convergence of the mean squared error
\[
\text{MSE} = \frac{1}{L} \sqrt{\sum_{i=1}^{L} [Y(i) - Y_{\text{exact}}(i)]^2}
\]
of the solution and its derivatives with increasing shape parameter and number of the nodes for case (i).

**FIGURE 8.11** Nodal values of the solution and its derivatives in Example 8.3.
Example 8.4
As a third example we study the temperature distribution in the plane inhomogeneous anisotropic body shown in Fig. 8.14 having a conductivity matrix

\[
D = \begin{bmatrix}
(x+y+2)^2 & (x-y)^2 \\
(x-y)^2 & (x+2y+2)^2
\end{bmatrix}
\]  \hspace{1cm} (8.116)

We look for the steady state response under the internal heat source \( f = 18(x^2 + y^2) - 16 \) and the boundary conditions shown in
The temperature distribution is governed by the boundary value problem

\[
\nabla \cdot (D \nabla u) + f = 0 \quad \text{in } \Omega
\]

\[
u = \bar{u} \quad \text{on } \Gamma_u \quad (8.117)
\]

\[
\nabla u \cdot \mathbf{m} = -\bar{q}_m \quad \text{on } \Gamma_q \quad (8.118a)
\]

where

\[
\bar{u} = x^2 + y^2 - 5xy \quad \text{and} \quad \bar{q}_m = [(2x - 5y)(2x + y + 2)^2 + (2x - 5y)(x - y)]
\]

By virtue of Eq. (8.116), Eq. (8.117) is written as

\[
(2x + y + 2)^2 u_{xx} + 2(x - y)^2 u_{xy} + (x + 2y + 2)^2 u_{yy} + (6x + 6y + 8) u_x + (6x + 6y + 8) u_y = -18(x^2 + y^2) + 16
\]

in \( \Omega \) \quad (8.119)

The problem has an analytical solution

\[
u_{\text{exact}} = x^2 + y^2 - 5xy \quad (8.120)
\]

The solution was computed using \( N = 180 \) constant boundary elements and \( L = 103 \) domain nodal points distributed uniformly (Fig. 8.15). Numerical results for the temperature \( u \) and the fluxes \( q_x = - (k_{xx} u_x + k_{xy} u_y) \), \( q_y = - (k_{yx} u_x + k_{yy} u_y) \) are given in Table 8.7 as compared with the exact ones. Table 8.8 shows the mean squared error

\[
\text{MSE} = \frac{1}{L} \sqrt{\sum_{i=1}^{L} [Y(i) - Y_{\text{exact}}(i)]^2}
\]

for the solution and its derivatives for different values of the shape parameter \( c \). Finally, Fig. 8.16 show the contours of the temperature distribution.
FIGURE 8.15 Nodal points in Example 8.4 \((N = 180, L = 103)\).

TABLE 8.7 Temperature and Fluxes Along the Line \(x = 0.25\) in Example 8.4. \((N = 180, L = 103, c = 1)\). Upper Value: Computed; Lower Value: Exact

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(u)</th>
<th>(q_x)</th>
<th>(q_y)</th>
</tr>
</thead>
<tbody>
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<td>6.119</td>
<td></td>
</tr>
<tr>
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</tr>
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<td>4.005</td>
<td></td>
</tr>
<tr>
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<td>-0.3125</td>
<td>18.014</td>
<td>2.766</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3125</td>
<td>18.015</td>
<td>2.765</td>
<td></td>
</tr>
<tr>
<td>0.583</td>
<td>-0.3264</td>
<td>22.983</td>
<td>1.242</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3264</td>
<td>22.984</td>
<td>1.241</td>
<td></td>
</tr>
<tr>
<td>0.667</td>
<td>-0.3264</td>
<td>28.396</td>
<td>-0.576</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3264</td>
<td>28.398</td>
<td>-0.578</td>
<td></td>
</tr>
<tr>
<td>0.750</td>
<td>-0.3125</td>
<td>34.264</td>
<td>-2.700</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3125</td>
<td>34.265</td>
<td>-2.703</td>
<td></td>
</tr>
</tbody>
</table>
THE BEM FOR COUPLED SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Systems of coupled second order elliptic partial differential equations arise in several fields of engineering and mathematical physics, for example, plane elasticity, large deflections of elastic membranes. These equations can be solved using the standard BEM only if the fundamental solution of the system of equations can be established, for example, for Navier equations (see Chapter 7). However, this is not possible if the equations have variable coefficients as in inhomogeneous bodies. Therefore, recourse to the DRM or AEM is inevitable. The DRM could be employed, but only for certain form of equations, while the AEM can treat the general form of the coupled second order equations, for example, the equations governing the deformation of

**TABLE 8.8** Mean Square Error MSE of the Solution and Its Derivatives in Example 8.4

<table>
<thead>
<tr>
<th>c</th>
<th>u</th>
<th>u_x</th>
<th>u_y</th>
<th>u_xx</th>
<th>u_yy</th>
<th>u_xy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.22</td>
<td>0.74</td>
<td>0.82</td>
<td>3.40</td>
<td>5.31</td>
<td>5.10</td>
</tr>
<tr>
<td>0.5</td>
<td>0.19</td>
<td>0.64</td>
<td>0.74</td>
<td>3.19</td>
<td>5.00</td>
<td>4.81</td>
</tr>
<tr>
<td>1.0</td>
<td>0.18</td>
<td>0.64</td>
<td>0.71</td>
<td>3.22</td>
<td>4.61</td>
<td>4.61</td>
</tr>
<tr>
<td>1.5</td>
<td>0.20</td>
<td>0.69</td>
<td>0.78</td>
<td>3.34</td>
<td>5.18</td>
<td>4.92</td>
</tr>
</tbody>
</table>

**FIGURE 8.16** Contours of temperature distribution in Example 8.4.

8.5 THE BEM FOR COUPLED SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS
inhomogeneous anisotropic elastic bodies. Hence, only the AEM is presented in the subsequent sections.

8.5.1 The AEM for the plane elastostatic problem

In this section the AEM is illustrated by applying it to solve the plane elasticity problem, first for homogeneous isotropic bodies (Navier’s equations) and then for inhomogeneous anisotropic bodies.

**Homogeneous isotropic elastic body**

We consider the plane stress problem for a homogeneous isotropic plane body of uniform (unit) thickness made of linear elastic material occupying the two-dimensional, in general multiply connected, domain \( \Omega \) with boundary \( \Gamma = \bigcup_{k=0}^{K} \Gamma_k \) in \( xy \)-plane (Fig. 8.17). Its response is governed by Eqs. (7.36) (Section 7.2), namely

\[
\nabla^2 u + \frac{1 + \nu}{1 - \nu} \left(u_{xx} + u_{xy}\right) + \frac{1}{G} f_x = 0 \quad \text{in } \Omega \quad (8.121a)
\]

\[
\nabla^2 v + \frac{1 + \nu}{1 - \nu} \left(v_{xy} + v_{yy}\right) + \frac{1}{G} f_y = 0 \quad \text{in } \Omega \quad (8.121b)
\]

The pertinent boundary conditions on a part of the boundary may be of the following type:

i. \( u = \overline{u}, \quad v = \overline{v} \) \quad (8.122a)

ii. \( u = \overline{u}, \quad t_y = \overline{t_y} \) \quad (8.122b)

iii. \( t_x = \overline{t_x}, \quad v = \overline{v} \) \quad (8.122c)

iv. \( t_x = \overline{t_x}, \quad t_y = \overline{t_y} \) \quad (8.122d)

**Figure 8.17** Domain \( \Omega \) and boundary \( \Gamma = \bigcup_{k=0}^{K} \Gamma_k \).
where \( u = u(x, y) \), \( v = v(x, y) \) designate the displacement components, \( f_x, f_y \) the body forces, and \( t_x, t_y \) the boundary tractions given as

\[
t_x = \lambda^*(u_x + v_y)n_x + \mu(u_x n_x + v_y n_y) + \mu u_n
\]

(8.123a)

\[
t_y = \lambda^*(u_x + v_y)n_y + \mu(u_y n_x + v_y n_y) + \mu v_n
\]

(8.123b)

and \( \mu, \lambda^* \) the effective Lamé constants

\[
\mu = G = \frac{E}{2(1 + \nu)}, \quad \lambda^* = \frac{\nu E}{1 - \nu^2}
\]

(8.124)

The overbar in Eqs. (8.122) designates a prescribed quantity. Attention should be paid to boundary condition (iv), Eq. (8.122d). In this case, the boundary tractions cannot be prescribed arbitrarily, but they must ensure overall equilibrium of the body (cf. Section 7.2.1 “Boundary Conditions”).

Since Eqs. (8.121) are of the second order, it is convenient to use the analog equations

\[
\nabla^2 u = b^{(1)}(x)
\]

(8.125a)

\[
\nabla^2 v = b^{(2)}(x)
\]

(8.125b)

where \( b^{(1)}(x), b^{(2)}(x) \) are two unknown fictitious sources.

The solution of Eqs. (8.125) is given in integral form

\[
\varepsilon u(x) = \int_{\Omega} u^* b^{(1)} d\Omega - \int_{\Gamma} (u^* u_n - u_n u^*) ds \quad x \in \Omega \cup \Gamma
\]

(8.126a)

\[
\varepsilon v(x) = \int_{\Omega} u^* b^{(2)} d\Omega - \int_{\Gamma} (u^* v_n - u_n v^*) ds \quad x \in \Omega \cup \Gamma
\]

(8.126b)

Approximating the fictitious sources with RBF series

\[
b^{(i)} \approx \sum_{j=1}^{L} a^{(i)}_j \phi_j, \quad i = 1, 2
\]

(8.127)

and applying the procedure described in Section 8.4.2 give

\[
H u - G u_n = T \alpha_1
\]

(8.128a)

\[
H v - G v_n = T \alpha_2
\]

(8.128b)

\[
u_{,kl} = -G_{,kl}u_n + \hat{H}_{,kl} u + (\hat{U}_{,kl} + G_{,kl} \hat{Q} - \hat{H}_{,kl} \hat{U}) \alpha_1, \quad k, l = 0, x, y
\]

(8.129a)

\[
v_{,kl} = -G_{,kl}v_n + \hat{H}_{,kl} v + (\hat{U}_{,kl} + G_{,kl} \hat{Q} - \hat{H}_{,kl} \hat{U}) \alpha_2, \quad k, l = 0, x, y
\]

(8.129b)

\[
b_1 = \Phi \alpha_1, \quad b_2 = \Phi \alpha_2
\]

(8.130a, b)
where $\mathbf{u}, \mathbf{u}_n, \mathbf{v}, \mathbf{v}_n$ represent the vectors of the boundary nodal values of $u, u_n, v, v_n$; $\mathbf{u}_{kl}, \mathbf{v}_{kl}$ the vectors of the interior nodal values of $u, v$ and their derivatives, $\mathbf{b}_1, \mathbf{b}_2$ the vectors of values of the fictitious sources at the interior nodes, and $\alpha_1, \alpha_2$ the vectors of the RBF series coefficients. All other matrices and vectors have been defined in Section 8.4.2.

Equations (8.128) represent a set of $2N$ equations for the boundary quantities $u, u_n, v, v_n$. Another set of $2N$ equations for these quantities is obtained from the boundary conditions (8.122), which may be combined as

$$\gamma_1 u + \gamma_2 t_x = \gamma_3$$  \hspace{1cm} (8.131a)

$$\delta_1 v + \delta_2 t_y = \delta_3$$  \hspace{1cm} (8.131b)

in which the coefficients $\gamma_1, \gamma_2$ and $\delta_1, \delta_2$ take the values 0 or 1, while $\gamma_3$ takes the values $\bar{u}$ or $\bar{t}_x$ and $\delta_3$ the values $\bar{v}$ or $\bar{t}_y$.

The tractions $t_x, t_y$ can be expressed in terms of $u_n, u_t$ and $v_n, v_t$ using Eqs. (8.123) (cf. Eq. 7.118). Thus, we may write

$$\left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\} = C \left[ \begin{array}{cc} n_x & \frac{1 - \nu}{2} n_y \\ \nu n_y & 1 - \nu n_x \end{array} \right] \left\{ \begin{array}{c} u_x \\ u_y \end{array} \right\} + C \left[ \begin{array}{cc} 1 - \nu n_y & \nu n_x \\ \frac{1 - \nu}{2} n_x & 1 - \nu n_y \end{array} \right] \left\{ \begin{array}{c} v_x \\ v_y \end{array} \right\}$$

$$= C \left[ \begin{array}{cc} n_x & \frac{1 - \nu}{2} n_y \\ \nu n_y & 1 - \nu n_x \end{array} \right] \mathbf{R}^T \left\{ \begin{array}{c} u_n \\ u_t \end{array} \right\} + C \left[ \begin{array}{cc} 1 - \nu n_y & \nu n_x \\ \frac{1 - \nu}{2} n_x & 1 - \nu n_y \end{array} \right] \mathbf{R}^T \left\{ \begin{array}{c} v_n \\ v_t \end{array} \right\}$$

(8.132)

where $C = E/(1 - \nu^2)$, and $\mathbf{R}$ the transformation matrix, that is,

$$\mathbf{R} = \left[ \begin{array}{cc} n_x & n_y \\ -n_y & n_x \end{array} \right] = \left[ \begin{array}{cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right], \quad \alpha = \angle x, \mathbf{n}$$  \hspace{1cm} (8.133)

Equation (8.132) may be written as

$$\left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\} = \left[ \begin{array}{cc} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array} \right] \left\{ \begin{array}{c} u_n \\ u_t \end{array} \right\} + \left[ \begin{array}{cc} \overline{T}_{11} & \overline{T}_{12} \\ \overline{T}_{21} & \overline{T}_{22} \end{array} \right] \left\{ \begin{array}{c} v_n \\ v_t \end{array} \right\}$$

(8.134)
where

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} = C \begin{bmatrix}
\frac{1 - \nu}{2} n_x & \frac{1 - \nu}{2} n_y \\
\nu n_y & \frac{1 - \nu}{2} n_x
\end{bmatrix} R^T
\]  
(8.135a)

\[
\begin{bmatrix}
\overline{T}_{11} & \overline{T}_{12} \\
\overline{T}_{21} & \overline{T}_{22}
\end{bmatrix} = C \begin{bmatrix}
\frac{1 - \nu}{2} n_y & \nu n_x \\
\frac{1 - \nu}{2} n_x & n_y
\end{bmatrix} R^T
\]  
(8.135b)

Applying Eqs. (8.131) at the \(N\) boundary nodal points yields

\[
\begin{bmatrix}
\gamma_1 & 0 \\
0 & \delta_1
\end{bmatrix} \{u\} + \begin{bmatrix}
\gamma_2 & 0 \\
0 & \delta_2
\end{bmatrix} \{t\} = \begin{bmatrix}
\gamma_3 \\
\delta_3
\end{bmatrix}
\]  
(8.136)

where \(\gamma_1, \gamma_2, \delta_1, \delta_2\) are \(N \times N\) diagonal matrices and \(\gamma_3, \delta_3\) are \(N \times 1\) vectors.

Equation (8.134) is now applied at the boundary nodes and introduced into Eq. (8.136). Then replacing the tangential derivatives \(u_t, v_t\) with finite differences using the matrix \(d\), Eq. (8.83), reduces the boundary conditions in the form

\[
\beta_{11} u + \beta_{12} u_n + \beta_{13} v + \beta_{14} v_n = \gamma_3
\]  
(8.137a)

\[
\beta_{21} u + \beta_{22} u_n + \beta_{23} v + \beta_{24} v_n = \delta_3
\]  
(8.137b)

where \(\beta_{ij}\) are \(N \times N\) known matrices given as

\[
\beta_{11} = \gamma_1 + \gamma_2 T_{12} d, \quad \beta_{12} = \gamma_2 T_{11}
\]  
(8.138a, b)

\[
\beta_{13} = \gamma_2 \overline{T}_{12} d, \quad \beta_{14} = \gamma_2 \overline{T}_{11}
\]  
(8.138c, d)

\[
\beta_{21} = \delta_2 T_{22} d, \quad \beta_{22} = \delta_2 T_{21}
\]  
(8.138e, f)

\[
\beta_{23} = \delta_1 + \delta_2 \overline{T}_{22} d, \quad \beta_{24} = \delta_2 \overline{T}_{21}
\]  
(8.138g, h)

Equations (8.128) and (8.137) are combined and solved for \(u, u_n, v, v_n\). This yields

\[
\begin{bmatrix}
u \\
u_n
\end{bmatrix} = Z^{(1)} \alpha_1 + e^{(1)}
\]  
(8.139a)

\[
\begin{bmatrix}
v \\
v_n
\end{bmatrix} = Z^{(2)} \alpha_2 + e^{(2)}
\]  
(8.139b)
where $Z^{(1)}$, $Z^{(2)}$ and $e^{(1)}$, $e^{(2)}$ are the upper and lower half of the matrices $Z$ and $e$, respectively, that is,

\[
Z = \begin{bmatrix} Z^{(1)} \\ Z^{(2)} \end{bmatrix} = \begin{bmatrix} H & -G & 0 & 0 \\ 0 & 0 & H & -G \\ \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix}^{-1} \begin{bmatrix} T & 0 \\ 0 & T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]  
(8.140a)

\[
e = \begin{bmatrix} e^{(1)} \\ e^{(2)} \end{bmatrix} = \begin{bmatrix} H & -G & 0 & 0 \\ 0 & 0 & H & -G \\ \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_1 \end{bmatrix}
\]  
(8.140b)

Equations (8.129) by virtue of Eqs. (8.139) and (8.140) become

\[
u_{,kl} = S^{(1)}_{,kl} \alpha_1 + z^{(1)}_{,kl} 
\]  
(8.141a)

\[
v_{,kl} = S^{(2)}_{,kl} \alpha_2 + z^{(2)}_{,kl} 
\]  
(8.141b)

where

\[
S^{(i)}_{,kl} = [\tilde{H}_{,kl} - G_{,kl}]Z^{(i)} + (\tilde{U}_{,kl} + G_{,kl} \tilde{Q} - \hat{H}_{,kl} \hat{U})
\]  
(8.142a)

\[
z^{(i)}_{,kl} = [\tilde{H}_{,kl} - G_{,kl}]e^{(i)}
\]  
(8.142b)

are known matrices and vectors, respectively ($i = 1, 2$, $k, l = 0, x, y$).

Next collocating Eqs. (8.121) at the $L$ internal nodes gives

\[
b_1 + \frac{1 + \nu}{1 - \nu} (u_{,xx} + v_{,xy}) + \frac{1}{G} f_x = 0 
\]  
(8.143a)

\[
b_2 + \frac{1 + \nu}{1 - \nu} (u_{,xy} + v_{,yy}) + \frac{1}{G} f_y = 0 
\]  
(8.143b)

where $b_1, b_2, u_{,xx}, u_{,xy}, v_{,xy}, v_{,yy}, f_x, f_y$ are the vectors containing the values of the respective quantities at the $L$ internal nodes.

Using Eqs. (8.130) and (8.141) to substitute the fictitious sources and the derivatives in Eq. (8.143) gives

\[
A_{11} \alpha_1 + A_{12} \alpha_2 = p_1 
\]  
(8.144a)

\[
A_{21} \alpha_1 + A_{22} \alpha_2 = p_2 
\]  
(8.144b)
where

\[
A_{11} = \frac{1 - \nu}{1 + \nu} \Phi + S_{xx}^{(1)} \quad (8.145a)
\]

\[
A_{12} = S_{xy}^{(2)} \quad (8.145b)
\]

\[
A_{21} = S_{xy}^{(1)} \quad (8.145c)
\]

\[
A_{2} = \frac{1 - \nu}{1 + \nu} \Phi + S_{yy}^{(2)} \quad (8.145d)
\]

\[
p_1 = -\frac{1}{G} \frac{1 - \nu}{1 + \nu} f_x - z_{xx}^{(1)} - z_{xy}^{(2)} \quad (8.145e)
\]

\[
p_2 = -\frac{1}{G} \frac{1 - \nu}{1 + \nu} f_y - z_{xy}^{(1)} - z_{yy}^{(2)} \quad (8.145f)
\]

Equations (8.144a,b) constitute a system of \(2L\) equations, which can be solved to obtain \(2L\) coefficients \(\alpha_1, \alpha_2\). The displacements and their derivatives at the internal nodes are obtained using Eqs. (8.141). Moreover, the solution and its derivatives at any point \(x \in \Omega\) can be evaluated using also Eqs. (8.141), but in this case the involved matrices must be evaluated with respect to the point \(x\).

**EXAMPLES**

On the basis of the procedure presented in the Section “Homogenous Isotropic Elastic Body”, a FORTRAN code has been written for the solution of the boundary value problem (8.121)—(8.122). The employed RBFs \(\phi_j\) are MQs. Certain example problems are presented which demonstrate the efficiency and accuracy of the AEM.

**Example 8.5**

In this example the deformation and the state of stress of the pipe of Example 7.3 is determined using the AEM with RBFs of MQ type as presented in the Section “Homogenous Isotropic Elastic Body”. The obtained numerical results are shown in Figs. 8.18 through 8.23 as compared with the BEM solution (see Chapter 7), the AEM with domain discretization for the approximation of the fictitious sources (see [19], Chapter 3) and the FEM. The BEM results have been obtained with \(N = 660\) constant boundary elements, the AEM-MQs results with \(N = 660\) constant boundary elements and \(L = 488\) domain nodal points, the AEM-Domain results with \(N = 660\) constant boundary elements and \(L = 488\) domain nodal points resulting from 786 linear triangular elements, while the FEM results with 8188 quadrilateral elements using the NASTRAN code.
**FIGURE 8.18** Distribution of traction $t_y$ along the boundary $y = 0$ in Example 8.5.

**FIGURE 8.19** Distribution of the displacement $v$ along the upper boundary $y = 1.5$ in Example 8.5.

**FIGURE 8.20** Distribution of the displacement $v$ along the line $y = 1.0$ in Example 8.5.
**Figure 8.21** Distribution of the displacement $u$ along the boundary $y = 1.5$ in Example 8.5.

**Figure 8.22** Distribution of the stress $\sigma_x$ along the boundary $y = 1.5$ in Example 8.5.

**Figure 8.23** Distribution of the stress $\sigma_x$ along the line $x = 1.25$ in Example 8.5.
INHOMOGENOUS ANISOTROPIC ELASTIC BODY

We consider an inhomogeneous anisotropic plane body of uniform (unit) thickness under plane stress \( (\sigma_z = \tau_{xz} = \tau_{yz} = 0) \) made of linear elastic material occupying the two-dimensional, in general multiply connected, domain \( \Omega \) with boundary \( \Gamma = \bigcup_{k=0}^{K} \Gamma_k \) in \( xy \)-plane (cf. Fig. 8.17). The governing equations result by taking the equilibrium of an element of the plane body.

Thus we have

\[
\hat{\nabla}^T \mathbf{\sigma} + \mathbf{f} = 0 \tag{8.146}
\]

where

\[
\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \tag{8.147a, b, c}
\]

are the displacement, the stress and the body force vector, respectively, and \( \hat{\nabla} \) is the differential operator defined as

\[
\hat{\nabla} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \tag{8.148}
\]

Moreover, the constitutive relations and the kinematic relations (cf. Eqs. (7.32) and (7.33)) are written as

\[
\mathbf{\sigma} = \mathbf{C} \mathbf{\varepsilon} \tag{8.149}
\]

\[
\mathbf{\varepsilon} = \hat{\nabla} \mathbf{u} \tag{8.150}
\]

where \( \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T \) represents the strain vector and

\[
\mathbf{C} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix} \tag{8.151}
\]

the elastic constitutive matrix, which is position dependent, \( C_{ij} = C_{ij}(x, y) \), symmetric \( C_{ij} = C_{ji} \) and invertible, \( \det \mathbf{C} \neq 0 \).

Equation (8.146) by virtue of Eqs. (8.149)–(8.151) is written as

\[
\hat{\nabla}^T \mathbf{C} \hat{\nabla} \mathbf{u} + \mathbf{f} = 0 \tag{8.152}
\]
which in terms of the displacements components become

\[
L_{11}(u) + L_{12}(v) + f_x = 0 
\tag{8.153a}
\]

\[
L_{21}(u) + L_{22}(v) + f_y = 0 
\tag{8.153b}
\]

where

\[
L_{11} = C_{11} \frac{\partial^2}{\partial x^2} + 2C_{13} \frac{\partial^2}{\partial x \partial y} + C_{33} \frac{\partial^2}{\partial y^2} 
+ (C_{11,x} + C_{13,y}) \frac{\partial}{\partial x} + (C_{13,x} + C_{33,y}) \frac{\partial}{\partial y} 
\tag{8.154a}
\]

\[
L_{12} = C_{13} \frac{\partial^2}{\partial x^2} + (C_{12} + C_{33}) \frac{\partial^2}{\partial x \partial y} + C_{23} \frac{\partial^2}{\partial y^2} 
+ (C_{13,x} + C_{33,y}) \frac{\partial}{\partial x} + (C_{12,x} + C_{23,y}) \frac{\partial}{\partial y} 
\tag{8.154b}
\]

\[
L_{21} = C_{13} \frac{\partial^2}{\partial x^2} + (C_{12} + C_{33}) \frac{\partial^2}{\partial x \partial y} + C_{23} \frac{\partial^2}{\partial y^2} 
+ (C_{13,x} + C_{12,y}) \frac{\partial}{\partial x} + (C_{33,x} + C_{23,y}) \frac{\partial}{\partial y} 
\tag{8.154c}
\]

\[
L_{22} = C_{33} \frac{\partial^2}{\partial x^2} + 2C_{23} \frac{\partial^2}{\partial x \partial y} + C_{22} \frac{\partial^2}{\partial y^2} 
+ (C_{33,x} + C_{23,y}) \frac{\partial}{\partial x} + (C_{23,x} + C_{22,y}) \frac{\partial}{\partial y} 
\tag{8.154d}
\]

The boundary conditions are given by Eqs. (8.122).

The boundary tractions are expressed in terms of the stress components

\[
t_x = \sigma_x n_x + \tau_{xy} n_y = \bar{t}_x 
\tag{8.155a}
\]

\[
t_y = \tau_{xy} n_x + \sigma_y n_y = \bar{t}_y 
\tag{8.155b}
\]

which by virtue of Eqs. (8.149) and (8.150) are written in terms of the displacements

\[
(C_{11,n_x} + C_{13,n_y}) u_x + (C_{13,n_x} + C_{33,n_y}) u_y 
+ (C_{13,n_x} + C_{33,n_y}) v_x + (C_{12,n_x} + C_{23,n_y}) v_y = \bar{t}_x 
\tag{8.156a}
\]

\[
(C_{13,n_x} + C_{12,n_y}) u_x + (C_{33,n_x} + C_{23,n_y}) u_y 
+ (C_{33,n_x} + C_{23,n_y}) v_x + (C_{23,n_x} + C_{22,n_y}) v_y = \bar{t}_y 
\tag{8.156b}
\]

where \( n(n_x, n_y) \) is the unit vector normal to the boundary.
If the body is orthotropic, the constitutive matrix for plane stress may be written in terms of the elastic moduli and Poisson’s ratios as

\[
C = \begin{bmatrix}
E_1 & \nu_{21}E_1 & 0 \\
\frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} & \frac{E_2}{1 - \nu_{12}\nu_{21}} & 0 \\
0 & 0 & G_{12}
\end{bmatrix}
\]

subject to the constraint \(\nu_{21}E_1 = \nu_{12}E_2\).

For an isotropic body it is \(E_1 = E_2 = E\) and \(\nu_{21} = \nu_{12} = \nu\) and the constitutive matrix is simplified as

\[
C = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1}{2}(1 - \nu)
\end{bmatrix}
\]

The AEM solution adheres to the same steps as in the Section “Homogenous Isotropic Elastic Body”. Therefore Eqs. (8.126)–(8.130) apply to this case, too.

The boundary tractions are expressed in terms of the normal and tangential derivatives as

\[
\begin{align*}
\{t_x, t_y\} &= \begin{bmatrix}
C_{11}n_x + C_{13}n_y \\
C_{13}n_x + C_{12}n_y \\
C_{33}n_x + C_{23}n_y
\end{bmatrix}R^T \begin{bmatrix}
u_n \\
u_t \\
v_n
\end{bmatrix} \\
&+ \begin{bmatrix}
C_{13}n_x + C_{33}n_y \\
C_{12}n_x + C_{23}n_y \\
C_{23}n_x + C_{22}n_y
\end{bmatrix}R^T \begin{bmatrix}
u_n \\
u_t \\
v_n
\end{bmatrix}
\end{align*}
\]

Equation (8.159) is now applied to the boundary nodes and subsequently introduced into Eq. (8.136). Then replacing the tangential derivatives \(u_t, v_t\) with finite differences using the matrix \(d\), Eq. (8.83), gives the boundary conditions in the form

\[
\begin{align*}
\bar{\beta}_{11}u + \bar{\beta}_{12}u_n + \bar{\beta}_{13}v + \bar{\beta}_{14}v_n &= \gamma_3 \\
\bar{\beta}_{21}u + \bar{\beta}_{22}u_n + \bar{\beta}_{23}v + \bar{\beta}_{24}v_n &= \delta_3
\end{align*}
\]

where \(\bar{\beta}_{ij}\) are \(N \times N\) known matrices. They are derived as the matrices \(\beta_{ij}\) in Eqs. (8.138). Their expressions are not given here as it is more convenient to derive them in the computer program for the numerical implementation of the solution procedure. Therefore, Eqs. (8.139)–(8.142) are valid, too, provided that the matrices \(\beta_{ij}\) in Eq. (8.140) are replaced with \(\bar{\beta}_{ij}\).
The last step of the solution procedure is to apply Eqs. (8.153) to the $L$ internal nodes. Thus we obtain

$$L_{11}(u) + L_{12}(v) + f_x = 0 \quad (8.161a)$$
$$L_{21}(u) + L_{22}(v) + f_y = 0 \quad (8.161b)$$

Then using Eqs. (8.141) to substitute the derivatives indicated by the operators $L_{ij}$ produces the system of equations

$$A_{11}\alpha_1 + A_{12}\alpha_2 = p_1 \quad (8.162a)$$
$$A_{21}\alpha_1 + A_{22}\alpha_2 = p_2 \quad (8.162b)$$

where

$$A_{11} = C_{11}S_{xx}^{(1)} + 2C_{13}S_{xy}^{(1)} + C_{33}S_{yy}^{(1)} + (C_{11,x} + C_{13,y})S_{xy}^{(1)} + (C_{13,x} + C_{33,y})S_{yy}^{(1)} \quad (8.163a)$$
$$A_{12} = C_{13}S_{xx}^{(2)} + (C_{12} + C_{33})S_{xy}^{(2)} + C_{23}S_{yy}^{(2)} + (C_{13,x} + C_{33,y})S_{xy}^{(2)} + (C_{12,x} + C_{23,y})S_{yy}^{(2)} \quad (8.163b)$$
$$A_{21} = C_{13}S_{xx}^{(1)} + (C_{12} + C_{33})S_{xy}^{(1)} + C_{23}S_{yy}^{(1)} + (C_{13,x} + C_{12,y})S_{xy}^{(1)} + (C_{33,x} + C_{23,y})S_{yy}^{(1)} \quad (8.163c)$$
$$A_{22} = C_{33}S_{xx}^{(2)} + 2C_{23}S_{xy}^{(2)} + C_{22}S_{yy}^{(2)} + (C_{33,x} + C_{23,y})S_{xy}^{(2)} + (C_{23,x} + C_{22,y})S_{yy}^{(2)} \quad (8.163d)$$

$$p_1 = -f_x - \left[ C_{11}z_{xx}^{(1)} + 2C_{13}z_{xy}^{(1)} + C_{33}z_{yy}^{(1)} + (C_{11,x} + C_{13,y})z_{xy}^{(1)} \right]$$
$$- \left[ C_{13}z_{xx}^{(2)} + (C_{12} + C_{33})z_{xy}^{(2)} + C_{23}z_{yy}^{(2)} + (C_{13,x} + C_{33,y})z_{xy}^{(2)} \right] \quad (8.163e)$$

$$p_2 = -f_y - \left[ C_{13}z_{xx}^{(1)} + (C_{12} + C_{33})z_{xy}^{(1)} + C_{23}z_{yy}^{(1)} + (C_{13,x} + C_{12,y})z_{xy}^{(1)} \right]$$
$$- \left[ C_{33}z_{xx}^{(2)} + 2C_{23}z_{xy}^{(2)} + C_{22}z_{yy}^{(2)} + (C_{33,x} + C_{23,y})z_{xy}^{(2)} \right] \quad (8.163f)$$
in which $C_{11}, C_{12}, \ldots C_{11,x}, \ldots$ are $L \times L$ diagonal matrices containing the values of the respective quantities at the internal nodes. The $S^{(i)}_{kl}$ and the vectors $z^{(i)}_{kl}$ ($i = 1, 2, k, l = 0, x, y$) are given by Eqs. (8.142). For the homogeneous isotropic body the stiffness matrix is given by Eqs. (8.158), and it is readily shown that Eqs. (8.163) reduce to (8.145).

Equations (8.162) are solved to give the vectors $\alpha_1, \alpha_2$. Then the displacements and their derivatives are obtained from Eqs. (8.141).

EXAMPLES
On the basis of the procedure presented in the Section “Inhomogenous Anisotropic Elastic Body” a FORTRAN code has been written for the solution of Eqs. (8.153) under the boundary conditions (8.131). The employed RBFs $\phi_j$ are MQs. This computer program uses constant boundary elements and has been given the name AEMELB. The electronic version of the program is given on this book’s companion website. Certain examples problems are presented which demonstrate the efficiency and accuracy of the AEM for the solution of inhomogeneous anisotropic elasticity problems.

Example 8.6
In this example the deformation of the inhomogeneous orthotropic rectangular plane body of Fig. 8.24 with unit thickness ($h = 1$) is studied. The body is subjected to a uniform normal traction along the two opposite sides. The other two edges are free to move tangentially, while they are restraint in the normal direction.

The material parameters are given as

$$C_{11} = \frac{8(10 - 9\xi^2)}{8(10 - 9\xi^2)(1 + 9\xi^2) - 5} E$$

$$C_{22} = \frac{0.8(10 - 9\xi^2)^2(1 + 9\xi^2)}{8(10 - 9\xi^2)(1 + 9\xi^2) - 5} E$$

$$C_{12} = 0.25 C_{11}, \quad C_{33} = \frac{2}{5(10 + 9\xi^2)} E, \quad C_{13} = C_{23} = 0$$

**FIGURE 8.24** Plane inhomogeneous orthotropic body in Example 8.6.
where $\xi = 2x/a$. The problem admits the exact solution [25]

$$u_{\text{exact}}(x, y) = N_x \frac{a}{2E} \left( 3\xi^3 + \xi - \frac{5}{48\sqrt{10}} \ln \frac{\sqrt{10} + 3\xi}{\sqrt{10} - 3\xi} \right)$$

(8.165)

Numerical results were obtained for $a = 2$ m, $b = 1$ m, $N_x = h t_x$, and $E = 100$ MPa using $N = 298$ boundary elements, $L = 200$ domain nodal points and $c = 0.1$. Fig. 8.25 shows the computed displacement $u(x, 0)$ versus the exact one.

**Example 8.7**

In this example the inhomogeneous orthotropic square plane body of Fig. 8.26 is studied. The body is fixed along the side $x = 0$ and subjected to the uniform normal traction along the side $x = a$. The other two edges are free to move tangentially, while they are restraint in the normal direction.

**FIGURE 8.25** Displacement $u(x, 0)$ in the plane body in Example 8.6.

**FIGURE 8.26** Plane inhomogeneous orthotropic square body in Example 8.7.
The material parameters are given as

\[ C_{11} = 6.14(1 + x)^2, \quad C_{12} = 2.14(1 + x)^2, \quad C_{13} = 0 \]
\[ C_{22} = 5.96(1 + x)^2, \quad C_{33} = 1.64(1 + x)^2 \]
\[ C_{13} = C_3 = 0 \]

The problem admits the exact solution \cite{26}

\[ u_{\text{exact}} = \frac{x}{6.14(1 + x)} \quad (8.167) \]

Numerical results were obtained for \( a = 1, \ P = 1 \) using \( N = 200 \) boundary elements, \( L = 81 \) domain nodal points and \( c = 0.1 \). The computed results are shown in Figs. 8.27 and 8.28 as compared with the exact ones.
8.6 REFERENCES


PROBLEMS

8.1. Use the DRM and AEM to solve the boundary value problem
\[ \nabla^2 u + u_{xx} + u_{xy} - 4u = -2[(1 + a^2 + x - x^2)(b^2 - y^2) + (1 + b^2 + y - y^2)(a^2 - x^2)] \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \Gamma \]

The domain \( \Omega \) is the rectangle \(-a \leq x \leq a, -b \leq y \leq b\). Compare the results with the exact solution \( u_{\text{exact}} = (a^2 - x^2)(b^2 - y^2) \).

8.2. Solve the boundary value problem below for the domain of Fig. P8.2

\[(1 + e^{2y})u_{xx} - 2xe^y u_{xy} + (4x^2 + 1)u_{yy} - xe^y u_x - e^y u_y - (x^2 + e^y)u = 2 + e^y - x^4 \quad \text{in } \Omega \]
\[(1 + e^{2y})u_n - xe^y u_t = -xe^{2y} - 2x \quad \text{on } AA' \quad \text{(a)} \]
\[(1 + e^{2y})u_n - xe^y u_t = xe^{2y} + 2x \quad \text{on } DD' \quad \text{(b)} \]
\[u = x^2 + e^y \quad \text{on } ABCD \text{ and } A'B'C'D' \quad \text{(c)} \]

The problem admits an exact solution \( u = x^2 + e^y \).
8.3. A cantilever of length \( L = 3.0 \text{ m} \), height \( b = 1.0 \text{ m} \), and thickness \( h = 0.1 \text{ m} \) (Fig. P8.3) is subjected to the uniform load \( P = 1 \text{ kN/m} \). Determine the displacements \( u \) and \( v \), and the stresses \( \sigma_x \), \( \sigma_y \), and \( \tau_{xy} \) along the cross-section \( x = L/2 \). Use \( E = 2 \times 10^4 \text{ kN/m}^2 \), \( \nu = 0.2 \). Compare the results with those obtained from an analytical solution based on the technical theory of beams including shear deformation.

\[ P = -1.0 \text{ kN/m} \]

**FIGURE P8.3**

8.4. Determine the displacement field of the plane inhomogeneous body with unit thickness shown in Fig. P8.4 governed by the system of equations

\[
\begin{align*}
\nabla^2 u + \frac{1 + \nu}{1 - \nu} (u_{xx} + v_{xy}) + \left( \frac{\lambda^*}{\mu} + \frac{2\mu_x}{\mu} \right) u_{x} + \frac{\mu_x}{\mu} u_{y} + \frac{\mu_y}{\mu} v_{x} + \frac{\lambda^*}{\mu} v_{xy} &= -\frac{b_x}{\mu} \\
\nabla^2 v + \frac{1 + \nu}{1 - \nu} (u_{xy} + v_{yy}) + \left( \frac{\lambda^*}{\mu} \right) u_{y} + \frac{\mu_x}{\mu} u_{y} + \frac{\mu_y}{\mu} v_{x} + \left( \frac{\lambda^*}{\mu} + \frac{2\mu_y}{\mu} \right) v_{xy} &= -\frac{b_y}{\mu}
\end{align*}
\]

Consider mixed boundary conditions

\[ u = xy + 0.2y^2, \quad v = xy + 0.2x^2, \quad \text{on DEAB} \]

\[ t_x = E(\nu x + y)/(1 - \nu^2), \quad t_y = 0.7E(x + y)/(1 + \nu), \quad \text{on BC} \]

\[ t_x = 0.7E(x + y)/(1 + \nu), \quad t_y = E(x + \nu y)/(1 - \nu^2), \quad \text{on CD} \]

Use the following data:

\[ E = E_0(1 + x)^2, \quad E_0 = 2 \times 10^5 \text{ kN/m}^2, \quad \nu = 0.2, \]

\[ \lambda^* = \nu E_0(1 + x)^2/(1 - \nu^2), \quad \mu = E_0(1 + x)^2/2(1 + \nu). \]

\[ b_x = E_0(1 + x)[ -0.1(1 + x)(7 + 3\nu) - 2(\nu x + y)]/(1 - \nu^2) \]
\[ b_y = E_0(1 + x)[ - 0.1(1 + x)(7 + 3\nu) - 1.4(x + y)(1 - \nu)]/(1 - \nu^2) \]

**FIGURE P8.4** Inhomogeneous plane body.