PARTICLE DEPOSITION IN FIBROUS MEDIA WITH DENDRITE-LIKE PATTERN: A PRELIMINARY MODEL

A. C. PAYATAKES
Department of Chemical Engineering, University of Houston, Houston, TX 77004, U.S.A.

and

CHI TIEN
Department of Chemical Engineering and Materials Science, Syracuse University, Syracuse, NY 13210, U.S.A.

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Abstract—A preliminary model of the formation of chain-like particle agglomerates on fibers during filtration of aerosols in fibrous media is proposed. Such dendrite-like growth has been observed experimentally and occurred beyond the initial period of filtration. Unlike most of the previous studies which are confined to clean filters, the present work is intended for the description of filtration performance (both filtration efficiency and pressure drop) over the entire loading period.

NOTATION

\( A \) cross-sectional area of filter
\( a_f \) fiber radius
\( a_p \) particle radius
\( b \) outer radius of the Kuwabara unit cell
\( d \) distance between successive layers of dendrite particles
\( F_f \) drag force per unit length acting on clean fiber
\( F_{pk} \) drag force acting on a particle in the \( k \)th layer of the dendrite
\( f_k^p, f_k^n \) drag force correction factors for tangential and normal flow, respectively
\( f \) index
\( K \) Kuwabara's constant defined by equation (12)
\( k \) index
\( L \) length of fiber contained in a filter element of thickness \( \delta x \)
\( m_k \) expected particle number of the \( k \)th layer of the dendrite
\( m_0 \) rate of increase of \( m_k \), equation (38)
\( N(t,\theta) \) frequency function of number of dendrites per unit length of fiber at time \( t \) and angle \( \theta \)
\( N(t) \) number of dendrites per unit length of fiber at time \( t \)
\( n \) number of particles per unit volume of gas-solid suspension
\( p \) pressure
\( r \) radial coordinate
\( t \) time measured from the instant of deposition of the first particle of the dendrite
\( U \) approach velocity to fiber; also mean interstitial velocity through the filter
\( U_s \) superficial velocity
\( v_r, v_\theta \) radial and angular fluid velocity components
\( X_k \) constants, defined by equation (24)
\( x \) distance of a point measured from the face of the filter

Greek letters
\( \alpha \) rate of particles approaching a clean fiber per unit length
\( \alpha_0 \) rate constant for particle deposition on a particle in the \( k \)th layer of a dendrite
\( \beta \) function given by equation (30)
\( \delta \): packing density
\( \delta P \) pressure drop and initial pressure drop across a fibrous filter with differential thickness \( \delta x \)
\( \delta P_0 \) initial porosity of fibrous medium
\( \eta_0 \) single clean fiber efficiency
\( \theta \) angular coordinate measured from the front stagnation point
\( \theta^* \) effective value of \( \theta \), equation (25)
\( \theta_0 \) angular position of dendrite
\( \lambda \) filter coefficient, defined by equation (31)
\( \lambda_{co} \) value of \( \lambda \) for a clean filter
\( \rho \) maximum number of particles in the \( (k+1) \)st layer which can be attached to the same particle of the \( k \)th layer
\( \sigma \) volume of deposited matter per unit volume of filter
\( \tau \) time measured from the beginning of aerosol flow through the filter
\( \chi(t; \tau, \theta) \) frequency function of dendrite age at time \( \tau \) and angular position \( \theta \)

**INTRODUCTION**

Filtration in fibrous media is one of the more effective means for the collection of particulate matters from solid-gas suspensions, especially for particles of submicron size. This process should not be confused with filtration on fabric filters (baghouse filtration), since in the former case the particles deposit mainly on sites in the interior of the fibrous mat, whereas in the latter case the particles form an external cake on the face of the fabric. The main advantages of filtration in fibrous media are high filtration efficiency and flow rates which are substantially higher than those obtained in baghouse filtration with the same pressure drop. Despite these attractive features this process has not yet been applied on an industrial scale. The main reason of its lack of wide industrial application has been the failure in developing effective methods for repetitive cleaning of such filters over long periods of time. Cleaning methods attempted in the past either require excessive energy consumption (for example, cleaning by blowback) or result in structural damage and early disintegration of the filter elements (for example, cleaning by reverse shock wave). For this reason its application has been largely confined to cases which require reliable, highly efficient removal but allow the disposal of the filter element after its use, such as respirator masks, emergency filtration systems for radioactive particle leaks, etc.

In recent years, the need for effective methods for removal of fine particles from industrial effluents has become more imperative. Particulate pollutants of submicron size have been identified as being most hazardous from a public health point of view. In addition, new technological developments such as coal conversion processes (both gasification and liquefaction) need efficient and effective removal of fine particles from both their product and process streams to meet the ecological, as well as engineering requirements. It is not surprising that investigation in the area of fluid-particle separations, especially those methods pertaining to fine particles, have been identified as being critical to the development of future fossil fuel energy sources (Katz et al., 1974).

In view of its growing need and importance, filtration in fibrous media has attracted renewed interest in industry. Proprietary work in material preparation for filter elements and development of ingenious methods of cleaning are likely to bear fruit in the near future and thus make possible the use of fibrous filters on industrial scale applications. In order to realize the full potential of these developments, a quantitative understanding of the fibrous filter performance in terms of particle deposition over the entire loading period is obviously needed.

Investigations (both theoretical and experimental) on aerosol filtration have been extensive, especially in recent years and they are too numerous to cite individually. Generally speaking, most of the theoretical studies were made with the use of the single collector (fiber) concept and were confined to the initial filtration period, i.e. for relatively clean filter beds; thus they do not deal with the much more important period when deposition is at an advanced stage. Experimental observations indicate (Leers, 1957; Billings, 1966) that over relatively long periods of time aerosol particles deposit preferentially on already deposited particles. Accordingly, over long loading periods, filtration results in the formation and growth of chain-like agglomerates (dendrites) on the fibers, which profoundly affect the filtration efficiency and pressure drop. Furthermore, effective filter cleaning requires the breaking-off of these dendrites from fibers and in some cases the breaking-off of the dendrites themselves.
The purpose of this work is to formulate a dendrite growth model of aerosol filtration in fibrous media with the hope that it can be used as a basis for the study of filtration in fibrous media over long loading periods. Particular emphasis is placed on its application to the estimation of particle collection efficiency, and pressure drop. As a preliminary model, the formulation has to be given on a simplified basis. The results obtained are not sufficient to draw definite conclusions, but they do justify guarded optimism about its basic approach. Through further refinement and modification, it is hoped that the model can be developed to the point where it can be fully exploited on a practical engineering level and used as a basis of rational design.

PREVIOUS WORK

The phenomenon of particle dendrite growth has been observed experimentally by a number of investigators (Watson, 1946; Leers, 1957; Billings, 1966) although none of these investigators were particularly concerned with the quantitative history of the individual dendrite growth. Radushkevich (1964) made the first attempt to model the growth of particle clusters on collectors. In his analysis, it is assumed that a given dendrite can be characterized by the number of particles it contains. No distinction is made among particles as to their respective positions in a given dendrite. Consequently, no predictions can be made about the dendrite configuration, a factor of primary importance for the determination of the effect of the dendrite on both filtration efficiency and resistance to flow. In addition, the fact that a young dendrite has generally a slender structure extending away from the collector surface into the bulk flow and the fact that the probability of new particle additions depends on the site of deposition along the dendrite suggest that the configuration of the dendrite should be an important factor in determining its rate of growth. The dendrite configuration and its relation to the dendrite growth rate, filtration efficiency and resistance to flow are taken into account in the present study, thus leading to a substantially more realistic model of the deposition process.

THEORY

The model presented here is based on the limited experimental observations available to date (Watson, 1946; Leers, 1957; Billings, 1966). By necessity, a large number of assumptions are introduced. Nevertheless, the formulation is of sufficiently general nature. The objective here is to present a basically sound framework capable of further refinement and improvement which will be introduced as more accurate experimental data become available.

Mathematical description of a single dendrite

As a preliminary model, we restrict our attention to the special case when all particles of the dendrite are of the same size. In other words, the particles in the solid–gas suspension are uniform.

Assume that at time \( t = 0 \) the first particle of a dendrite under formation is deposited on a given site of the collector. For convenience, a cylindrical collector with radius \( a_f \) will be considered (Fig. 1), although the same derivation can be easily extended to collectors of other configurations. Let \( \theta_0 \) be the value of the angular coordinate at the site of deposition, measured from the forward stagnation point.

In order to express the growth process and to describe dendrite configuration properly, the following convention is adopted. The space adjacent to the collector surface is divided in layers of thickness \( d \) by planes which are all parallel to a plane tangential to the collector surface at \( \theta = \theta_0 \), (Fig. 2) numbering them in ascending order, i.e. the first layer is immediately adjacent to the collector surface. The dendrite configuration is idealized with the convention that if a particle of the dendritic structure has at least half of its volume in the \( k \)th layer, it is assumed to lie entirely in the \( k \)th layer, (Fig. 2). Furthermore, it is assumed that particles belonging to the same layer are equivalent to each other in every respect. This is a reasonable assumption in view of the fact
that particle dendrites tend to be slender, containing only a few particles in each layer. Thus the idealized dendrite is described mathematically by a set of numbers \([m_k(t; \theta_0); k = 1, 2, 3, \ldots]\) where \(m_k\) is the number of particles of the \(k\)th layer. Clearly, \(m_k\) is a function of both time, \(t\), and site of deposition, \(\theta_0\). This approximation of the dendrite structure is fairly reasonable, since if it is set
\[
d = 2a_p,\]
where \(a_p\) is the particle radius, then in the process of idealizing the dendrite no particle in the chain is displaced in the radial direction by more than \(d/2 = a_p\). Accordingly, the problem of the dendrite growth is reduced to that of obtaining \([m_k(t; \theta_0); k = 1, 2, 3, \ldots]\). Rigorously, \([m_k; k = 1, 2, 3, \ldots]\) are a set of integer valued random processes and it can be argued that they should be studied as such. This approach, although unquestionably elegant and proper, would be exceedingly difficult. Instead, the problem will be studied in a less rigorous manner which, however, retains the important characteristics of the growth process. The problem is to obtain analytical expressions for \([m_k(t; \theta_0); k = 1, 2, 3, \ldots]\) under a given set of operating conditions.

**Dendrite rigidity**

It will be assumed that the particle dendrites are sufficiently rigid to retain their shape and orientation despite the fact that the distributed drag force acting on each of them gives rise to a moment, which would otherwise deform the dendrite and force it to lie almost parallel to the direction of flow. The origin of this rigidity lies in the fact that particles are never entirely smooth. As a result particle-particle contacts are never single-point contacts, but rather \(j\)-point contacts where \(j \geq 3\). The adhesion forces at these \(j\)-point contacts "lock" the two particles in their relative positions. This phenomenon accounts for the rigidity of short chains of small particles. In addition to this factor, if the particles forming the dendrites carry substantial electrical charges, mutual repulsion between neighboring dendrites could contribute significantly in keeping the...
dendrites nearly normal to the collector surface. However, in the considerations of the present work it is assumed that electrostatic effects are negligible.

**Rate of dendrite growth**

It will be assumed that \( m_k(t; \theta_0) \) is a real (not necessarily integer) number representing the expected rather than actual number of particles in the \( k \)th layer of a dendrite growing from the position \( \theta = \theta_0 \) at the collector surface at time \( t \). It is further assumed that

\[
m_k(t; \theta_0) = 1 \quad t > 0.
\]

(2)

This assumption is consistent with experimental data (Billings, 1966).

As time progresses, particles colliding with the outer half of any particle of the \( k \)th layer of the dendrite are assumed to become members of the \( (k+1) \)st layer. This, in essence, is the mechanism of the outward growth of the dendrite. For a given value of \( m_k \) there is an upper limit to the number of particles that can be accommodated in the \( (k+1) \)st layer. More specifically, it is assumed that at a given instant \( t \) the maximum possible value of \( m_{k+1} \) is \( \rho m_k \), that is

\[
m_{k+1} \leq \rho m_k
\]

(3)

where \( \rho \) is a constant. From geometrical considerations, the value of \( \rho \) could be of the order of 3–4, but examination of the photographs obtained by Billings (1966) suggests that a more reasonable value of \( \rho \) would be \( \rho = 2 \). Further experimental evidence is required for the elucidation of this point.

The rate of dendrite growth is assumed to be

\[
\frac{dm_k(t; \theta_0)}{dt} = 0
\]

(4)

\[
\frac{dm_k(t; \theta_0)}{dt} = \alpha \phi_{k-1}(\theta_0)m_{k-1}(t; \theta_0)\left[1 - \frac{m_k(t; \theta_0)}{\rho m_{k-1}(t; \theta_0)}\right]
\]

for \( k = 2, 3, 4, \ldots \)

(5)

\[
m_1(0; \theta_0) = 1, \{m_k(0; \theta_0) = 0; k = 2, 3, 4, \ldots \}.
\]

(6)

Equation (4) is obvious; it is the natural consequence of equation (2). In the rate expression of equation (5), the term \( \alpha \phi_{k-1}(\theta_0) \) is the rate of deposition of particles on the outer half of a single particle of the \( (k-1) \)st layer.\footnote{It is assumed that particles colliding with the inner half of a particle in the \( (k-1) \)st layer also collide with the outer half of the adjacent particle in the \( (k-2) \)nd layer, and are accounted for accordingly.} Note the implicit assumption that there is no preferential deposition on any of the \( m_{k-1} \) particles of the \( (k-1) \)st layer.

The quantity in the bracket,

\[
1 - \frac{m_k(t; \theta_0)}{\rho m_{k-1}(t; \theta_0)},
\]

is the fraction of sites available for particle addition in the \( k \)th layer. The quantity \( \alpha \) is defined as the number of particles approaching the cylindrical collector of radius \( a_f \) (normal to the direction of flow) per unit length per unit time, or

\[
\alpha = 2a_f U n,
\]

(7)

where \( U \) is the average interstitial velocity and \( n \) is the particle concentration (number concentration) of the approaching gas stream. It follows that \( \phi_{k-1} \) is the fraction of \( \alpha \) expected to collide with a particle occupying the \( (k-1) \)st layer of the dendrite to become a member of the \( (k) \)th layer and can be determined from particle trajectory calculations assuming the various factors underlying particle deposition are clearly understood.

Equation (5) can be rewritten as

\[
\frac{dm_k(t; \theta_0)}{dt} = \alpha \phi_{k-1}(\theta_0)m_{k-1}(t; \theta_0) - \frac{\alpha}{\rho} \phi_{k-1}(\theta_0)m_k(t; \theta_0) \quad \text{for } k = 2, 3, 4, 5, \ldots
\]

(8)
The system of equations (4 and 8) can be solved with initial conditions given by equation (6) to yield

\[ m_1 = 1 \] (8a)

\[ m_2 = \rho \left[ 1 - \exp \left( -\frac{x \phi_1}{\rho} t \right) \right] \] (8b)

\[ m_3 = \rho^2 \left[ 1 - \frac{\phi_2}{(\phi_2 - \phi_1)} \exp \left( -\frac{x \phi_1}{\rho} t \right) - \frac{\phi_1}{(\phi_1 - \phi_2)} \exp \left( -\frac{x \phi_2}{\rho} t \right) \right] \] (8c)

\[ m_4 = \rho^3 \left[ 1 - \frac{\phi_2 \phi_3}{(\phi_2 - \phi_1)(\phi_3 - \phi_1)} \exp \left( -\frac{x \phi_1}{\rho} t \right) - \frac{\phi_1 \phi_2}{(\phi_1 - \phi_2)(\phi_3 - \phi_2)} \exp \left( -\frac{x \phi_3}{\rho} t \right) \right] \] (8d)

or, in general,

\[ m_k = \rho^{k-1} \left[ 1 - \sum_{j=1}^{k-1} \left\{ \prod_{j \leq i \leq k-1} \frac{\phi_i}{(\phi_i - \phi_j)} \right\} \exp \left( -\frac{x \phi_j}{\rho} t \right) \right] \] (8e)

for \( k = 3, 4, \ldots \)

It can be shown easily that equations (8a–e) satisfy the system of equations (4, 8 and 6) by keeping in mind that if \( \phi_k \) are such that for \( i \neq j, \phi_i \neq \phi_j \), then the following relation is an identity

\[ \sum_{j=1 \mid j \neq i}^{k} \prod_{j \neq i}^{k} \frac{\phi_j - \phi_i}{\phi_j - \phi_i} = 1 \] (9)

for any \( \phi \).

Estimation of the rate constants, \( a \phi_k \). In order to study the growth pattern and the behavior of the system of equations (8), the numerical values of the rate constants \( a \phi_k \) are required. Their estimation, in principle, can be made from particle trajectory calculations. However, for certain limiting cases where particle capture takes place by a single dominant mechanism, explicit expressions for \( a \phi_k \) may be possible. In the following, we shall confine our discussion to the case of particle capture by interception only in order to demonstrate the approach more clearly.

The starting point of the estimation is the use of Kuwabara’s model for the characterization of the flow field near individual fibers in a fibrous filter. The stream function, \( \psi \), is given as (Kuwabara, 1959).

\[ \psi = \frac{U r}{2K} \left[ 2 \ln \frac{r}{a_f} - 1 + \frac{a_f}{r} \left( 1 - \frac{r}{2} \right) - \frac{\gamma}{2} \left( \frac{r}{a_f} \right)^2 \right] \sin \theta \] for \( a_f \leq r \leq b \) (10)

with

\[ b = \frac{a_f}{\sqrt{\gamma}} \] (11)

and

\[ K = -\frac{1}{4} \ln \gamma - \frac{3}{2} + \gamma - \frac{\gamma^2}{4} \] (12)

where

\( \gamma \) = fraction of bed volume occupied by fibers (packing density) = 1 - \( \epsilon_0 \)
\( \epsilon_0 \) = initial filter porosity
\( U \) = average interstitial velocity = \( U_s (1 - \gamma) \)
\( U_s \) = superficial velocity (flow rate per unit filter area).
The velocity components \( v_r \) and \( v_\theta \) are given by

\[
\begin{align*}
v_r &= -\frac{U}{2K} \left[ 2 \ln \frac{r}{a_f} - 1 + \gamma - \left( \frac{a_f}{r} \right)^2 \left( 1 - \frac{\gamma}{2} \right) - \frac{3\gamma}{2} \left( \frac{r}{a_f} \right)^2 \right] \cos \theta \quad (13) \\
v_\theta &= \frac{U}{2K} \left[ 2 \ln \frac{r}{a_f} + 1 + \gamma - \left( \frac{a_f}{r} \right)^2 \left( 1 - \frac{\gamma}{2} \right) - \frac{3\gamma}{2} \left( \frac{r}{a_f} \right)^2 \right] \sin \theta. \quad (14)
\end{align*}
\]

The coordinate system is shown in Fig. 1.

To estimate the value of \( \alpha \phi_k \) due to direct interception alone consider a particle occupying the \( k \)th layer of the dendrite (Fig. 3). In the case when direct interception is the sole mechanism, particle trajectories coincide (at steady state) with streamlines. The following two assumptions are made. First, the flow around the fiber is not affected by the presence of dendrites. Second, if a circle with center coinciding with that of the particle in the \( k \)th layer and radius \( 2a_p \) is drawn normal to the direction of flow, then the outer half of this circle [shaded area, \( A_k \), in Fig. 3(b)] is the locus of the centers of all particles captured by the dendrite particle under consideration, and which become members of the \( (k + 1) \)st layer. Accordingly, if \( q_k \) is the flow rate through the area \( A_k \),

\[
q_k \approx \frac{1}{2} \pi (2a_p)^2 v_\theta (a_f + 2ka_p, \theta_0) \quad (15)
\]

and

\[
\alpha \phi_k = nq_k = 2\pi a_p^2 n v_\theta (a_f + 2ka_p, \theta_0)
\]

\[
= \frac{\pi a_p^2 U n}{K} \left[ 2 \ln \left( 1 + 2k \frac{a_p}{a_f} \right) + 1 + \gamma - \left( 1 - \frac{\gamma}{2} \right) \left( 1 + 2k \frac{a_p}{a_f} \right)^2 \right. \\
&\left. - \frac{3\gamma}{2} \left( 1 + 2k \frac{a_p}{a_f} \right)^2 \right] \sin \theta_0. \quad (16)
\]

The assumption that the presence of the dendrites does not have an appreciable effect on the flow field may not be justified when the value of \( 2ka_p \) is of magnitude comparable to that of \( a_f \).

**Example calculation.** Three specific examples on the growth of particle dendrites are presented based on equations (8 and 16) and the parameter values used are given in Table 1. The values of \( m_k(\tau; \theta_0) \) were calculated for three different particle concentrations, \( n = 500, 1000 \) and \( 1500 \) particles/cm³, and are shown in Figs. 4, 5 and 6, respectively. For the conditions in Table 1 the dominant deposition mechanisms are [see, e.g. Emi et al. (1973)] interception and inertial impaction. Since equation (16)

\( ^{\dagger} \theta \) measured clockwise from the stagnation point.
Table 2. Values of $\alpha \phi_k$ based on the parameter values in Table 1 and equation (16)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n = 500 \text{ cm}^{-3}$</th>
<th>$n = 1000 \text{ cm}^{-3}$</th>
<th>$n = 1500 \text{ cm}^{-3}$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.593</td>
<td>3.185</td>
<td>4.778</td>
</tr>
<tr>
<td>2</td>
<td>2.613</td>
<td>5.227</td>
<td>7.840</td>
</tr>
<tr>
<td>3</td>
<td>3.339</td>
<td>6.679</td>
<td>10.018</td>
</tr>
<tr>
<td>4</td>
<td>3.885</td>
<td>7.771</td>
<td>11.656</td>
</tr>
<tr>
<td>5</td>
<td>4.641</td>
<td>9.283</td>
<td>13.924</td>
</tr>
<tr>
<td>6</td>
<td>4.903</td>
<td>9.805</td>
<td>14.708</td>
</tr>
<tr>
<td>7</td>
<td>5.106</td>
<td>10.212</td>
<td>15.318</td>
</tr>
<tr>
<td>8</td>
<td>5.260</td>
<td>10.519</td>
<td>15.779</td>
</tr>
<tr>
<td>9</td>
<td>5.371</td>
<td>10.741</td>
<td>16.112</td>
</tr>
</tbody>
</table>

Fig. 4. Calculated profiles of $m_k$ for the conditions given in Table 1 and $n = 500 \text{ particles/cm}^3$

Fig. 5. Calculated profiles of $m_k$ for the conditions given in Table 1 and $n = 1000 \text{ particles/cm}^3$
does not account for inertial impaction, the calculated values of $k$ given in Table 2 and the corresponding values of $m_k$ (Figs. 4–6) should be considered as somewhat underestimated.

By rounding the values of $m_k$ at various times to their nearest integers, one can obtain the configuration of the growing dendrites which are given in Figs. 7–9. Based on these results the following observations can be made.

a. The idealized dendrite configurations predicted theoretically are in agreement with those observed experimentally. For comparison, a photograph of particle dendrites on a single fiber (obtained by Billings, 1966) is shown in Fig. 10. The conditions

Fig. 6. Calculated profiles of $m_k$ for the conditions given in Table 1 and $n = 1500$ particles/cm$^3$.

Fig. 7. Theoretically predicted particle dendrite growth for the conditions in Table 1 and $n = 500$ particles/cm$^3$. 
under which the dendrites in Fig. 10 were formed were different than those used in the sample calculations of the present study, but not so as to make the comparison invalid.

b. The theoretically calculated waiting times between particle additions are in the same range with the corresponding values obtained from rough estimates based on Billings' experimental observations.

c. The rate of particle dendrite growth is relatively slow at the beginning but increases rapidly with new particle additions. For example, in the case of Fig. 9 there is only one new particle addition to the forming dendrite during the first 15 min. During the following 15 min, however, there are three new additions, and in the next 15 min there are ten new additions. This rapid rate of growth accelerates even further leading within relatively short time to a "population explosion", as can be seen in Fig. 6. During the explosive growth period the dendrite will either be re-entrained and subsequently be wedged at a downstream region of the filter, or it will intermesh with neighboring fibers and/or dendrites extending from nearby fibers leading ultimately to the formation of matrix-like particle clusters. Unfortunately, the model at its present
Fig. 10. Dendrites of polystyrene latex particles of 0.65 μm radius deposited on a single glass fiber of 4.35 μm radius (from Billings, 1966).
stage of development cannot account for these phases of the phenomenon, although
extension of the model to include these phases is not beyond reach.
d. The theoretically predicted dendrite growth rate explains the substantial increases
due to particle deposition in both collection efficiency and resistance to flow. Indeed,
a dendrite is a collector of particles with a rapidly increasing efficiency and drag
force. The overall effect of the presence and growth of dendrites on the fibers compos-
ing a filter is the rapid increase of overall filter efficiency and pressure drop.
e. The growth history of a dendrite depends strongly (and non-linearly) on the number
concentration of particles, \( n \), in the aerosol, (see Figs. 4 - 6).
f. Based on equation (16) it can be seen that the rate of growth and the configuration
of a dendrite depend on several other factors besides \( n \), namely mean interstitial
gas velocity, \( U \), particle radius, \( a_p \), fiber radius, \( a_f \), etc. For a complete study of
these effects development of a method for the calculation of \( \alpha \phi_k \) is required which
is based on all particle deposition mechanisms. Development of such a method is
part of the theoretical work planned within the framework of the present model.

**PREDICTION OF PRESSURE DROP INCREASE**

One of the main factors to be considered in the design and operation of filtration
systems is the effect of particle deposition on the pressure drop along the filter. For
a constant flow rate system the required pressure drop increases during the filtration
cycle. In fact, this factor often determines the length of the loading period and the
frequency of filter cleaning. In the following a method for the order of magnitude predic-
tion of the increase in pressure drop is presented. This method can be refined to yield
more accurate predictions, and this is pursued in a forthcoming publication.

Calculation of the drag exerted on a fiber covered with dendrites requires knowledge
of the spatial distribution of the dendrites, of the age distribution of the dendrite popula-
tion and of the drag force acting on each dendrite. Let \( \tau \) be the time measured from
the beginning of the aerosol flow through the filter, whereas time \( t \) is used to measure
the age of a dendrite starting from the instant of deposition of its first particle. Now,
let \( N(\tau, \theta) \delta \theta \) be the number of dendrites per unit length of fiber\(^†\) between \( \theta \) and \( \theta + \delta \theta \)
at time \( \tau \), and let \( \chi(t;\tau,\theta)\delta t \) be the fraction of the population of dendrites at time
\( \tau \) and at angle \( \theta \) with age between \( t \) and \( t + \delta t \). The ratio of the pressure drop across
a differential filter element of thickness \( \delta x \) at an advanced stage of particle deposition,
\( \delta P(\tau) \), to that across a clean filter element, \( \delta P_0 \), is given approximately as

\[
\frac{\delta P(\tau)}{\delta P_0} = \frac{F_f + \int_0^\pi \left\{ \int_0^\tau \sum_{k=1}^{M(\tau,\theta)} m_k(t;\theta)\chi(t;\tau,\theta)F_{pk}(\theta)\,dt \right\} N(\tau,\theta) \, d\theta}{F_f},
\]

where \( F_f \) is the drag force per unit length of clean fiber, \( F_{pk} \) is the drag force on
a particle in the \( k \)th layer of a dendrite with its basis at \( \theta \), and \( M(\tau,\theta) \) is an integer
function defined by

\[
m_M(t;\theta) \geq 0.05, \quad m_M + 1(t;\theta) < 0.05.
\]

At this stage of development of the model, \( \chi(t;\tau,\theta) \) and \( N(\tau,\theta) \) are not available.
Further modeling work guided and substantiated by experimental data is necessary
in this area. In order to obtain an order of magnitude estimation of the drag force
it will be assumed that the fiber is covered with uniform dendrites, each composed
of \( M^*(\tau) \) layers of particles at time \( \tau \), and that \( t = \tau \). The number of dendrites per
unit length of fiber, \( \bar{N}(\tau) \) is given by

\[
\bar{N}(\tau) = \int_0^\pi N(\tau,\theta) \, d\theta.
\]

\(^†\) Taking into account both sides of fiber.
Under these assumptions

\[
\frac{\delta P}{\delta P_0} = \frac{F_f + \int_0^{\pi} \left[ \sum_{k=1}^{M(t)} m_k(t;\theta) F_{pk}(\theta) \right] N(t,\theta) d\theta}{F_f}.
\]  

(20)

The drag force on the clean fiber is given by the Kuwabara model as

\[
F_f = \frac{4\pi \mu U}{K}.
\]  

(21)

To obtain an estimate of \( F_{pk}(\theta) \) the contributions due to both the tangential and the radial flow have to be considered. The drag force \( F_{pk}(\theta) \), which is along the direction of the main flow, is given as

\[
F_{pk} = 6\pi \mu a_p v_r \left[ a_f + (2k - 1)a_p \right] f_k^p \sin \theta - v_r \left[ a_f + (2k - 1)a_p \right] f_k^l \cos \theta,
\]  

where \( f_k^c \) and \( f_k^l \) are correction factors accounting for the presence of the neighboring particles. The values of these factors are not known at the present time, although they can, in principle, be calculated (see Goren, 1970; O’Neill, 1968). In the absence of their precise values and in view of the fact that at this point we are interested only in an order of magnitude estimate, it is assumed that

\[
f_k^c = f_k^l = 1.
\]

(22)

With this assumption and substituting the expressions of \( v_r \) and \( v_c \) from equations (13 and 14) into equation (22) one obtains

\[
F_{pk} = \frac{3\pi \mu a_p U}{K} \sin^2 \theta \left[ 2 - (2 - \gamma)X_k^{-2} - \gamma X_k^2 \right]
\]  

\[+ \left[ 2 \ln X_k - 1 + \gamma + \left( 1 - \frac{\gamma}{2} \right) X_k^{-2} - \frac{\gamma}{2} X_k^2 \right] \]  

(23)

with

\[
X_k = 1 + (2k - 1) \frac{a_e}{a_f}.
\]  

(24)

If \( N(t,\theta) \) were known, one could substitute the expression for \( F_{pk} \) from equation (23) into equation (20) and determine \( \delta P/\delta P_0 \). As mentioned earlier, further modeling work is needed to determine \( N(t,\theta) \) and this is done in a forthcoming publication. For our immediate purpose, namely an order of magnitude estimation of \( \delta P/\delta P_0 \) we define an effective angle, \( \theta^* \), such that

\[
\int_0^{\pi} \sum_{k=1}^{M(t)} m_k(t;\theta) F_{pk}(\theta) N(t,\theta) d\theta = \tilde{N}(t) \sum_{k=1}^{M(t)} m_k(t;\theta^*) F_{pk}(\theta^*)
\]  

(25)

in which \( \theta^* \) is a function of \( t \), in general, and such that

\[
0 < \theta^*(t) < \pi.
\]  

(26)

Combining equations (20, 21, 23 and 25) one obtains

\[
\frac{\delta P}{\delta P_0} = \frac{F_f + \tilde{N}(t) \sum_{k=1}^{M(t)} m_k(t;\theta^*) F_{pk}(\theta^*)}{F_f}
\]  

\[= 1 + \frac{3}{4a_0} \tilde{N}_d \sum_{k=1}^{M(t)} m_k(t;\theta^*) \left\{ \sin^2 \theta^* \left[ 2 - (2 - \gamma)X_k^{-2} - \gamma X_k^2 \right] \right\}
\]  

\[+ \left[ 2 \ln X_k - 1 + \gamma + \left( 1 - \frac{\gamma}{2} \right) X_k^{-2} - \frac{\gamma}{2} X_k^2 \right].
\]  

(27)
Let $\sigma$ be the specific deposit, that is the volume of deposited matter per unit filter volume. Based on the above assumption, it can be shown easily that

$$\sigma = \sigma(t) = N(t) \frac{4\gamma d_r^3 \sum_{k=1}^{M^*} m_k(t; \theta^*)}{3d_f} \approx \frac{8\gamma U d_r^3 n \eta_0}{3d_f} \sum_{k=1}^{M^*} m_k(t; \theta^*),$$  \hspace{1cm} (28)

where $\eta_0$ is the single clean fiber efficiency and it is defined as

$$\eta_0 = \frac{\text{rate of particle deposition on a clean fiber per unit length}}{\text{rate of particles approaching a clean fiber per unit length}}.$$  \hspace{1cm} (29)

It is $\eta_0$ which has been the object of study of almost all previous theoretical investigations in the area of aerosol filtration. For this reason a further discussion of $\eta_0$ will not be given here. An excellent review of these works was given recently by Davies (1973). Combining equations (27 and 28) one obtains

$$\frac{\delta P}{\delta P_0} = 1 + \beta \sigma$$ \hspace{1cm} (30a)

with

$$\beta = \frac{9d_r^2}{16\gamma d_f \epsilon_0}$$ \hspace{1cm} (30b)

As $m_k$, $M^*$ and $\theta^*$ are functions of $t$, $\beta$ is also a function of $t$. Several investigators in the area of solid-liquid depth filtration through granular beds have postulated an equation similar in form to equation (30), but with $\beta$ as constant (for a review see Herzig et al., 1970). In the present case the dependence of $\beta$ on time, $t$, is shown in Fig. 11, for the conditions in Table 1, three different concentrations, and $\theta^* = \pi/3$. The corresponding values of $\sigma$ and $\delta P/\delta P_0$, with $\eta_0 = 0.019$ [calculated using the Kuwabara (1959) flow model], are plotted vs time in Fig. 12. It should be stressed that the actual value of $\theta^*$ is not known and may well be time dependent (although not strongly so). The value $\theta^* = \pi/3$ has been chosen to enable calculation of numerical values for $\beta$ in an order of magnitude sense. It is expected that this particular value, $\pi/3$, is close to the true value, as the concentration of dendrites on the upstream side of the fiber is larger than that on the downstream side. In order to keep the calculation

![Fig. 11. Calculated values of $\beta$ vs $t$ for the conditions in Table 1, $\theta^* = \pi/3$ and $n = 500, 1000$ and 1500 particles/cm$^{-3}$.](image)
within the range of validity of the model, the calculation of $\beta$ was discontinued at the time when $m_{10} \geq 0.5$.

**PREDICTION OF FILTRATION EFFICIENCY INCREASE**

The increase in filtration efficiency due to the growth of dendrites on the fibers of the filter can be calculated as follows. Consider a filter element with area $A$ and differential thickness $\delta x$. Let $\delta n$ be the change of $n$ across this filter element ($\delta n < 0$), and let a filter coefficient be defined as

$$i = \frac{1}{n} \frac{\delta n}{\delta x}.$$  \hspace{1cm} (31)

A particle “mass balance” can be written as

$$\text{[Total rate of particle capture]} = \{\text{Rate of particle capture on fibers}\} + \{\text{Rate of particle capture on dendrites}\}. \hspace{1cm} (32)$$

We have

$$\text{[Total rate of particle capture]} = -UA \delta n.$$ \hspace{1cm} (33)

Let $L$ be the total fiber length in the filter element under consideration. It can easily be shown that

$$L = \frac{\gamma A \delta x}{\pi \alpha_f^2}. \hspace{1cm} (34)$$

Then

$$\{\text{Rate of particle capture on fibers}\} = 2nUa_f Ln_0 = L\alpha_0.$$ \hspace{1cm} (35)

Finally,

$$\{\text{Rate of particle capture on dendrites}\} = L \left[ \int_0^\tau \int_0^{\pi \theta} \sum_{k=1}^{n_{10}} \frac{dm_k}{dt} \chi(t;\tau,\theta) d\theta \right] \delta n, \hspace{1cm} (37)$$

in which, using equation (8e),

$$\frac{dm_k}{dt} = m_k = \gamma \rho^{k-2} \sum_{j=1}^{k-1} \left[ \prod_{i=1}^{k-1} \left( \frac{\phi_i}{\phi_j} \right) \right] \phi_j \exp \left( -\frac{\alpha \phi_j t}{\rho} \right), \text{ for } k = 3, 4, \ldots \hspace{1cm} (38)$$
Combining equations (7, 37 and 38) one obtains

\[
\text{[Rate of particle capture on dendrites]} = 2La_f U_n \int_0^\infty \left[ \sum_{k=1}^{M(t)} \rho^{k-2} \chi(t, \tau, \theta) \sum_{j=1}^{k-1} \frac{\phi_j}{\prod_{l=j+1}^{k} (\phi_l - \phi_j)} \phi_j \exp \left( -\frac{z\phi_j}{\rho} \right) \right] dt \int_0^\infty \phi \left( N(\tau, \theta) d\theta \right).
\]

Combining equations (31–35 and 37) one gets

\[
\dot{\lambda} = -\frac{1}{\delta n} \frac{\partial n}{\partial x} = \frac{2\gamma}{\pi a_f} \left\{ \eta_0 + \int_0^\infty \int_0^\pi \left[ \sum_{k=1}^{M(t)} \tilde{m}_k(t; \theta) \chi(t; \tau, \theta) dt \right] N(\tau, \theta) d\theta \right\}.
\]

Letting \( \lambda_0 \) be the value of \( \lambda \) for a clean filter \( (\tau = 0) \), we have

\[
\lambda_0 = \frac{2\gamma}{\pi a_f} \eta_0.
\]

The expression on the right hand side of equation (42) is an increasing function of time, in qualitative agreement with Billings' (1966) experimental observations. However, in order to calculate this expression, \( \chi(t, \tau, \theta) \) and \( N(\tau, \theta) \) are needed. This underlines the need for accurate quantitative data on the growth of dendrites to guide and substantiate the modeling effort.

**IMPROVEMENT AND REFINEMENT OF MODEL**

The ultimate objective of any theoretical modeling work is to provide an adequate description of the relevant physical phenomenon, which can be used for predictive purposes. The preliminary version of the dendrite growth model presented above, therefore, is incomplete. A number of questions have been overlooked in the present analysis. Their resolution and incorporation into the model are obviously needed if the model is to become truly predictive. A brief outline of some of the more important issues is presented below.

**Dendrite spatial distribution and dendrite age distribution**

Information about the number of dendrites as a function of angular position, \( \theta \), and time \( \tau \), along the fiber is required for the study of the effect of particle retention on collection efficiency and pressure drop increases. The angular position can be shown to be of primary importance. For example, the value of \( F_{p,k} \) at \( \theta = \pi/2 \) is more than 10 times larger than that at the front stagnation point (i.e. \( \theta = 0 \)). This problem can be answered, at least in part, by using a time dependent probability function for the angular position of the dendrites, \( \chi(t, \tau, \theta) \), which can be considered to be directly related to the local particle flux which, in turn, can be determined from the trajectory calculation. It is also necessary to have some knowledge about the age distribution of the dendrites, \( \chi(t, \tau, \theta) \), since initiation of individual dendrite growth takes place at all times (see Fig. 10). As a possible approach to the study of this problem, consider that a dendrite is formed whenever a particle becomes attached to the fiber surface. Thus the increase in dendrite density can be related to the initial particle collection efficiency and the particle concentration. However, this has to be modified as filtration proceeds since beyond the initial period of filtration, particle collection can be effected through attachment to either the fiber or to a growing dendrite. This latter quantity, therefore, has to be subtracted from the total amount of captured particles in determining the formation of new dendrites. With this argument, one can, in principle, determine the spatial density of dendrites as well as their age distribution as functions of time. This, together with the growth equations discussed previously, would provide complete information about the spatial density, age distribution, and configuration of the dendrites.
Estimation of rate constants of dendrite growth, $\dot{x} \delta k$

In the sample calculations presented above, deposition is considered to be entirely due to interception. In fact, interception is just one of the several mechanisms of particle capture (it may even be one of the less important ones under certain conditions). A rigorous calculation of the rate constants, $\dot{x} \delta k$, including all the mechanisms (inertial, hydrodynamic, gravitational, electrostatic, etc.) can be made using the particle trajectory method. It would require extensive calculations and may not be justified here. Instead, one may calculate $\dot{x} \delta k$ for each individual mechanism and obtain an approximate estimation by summing up the individual contributions. This also has the added advantage of rendering a closed form expression and is, therefore, more convenient to use.

Correction factors of drag forces

In estimating the pressure drop increase, the drag force acting on the individual particles of the dendrite has to be considered. The use of Stoke's law is inappropriate because of the presence of objects in close proximity to each other and correction factors have to be developed. The problem which has to be considered is the flow over a string of spheres extending from a flat surface for both shear and stagnation flows. The calculation involved would be tedious but does not present any difficulties of a fundamental nature.

Prediction of dynamic behavior of deep fibrous filters

During the operation of a deep filter the particle concentration, $n$, is a function of position and time, $n = n(x,t)$. Consequently, the concentration of the aerosol entering a given filter element of differential thickness $\delta x$ is not constant, as it has been assumed in the derivations of the present work, but a function of time. In order to develop a model capable of predicting the dynamic behavior of a deep fibrous filter, this factor has also to be taken into consideration.

In conclusion, it is the authors' opinion that the dendrite growth model presented above provides a reasonably simple and yet realistic description of filtration in fibrous media. With further improvements which should include the resolution of some of the problems stated above, it can be developed as a truly predictive tool of the dynamic behavior of fibrous filters for solid-gas separations. It is also worth noting that recent experimental evidence (Fochtman, 1975) shows that dendrite growth takes place not only in aerosol systems but in liquid phase filtration as well.

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