A REVIEW OF HYSTERESIS THEORIES FOR ELASTOMERS*

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SUMMARY

The authors present an exhaustive and critical appraisal of current theories of hysteretic friction in this second review article on the friction of elastomeric materials. A physical interpretation of the hysteresis component of friction is given for single and multiple asperities, and its viscoelastic nature is demonstrated. Most of the paper is devoted to a detailed comparison of "elastic" and viscoelastic theories, with an overall appraisal of each contribution where appropriate. It is emphasized that the fundamental element in both the adhesional and hysteretic mechanisms is energy loss associated with relative motion at a sliding interface.

ZUSAMMENFASSUNG

In diesem zweiten Artikel zum Stand des Wissens auf dem Gebiet der Reibung von Elastomeren wird eine erschöpfende und kritisch wertende Darstellung der bekannten Theorien zur Hysterese-Reibung gegeben.


Der überwiegende Teil des Artikels ist einem detaillierten Vergleich "elastischer" und "viskoelastischer" Theorien gewidmet, wobei eine allgemeine Wertung jeder Theorie vorgelegt wird, sofern dies angemessen erschien.

Es wird betont, dass dem Adhäsions- wie dem Hysterese-"Mechanismus" grundsätzlich ein Energieverlust in Verbindung mit einer relativen Bewegung zu einer Gleitfläche zugrunde liegt.

RÉSUMÉ

Dans ce second article sur le frottement des matériaux élastomériques, les auteurs présentent un rapprochement compréhensif et critique des théories connues

* This paper is presented to commemorate the retirement of Professor Dr.-Ing. A. W. Hussmann of the Institut für Verbrennungskraftmaschinen und Kraftfahrzeuge, Technische Universität München in June 1974, on the occasion of his 68th birthday.
INTRODUCTION

The friction force generated between sliding bodies has two principal components commonly described as the adhesion and deformation terms respectively. As we have seen in an earlier review, the adhesion term is a surface effect which takes place within a thin layer (measurable in Angstrom units) at the sliding interface. In the case of metal sliding on metal, the deformation term is usually called the ploughing component of friction. In physical terms, the shearing of welded junctions first gives rise to the basic adhesional component of metal-on-metal friction, but complete sliding cannot yet occur without the asperities of the harder metal ploughing or grooving their way through the matrix of the softer metal. Energy is therefore expended not only in the basic shearing process at asperity tips, but also subsequently in the ploughing action. For the case of an elastomer sliding on a textured and rigid base surface, the adhesion term can be interpreted as a thermally-activated molecular stick-slip action occurring within a thin surface layer of the elastomer. In contrast with the ploughing action of metal-on-metal friction, the sliding elastomer "flows" readily over the rigid asperities of the base and conforms to their contour. The deformation component of friction produced by such flowing action is called hysteresis, and it is a characteristic feature of the frictional behaviour of viscoelastic bodies on rough surfaces.

The existence of hysteresis friction is a consequence of energy loss associated with internal damping effects within a viscoelastic body, and it normally appears in the form of heat. Under dry conditions, the adhesional contribution to total friction is so large that the hysteretic part of friction is almost always negligible. For wet conditions, however, the adhesional component is drastically reduced, whereas the hysteretic energy loss remains largely unaffected by wetness. In such cases, we cannot neglect hysteresis except in the case of perfectly smooth surfaces. We note that the existence of hysteretic losses has never been denied, whereas the reality of the adhesional mechanism has been disputed by at least one author.

The separation of the frictional mechanism for elastomers into adhesional and hysteretic components offers a particularly graphic and physical understanding of the phenomenon. For this reason, the separation of the two components is widely accepted—but we are not at all bound to adhere to this convention in interpreting frictional data. On the other hand, the fundamental and common element in both the adhesional and hysteretic mechanisms is energy loss associated
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with relative motion at the sliding interface. This energy is accompanied by either macroscopic or microscopic surface deterioration as a consequence of the energy transfer mechanism. If we divide the energy loss by some distance dimension, we define automatically a friction force. This concept is particularly useful if the energy loss mechanism takes place at or in the vicinity of a sliding interface, and the friction force so defined is taken to occur at the interface. In some cases, however, internal friction losses (or hysteresis) may occur within a body subjected to cyclic stress variation without the proximity of a sliding interface, and it becomes meaningless to speak of a friction force. The best example which comes to mind is the rolling behaviour of a pneumatic tyre on a rigid pavement surface. Here, the continual flexing of carcase and sidewalls dissipates energy as heat within the tyre, and we speak of internal friction or hysteresis. If braking action is now externally applied to the tyre, additional energy losses are caused near the tread/road boundary, and these can be conveniently expressed as the adhesional and hysteretic components of an interfacial friction force. We emphasise that energy loss is the fundamental factor in both the internal and external friction mechanisms, and the concept of a frictional force is a convenient means of expressing such energy loss only in the latter case.

Previous summaries of the literature on the hysteretic friction of polymers have been given by Rieger and Geyer, but without a detailed description of the contents of all cited papers. Here, the primary objective has been to make a significant contribution to the state of the art, following a brief survey of previous work. In this review article, the authors critically examine and compare in detail the most important hysteresis theories of friction in existence, so as to establish the present state of knowledge in this area.

SINGLE AND MULTIPLE ASPERITY SURFACES

The generation of hysteresis friction forces between a sliding elastomer and a rigid base surface requires that the latter exhibits distinct asperities on a macroscopic scale. In this manner, the "flowing" action of the elastomer over the base surface is readily envisaged. We now attempt to explain the nature of hysteresis friction for such a system. Consider first the interaction of a single symmetrical asperity with an elastomer slider, as shown in Fig. 1. The pressure distribution is drawn about an individual asperity of the surface (a) when no relative motion exists at the interface, and (b) in the presence of relative sliding. The absence of

![Fig. 1. Physical interpretation of the hysteresis component of friction.](image-url)
relative motion produces a symmetrical draping of the elastomer about the asperity. If the pressure distribution normal to the contour of the asperity is resolved in this case into vertical and horizontal components, it is seen that the summation of vertical pressures must be in equilibrium with the load $W$, and the horizontal components cancel. If, however, the elastomer is sliding with a finite velocity $V$ relative to the base surface, it tends to accumulate or "pile up" at the leading edge of the asperity and to break contact at a higher point on the downward slope. Thus, the contact arc moves backwards relative to the direction of sliding, thereby creating an unsymmetrical pressure distribution, as shown in the lower figure. Here, the horizontal pressure components give rise to a net force $F_{h,t}$ which opposes the sliding motion. The asymmetry effect is due to the mass or inertia of surface rubber elements in changing direction when they suddenly experience an obstacle in their path.

This simple visualisation of the hysteresis effect on a single asperity model indicates how a friction force evolves from pressure asymmetry, but it makes no mention of energy loss occurring in the process. In point of fact, we have explained the physical nature of hysteresis by concentrating more on the effect than the cause of the phenomenon, in the interests of simplicity and clarity. A more fundamental approach would involve considerations of the energy expended in the sliding process. Thus, within the forward arc of contact, work is done in deforming the elastomer and deflecting it from its straight-line travel path, whereas within the rearward contact arc almost all of this work is returned. The difference between the work expended and returned is the energy loss which gives rise to frictional forces.

In tribology studies, it is always convenient to consider a rough surface as a random assembly of idealised asperity shapes and sizes. The performance of each of the latter can usually be predicted by mathematical methods, and, indeed, this is the basic justification for using single-asperity models. There are at least three basic shapes for idealised asperities (namely, cones, cylinders and spheres), and these can be combined in different arrangements to approximate the nature of a random engineering surface. These basic shapes represent extremes of roundedness and sharpness which might be encountered in practice. In the case of hysteresis friction, however, it is not sufficient to consider individual asperities, irrespective of their shape. It is also necessary to consider multiple asperities in the basic theory, because of interference effects which become prominent and characterise the very nature of hysteresis at higher sliding speeds. We examine this effect qualitatively in the next section.

THE VISCOELASTIC NATURE OF HYSTERESIS

At zero sliding speed, the symmetrical pressure distribution about a regular or idealised asperity pressed into a viscoelastic plane creates no net side force, as can be seen in Fig. 1 and also in Fig. 2(A). When relative motion exists between asperity and elastomer, the contact arc ab in Fig. 2(A) moves backwards relative to the sliding direction and assumes a new position cd (see Fig. 2(B)). At the same time, the length of the arc cd is less than the original length ab. This can be regarded as an effective "stiffening" of the elastomer which occurs as the speed of
sliding is raised, and the reduction of contact length effectively adjusts the area of contact at each asperity according to the relationship:

\[ A_i = \frac{W}{M \rho} \]  

(1)

where \( M \) is the number of asperities over which the load \( W \) acts.

As speed is further increased, the delayed recovery of the elastomer on the downward slope of one asperity will decrease the accumulation on the positive slope of the succeeding asperity. Thus, the point \( c \) which perhaps moved to the left relative to the static position \( a \) at modest sliding speeds, now moves to the right to the position \( e \) at higher sliding speeds, as shown in Fig. 2(C). The points \( c \) and \( d \) therefore approach each other, and the contact arc at very high sliding speeds has not only a minimum value but it is virtually symmetrical. The hysteresis force due to pressure asymmetry is therefore reduced at very high speeds of sliding.

The physical mechanism of hysteresis friction described here accounts for the viscoelastic peak shown in the insert of Fig. 2, and it occurs at intermediate velocities of sliding. It has been tacitly assumed, of course, that the interface temperature \( T \) is constant at all sliding speeds, although it is generally agreed that an increase of sliding speed \( V \) implies a corresponding increase in \( T \).
Figure 3 shows that the effect of increasing temperature is to move the hysteresis peak to a higher velocity, and the required velocity shift $\Delta V$ corresponding to a given temperature increase $\Delta T$ may be calculated from the well-known WLF transformation method. In place of the abscissa $V$ in Fig. 3, we sometimes use frequency $\omega$, where $\omega = \frac{V}{\lambda}$ and $\lambda$ is the mean wavelength of surface asperities.

**CLASSIFICATION OF HYSTERESIS THEORIES**

Existing theories of hysteresis friction may be classified in any of three ways. Thus, we distinguish between:

(a) "Elastic" and viscoelastic theories,

(b) Single- and multiple-element models,

or

(c) Force and energy concepts.

The earlier concept of hysteresis applied elasticity theory to the rolling of spheres and cylinders on an elastomeric plane surface, and it was conjectured that a small fraction of the input elastic energy to the deformed elastomer must be dissipated in the form of hysteretic friction. Later theories assumed viscoelastic behaviour in the elastomer, usually by making use of some form of energy-dissipative mechanical model. The two basic mechanical models commonly used are known as Voigt and Maxwell models, as shown in Figs. 4(a) and (b) respectively.

![Fig. 4. Single element models of viscoelastic behaviour. (a) Voigt model. (b) Maxwell model. (c) Modified Voigt model. (d) Modified Maxwell model. (e) Maxwell-Voigt model.](image)

However, they are usually far too simplified to represent dynamic performance in actual viscoelastic materials, and modified forms are therefore used, as shown in Figs. 4(c), (d) and (e). As we shall see later, several existing viscoelastic theories make use of one or other of the models shown in Fig. 4, and they are conveniently described as single-element models. For a more realistic simulation of actual behaviour in materials, it is necessary to offer arrays of Voigt and Maxwell models in different combinations, as indicated in Fig. 5. Such relatively complex
representation we call multiple-element modelling. Because of the complexity of such models, few theories use such a precise simulation of material behaviour in sliding and rolling friction.

The concepts of force and energy in describing the causes of hysteresis friction have been dealt with previously in this review paper. We recall that the most basic explanation is energy loss in the vicinity of a sliding interface. At the same time, the force concept arising from pressure asymmetry offers the most convincing demonstration of the mechanism involved. We may regard the energy loss mechanism as giving rise to the typical delayed recovery of elastomeric material on the downstream side of asperities, and the latter in turn creates a hysteresis force from pressure asymmetry at the sliding interface.

We now examine critically the main theories of hysteresis friction.

"ELASTIC THEORY": ROLLING AND SLIDING

Rolling cylinder

Consider a rigid cylinder of radius $R$ pressed on to an elastomer by a load $W$ per unit width of cylinder, as shown in Fig. 6. According to the analysis by Hertz in 1881, the pressure $p$ at a point $x$ within the chord of contact is given by:

$$ p = \frac{2W}{\pi a} \left[ 1 - \left( \frac{x}{a} \right)^2 \right]^{\frac{1}{2}} $$

(2)

For small displacements in the vertical direction, the displacement of the elastic medium $w$ at point $x$ is as follows:

$$ w = w_0 - \frac{x^2}{2R} $$

(3)

where $w_0$ is the centre-line or maximum displacement.
In moving forward unit distance, the change in displacement at $x$ is:

$$\frac{\partial w}{\partial x} = -\frac{x}{R}$$

so that the work done by the pressure over a strip of width $dx$ will be $p \cdot dx(x/R)$. If we now substitute from eqn. (2) for $p$ and integrate from 0 to $a$, we obtain the work $\phi_1$ required to move forward unit distance as follows:

$$\phi_1 = \frac{1}{R} \int_0^a p(x) \cdot dx = \frac{2Wa}{3\pi R}$$

Analysis by Hertz has also shown that the semi-contact width $a$ is given by:

$$a = \frac{2}{\pi^{\frac{3}{2}}} \left[ WR \left( \frac{1-v^2}{E} \right) \right]^{\frac{1}{2}}$$

where $E$, $v$ are the Young's modulus and Poisson's ratio respectively for the elastomer. Substitution of eqn. (6) into eqn. (5) gives:

$$\phi_1 = \frac{W^{\frac{3}{2}}}{R^{\frac{3}{2}}} \left[ \frac{16}{9\pi^3} \left( \frac{1-v^2}{E} \right) \right]^{\frac{1}{2}}$$

If the elastic medium possessed ideal elastic qualities, the part behind the cylinder would yield the same amount of work $\phi_1$. The elastomer would then effectively restore to the cylinder the work done on it in the initial compressive stage, and no net energy would be expended. However, a constant fraction $\alpha$ of the input elastic energy $\phi_1$ is lost as a consequence of hysteresis within the elastomer, and this gives rise to a friction force $F$. Thus, we may write:

$$f_{\text{rolling}} = \frac{F}{W} = \frac{\alpha \phi_1}{W} = \frac{2\alpha}{3\pi} \left( \frac{a}{R} \right)$$

or

$$f_{\text{rolling}} = \frac{4\alpha}{3\pi} \left[ \frac{W}{R} \left( \frac{1-v^2}{E} \right) \right]^{\frac{1}{2}}$$
where $f_{\text{rolling}}$ is the coefficient of friction due to rolling. The value of $a$ is obtained by experiment.

We have neglected entirely the contribution of adhesional friction to the overall coefficient of rolling friction in these last two equations, so that the coefficient $f_{\text{rolling}}$ may be attributed to hysteresis. Experiments have shown, in fact, that adhesion is very small in rolling\(^{11}\) although not entirely negligible.

**Rolling sphere**

The case of a rigid sphere of radius $R$ which rests on an elastomer under the action of an external load $W$ is shown in Fig. 7. From the theory of Hertz, the pressure $p$ at radius $r$ and the radius of contact are given by:

\[
p = \frac{3W}{2\pi a^2} \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^{1/2}
\]

and:

\[
a = \left[ \frac{3W}{4\pi R} \left( \frac{1 - \nu^2}{E} \right) \right]^{1/2}
\]

The vertical displacement of the elastic body at location $r$ is:

\[
w = w_0 - \frac{r^2}{2R}
\]

where $r^2 = x^2 + y^2$. As before, the change in displacement at $(x, y)$ is $\partial w / \partial x = -x/R$, during which time the sphere moves forward unit distance and does work of an amount

\[
d\phi_2 = -p \, dx \, dy \left( \frac{\partial w}{\partial x} \right) = \frac{p}{R} \, x \, dx \, dy
\]

By substituting in eqn. (13) for $p$ from eqn. (10) and integrating between the
limits \( x=0 \) to \( (a^2-y^2)^{1/2} \) and \( y=-a \) to \( a \), we obtain:

\[
\phi_2 = \frac{1}{R} \int_{x=0}^{(a^2-y^2)^{1/2}} \int_{y=-a}^{a} px \, dx \, dy = \frac{3 Wa}{16R} \quad (14)
\]

Using eqn. (11) to eliminate \( a \) gives:

\[
\phi_2 = \frac{3}{16} \frac{W^{1/2}}{R^{3/2}} \left[ \frac{3}{4} \left( 1 - \frac{v^2}{E} \right) \right]^{1/4} \quad (15)
\]

As in the case of the cylinder, the coefficient of rolling friction for a sphere on an elastic medium is given by:

\[
f_{\text{rolling}} = \frac{F}{W} = \frac{\alpha \phi_2}{W} = \frac{3 \alpha}{16} \left( \frac{a}{R} \right) \quad (16)
\]

or

\[
f_{\text{rolling}} = \frac{3 \alpha}{16} \left[ \frac{3 W}{4R^2} \left( 1 - \frac{v^2}{E} \right) \right]^{1/4} \quad (17)
\]

If eqns. (16) and (8) are plotted as shown in Fig. 8, it is seen that the coefficient of rolling friction is proportional to the dimensionless contact area \((a/R)\) for both sphere and cylinder, with a 14% larger constant of proportionality for the case of the cylinder. Equations (8) and (16) also show that the frictional force \( F \) is proportional to \( W \), and these plots are given in Fig. 9.

If \( R \) is eliminated from eqn. (16) by substitution from eqn. (11), then:

\[
\phi_2 = \frac{9 W^2}{64 a^2} \left( \frac{1-v^2}{E} \right) \quad (18)
\]

By defining the mean or areal pressure \( \bar{p} = W/\pi a^2 \), we can then use eqn. (18) to define the coefficient of rolling friction as follows:

\[
f_{\text{rolling}} = \frac{\alpha \phi_2}{W} = \left[ \frac{9 \pi}{64} \left( 1 - \frac{v^2}{E} \right)^2 \right] \times \bar{p} \quad (19)
\]
which shows that \( f_{\text{rolling}} \) is directly proportional to the mean pressure in the contact zone. For rubber, the bracketed term in this equation has the numerical value \( 0.55 \times 10^{-6} \text{ m}^2/\text{kg} \) and \( \alpha = 0.7 \), so that

\[
f_{\text{rolling}} = 0.39 \times 10^{-6} \bar{p}
\]  

(19a)

These results have been confirmed by experiment\(^{10}\).

It must be stated that the above analysis for sphere and cylinder has very limited application, and it is valid for very low speeds of sliding. As speed increases to finite values, pressure asymmetry becomes a necessity to sustain the rolling mode, and this can only be explained by introducing a combination of dynamic and viscoelastic effects into the “elastic” medium. The theory inherently contradicts itself, since if elasticity is assumed there will be complete recovery behind sphere and cylinder and hence no coefficient of rolling friction. To explain this contradiction, the authors have used the term “elastic hysteresis” which is a further contradiction. The analysis presented must also necessarily be incomplete, relying on separate experiments to provide the fraction \( \alpha \) of input energy which is lost in the form of hysteretic friction.

**Sliding cone**

The final example of “elastic” theory is that of a lubricated cone sliding on an elastic solid under the action of a normal load \( W \). The circle of contact has a radius \( a \) (as shown in Fig. 10) and the contact surface is obviously a cone of maximum radius \( a \). The area \( ds \) is then given by:

\[
ds = r \cdot d\phi \cdot dl
\]

where

\[
dl = \cosec \theta \cdot dr
\]

and the horizontal component of force at location \( \phi \) is given by:

\[
dF = (p \cdot ds \cos \theta) \cos \phi
\]

(21)

Using eqn. (20) in eqn. (21) and integrating from \( \phi = -\pi/2 \) to \( \pi/2 \), the total horizontal force \( F \) in the direction of travel caused by interfacial pressure between cone and elastomer is given by:

\[
F = 2 \cot \theta \int_{0}^{\pi/2} \cos \phi d\phi \int_{0}^{a} pr \cdot dr
\]

\[
= 2 \cot \theta \int_{0}^{a} pr \cdot dr
\]

(22)

The load \( W \) is supported over the whole cone of contact, so that we may write:

\[
W = \int p \sin \theta \cdot ds = \int_{0}^{2\pi} d\phi \int_{0}^{a} pr \cdot dr = 2\pi \int_{0}^{a} pr \cdot dr
\]

(23)

By eliminating the integral from eqns. (22) and (23), we obtain:

\[
F = \frac{W}{\pi \cot \theta}
\]

(24)
Now, in sliding forward unit distance, the work $\phi$ is identical with $F$ so that (as in the case of the cylinder and sphere) a fraction $\alpha$ gives rise to the sliding coefficient, $f_{\text{sliding}}$, thus:

$$f_{\text{sliding}} = \frac{\alpha \phi}{W} = \frac{\alpha}{\pi} \cot \theta$$  \hspace{2cm} (25)

which shows that the coefficient of sliding friction depends only on the semicone angle $\theta$ and not on the pressure distribution. From Love's classical theory of elasticity, the mean pressure $\bar{p}$ is given by:

$$\bar{p} = \frac{E}{2(1-\nu^2)} \cot \theta$$  \hspace{2cm} (26)

so that from eqns. (25) and (26):

$$f_{\text{sliding}} = \frac{2(1-\nu^2)}{\pi E} \alpha \bar{p}$$  \hspace{2cm} (27)

For the case of rubber, $\nu = 0.5$, $E = 5.8 \times 10^5$ kg/m$^2$ and $\alpha = 0.8$ and:

$$f_{\text{sliding}} = 0.66 \times 10^{-6} \bar{p}$$  \hspace{2cm} (27a)

Experimental proof$^{12}$ of the validity of eqn. (27a) is shown in Fig. 11. As the semicone angle $\theta$ decreases, tearing and rupture of the elastic solid occurs, and departures from eqn. (27a) are inevitable.

The test speed underlying the experimental results in Fig. 11 was about 25 mm/min, so that we may assume a slight pressure asymmetry about the cone axis. It seems reasonable to neglect hydrodynamic and elastohydrodynamic effects because of the low speed, but the neglect of adhesion entirely in the analysis is controversial. Later tests using water as a lubricant$^{13}$ and a much higher sliding speed ($\sim 2$ m/s) show remarkably good agreement between theory and experiment, but at these speeds viscoelastic, elastohydrodynamic and temperature effects are significant and
their interaction is relatively complex. The theory for cones as presented above follows the same pattern as in the case of spheres and cylinders—namely, the assumption of elastic behaviour in the material ahead of the sliding cone, and the supposition that a fraction \( \alpha \) of the input energy is lost during the expansion process behind the cone. At very slow or creeping speeds of travel, the theory may be valid, although it appears less justifiable to the authors to neglect adhesion forces here than in the case of rolling elements.

MAIN VISCOELASTIC THEORIES

**Mechanical model theories**

We distinguish between the various mechanical model theories on the basis of the shape of asperities in the rigid base surface (i.e. sinusoidal, cylindrical or spherical). In the case of sinusoidal asperities, it is convenient to consider a multiple asperity model, since the mathematical definition of the surface is a continuous function of position. On the other hand, single or individual asperities are used for cylindrical and spherical shapes.

**Sinusoidal asperities**

Both Rieger\(^6\) and Kummer\(^14\) offer a similar simple Voigt model simulation of rubber sliding on a sinusoidal track, as shown in Fig. 12. The amplitude of the track may be described by the equation

\[
z = a(1 - \cos \omega t)
\]

where

\[
\omega = \frac{2\pi V}{\lambda}
\]

The frictionless rollers depicted in Fig. 12 symbolize the neglect of the adhesion term. We can write for the load \( W(x) \) sustained by the Voigt element\(^2\) at any position \( x \):

\[
W(x) = EL(z + z_0) + \eta L \dot{z}
\]

where \( E, \eta \) are the spring and damping moduli, \( z_0 \) a pre-load deflection, and
Fig. 12. Sinusoidal-asperity model for track roughness\textsuperscript{b, 14}.

$L$ a characteristic length dimension. By substituting for $z$ and $\dot{z}$ from eqn. (28) into eqn. (30), the instantaneous load $W$ becomes a function of time as well as position, and the total energy $E_d$ dissipated as hysteretic friction becomes:

$$E_d = \frac{1}{2}\pi a^2 \eta \omega L$$

By dividing by the wavelength $\lambda$ of the sinusoidal roughness, we obtain the mean hysteretic friction force $F_{\text{hyst}}$, thus:

$$F_{\text{hyst}} = E_d / \lambda = \pi a^2 (L/\lambda) \eta \omega$$

(32)\hspace{1cm} (32a)

$$= 2\pi^2 a^2 (L/\lambda) E \tan \delta$$

(32b)

We note that in eqn. (32b) the tangent modulus $\tan \delta$ for a Voigt model is given by the ratio $(\omega \eta / E)$.

Both Kummer\textsuperscript{14} and Rieger\textsuperscript{6} have indicated that the form of eqn. (32a) shows a constantly increasing hysteretic force $F_{\text{hyst}}$ as the sliding speed $V$ is raised, and indeed this is the case, since the viscosity term $\eta$ is assumed constant in the Voigt model representation of Fig. 12. This reasoning can be valid, however, only within a limited velocity range. At higher velocities, the bottom roller in Fig. 12 loses contact with the sinusoidal track on its downward slope, so that the friction force $F_{\text{hyst}}$ drops in value, thus exhibiting the characteristic viscoelastic peak. Kummer discusses this effect qualitatively, but fails to show any mathematical reasoning to account for such behaviour. On the other hand, Rieger has calculated precisely the instants $t_1$ and $t_2$ at which the roller leaves and contacts the track, at each sliding speed $V$. This is essentially obtained by putting $W(x) = 0$ in eqn. (30), and then solving the resultant equation in conjunction with eqn. (28). The limits of the integral in eqn. (31) are therefore changed as follows:

$$E_d = \int_{t_2}^{t_1 + 2\pi / \omega} W(t) \cdot \dot{z} dt$$

(33)
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from which the true hysteresis force \( F_{\text{hyst}} = \frac{E_d}{\lambda} \) may be calculated over the entire speed range, as indicated by the characteristic curve in Fig. 13.

It is to be noted that since \( E \) and \( \eta \) in the simple Voigt model of Fig. 12 have constant values, the tangent modulus also increases linearly with sliding velocity following eqn. (32b). Thus, there is a consistency in the various forms of eqn. (32). We normally anticipate, however, that the tangent modulus exhibits a viscoelastic peak\(^2\), as shown by the solid curve in Fig. 13. Rieger's analysis has therefore modified the simple Voigt model at higher sliding speeds (by allowing systematically for periods of non-contact between roller and track) so that an effective change in the values of \( E \) and \( \eta \) is tolerated.

An alternative and perhaps simpler approach to that of Rieger would be to assume complex Voigt behaviour\(^9\), and consider continuous roller contact at all sliding speeds. This method is described in detail in a later section. We note that although Kummer offered no mathematical model to explain high-speed effects, he has discussed qualitatively the decrease in dashpot viscosity \( \eta \) with increasing sliding speed \( V \), so that the product \( \eta V \) in eqn. (32a) may attain a constant value at the viscoelastic peak in Fig. 13. A third and simple explanation of speed effects is to rely on the form of eqn. (32b) for the hysteresis friction force, and then apply realistic or known speed variations to both \( E \) and \( \tan \delta \). All of these approaches will yield the typical viscoelastic characteristic in Fig. 13.

We must finally obtain expressions for the coefficient of hysteresis friction based on the model in Fig. 12. The normal load \( W(x) \) acting on the Voigt model has an average value \( \bar{W} \), which at low speed may be calculated as follows:

\[
\bar{W} = \frac{1}{I} \int_0^I W(t) \cdot dt = E L (a + z_0)
\]

(34)

where \( I = \frac{\lambda}{V} \) is the time taken to traverse one wavelength of the surface. For the case where no preloading of the Voigt element exists \( (z_0 = 0) \), we can define \( f_h \) using eqns. (32) and (34), thus:

\[
f_h = \frac{F_{\text{hyst}}}{\bar{W}} = \frac{\eta \omega}{E}
\]

(35)

\[
= 2\pi^2 \left( \frac{a}{\lambda} \right)^2 \tau V
\]

(35a)

\[
= 2\pi^2 \left( \frac{a}{\lambda} \right) \tan \delta
\]

(35b)

\* The complex Voigt model has spring and dashpot elements as in the case of the simple Voigt model of Fig. 12, but their values are both temperature and frequency dependent.
where $\tau = \eta/E$ is the relaxation time for the simple Voigt model. The same limitations regarding speed apply to these equations as in the case of the hysteresis force, $F_{\text{hyst}}$. At higher speeds, we must change the limits on the integral in eqn. (34) in computing the mean load $\bar{W}$, thus:

$$\bar{W} = \frac{1}{t} \int_{t_2}^{t_1} W(t) \cdot dt$$

Here, as in the case of the lower-speed model, $f_h$ exhibits a typical viscoelastic peak value.

**Cylindrical asperities**

Consider now the case of a rigid cylinder rolling on a viscoelastic plane, as shown in Fig. 14. Following Norman's theory, we assume initially a simple Voigt model as shown in Figs. 4(a) and 12. For convenience, the centre of the cylinder will be assumed fixed, and the viscoelastic plane moves past it with velocity $V$. Now, the time taken for the plane to acquire its maximum deformation $z_M$ is given by the relationship:

$$t = \frac{z_M}{\dot{z}_m} = \frac{a}{V}$$

(36)

where $\dot{z}_m$ represents the rate of deformation under the centre of the cylinder, and $a$ is the Hertzian contact radius. The viscoelastic plane experiences deformation by the rolling cylinder only once, and it is therefore incorrect to speak of a deformation frequency. (The latter term appears to be valid only in the case of a series of cylindrical elements, or a continuous roughness as in Fig. 12). However, the reciprocal of $t$ in eqn. (36) has the dimensions of a frequency, and we can therefore define:

$$\omega = \frac{1}{t} = \frac{V}{a}$$

(37)

Thus,

$$\tan \delta = \frac{\omega \eta}{E} = \frac{\eta V}{Ea} = \left(\frac{\eta V}{ER}\right) \frac{\beta}{\phi} = \gamma$$

(38)

Fig. 14. Cylinder rolling on viscoelastic plane.
We now divide each term* in eqn. (30) by $L'$, so that a pressure term $p$ appears on the left-hand side:

$$p(x) = \frac{E}{L} z + \frac{\eta}{L} \dot{z} \tag{39}$$

The deflection $z$ within the contact arc ab in Fig. 14 is written as:

$$z = (R^2 - x^2)^{\frac{1}{2}} - (R^2 - a^2)^{\frac{1}{2}}$$

and:

$$\dot{z} = -V \left( \frac{dz}{dx} \right) = \frac{Vx}{(R^2-x^2)^{\frac{1}{2}}} \tag{40}$$

By substituting for $z$ and $\dot{z}$ from this last equation in eqn. (39), the pressure $p$ for very light loads** takes the form:

$$\frac{pL}{E} = \frac{R}{2} \left\{ \phi^2 - \frac{x^2}{R^2} + \frac{2\beta x}{R} \ldots \right\} \tag{41}$$

The value of $x(= -x_0)$ which makes $p=0$ in this equation is given by:

$$\phi^2 - \lambda^2 + 2\beta = 0$$

or,

$$\frac{\lambda}{\phi} = (1 + \tan^2 \delta)^{\frac{1}{2}}$$

where $\lambda = -x_0/R$. This last equation may also be written as:

$$\left( \frac{x_0}{a} \right) = (1 + \tan^2 \delta)^{\frac{1}{2}} - \tan \delta \tag{42}$$

Thus, the contact asymmetry factor $(x_0/a)$ is a direct function of the tangent modulus $\tan \delta$ of the elastomeric material. Table I below is plotted from eqn. (42), and we see at once that the greater the value of $\tan \delta$, the sooner the separation point b in Fig. 14.

The coefficient of rolling friction due to hysteresis is determined in the following manner. Knowing the pressure from eqn. (41), we can write for the normal load $W$ and the friction force $F_{hyst}$

$$W = \int_{-x_0}^{a} pb \cdot dx \tag{43a}$$

$$F_{hyst} = \frac{M}{R} = \int_{-x_0}^{a} pbx \cdot dx \tag{43b}$$

** Table I

<table>
<thead>
<tr>
<th>$\tan \delta$</th>
<th>0*</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x_0/a$</td>
<td>1</td>
<td>0.905</td>
<td>0.82</td>
<td>0.744</td>
<td>0.616</td>
<td>0.52</td>
<td>0.414</td>
<td>0.362</td>
<td>0.32</td>
<td>0.236</td>
</tr>
</tbody>
</table>

* Perfectly elastic case (symmetrical "draping" effect).

* Neglecting $z_0$.

** $\phi = a/R \ll 1$ and $x/R \ll 1$. 
where \( b \) is the thickness of the cylinder perpendicular to the paper, \( M \) is the moment of the pressure forces about the point 0 in Fig. 14, and the lower limit \( x_0 \) in the integrals is taken from eqn. (42). Thus,

\[
f_1 = \frac{F_{\text{hyst}}}{W} = \phi \left\{ \gamma - \frac{3\gamma^2}{2[1 - \gamma^3 + (1 + \gamma^2)^2]} \right\}
\]

This equation may be rationalised still further\(^{15}\) so that \( f_h \) is expressed as a direct function of the tangent modulus. Thus,

\[
f_h = C_0 \left( \frac{W}{bER} \right)^{\frac{1}{3}} \tan \delta \quad (\text{for } \tan \delta < 0.2)
\]

\[
f = C_1 \left( \frac{W}{bER} \right)^{\frac{1}{3}} (\tan \delta)^{0.63} \pm 10\% \quad (\text{for } 0.2 \leq \tan \delta \leq 1.4)
\]

where \( C_0 = 2C_1 = \text{constant} \).

The form of eqn. (39) is such that pressure is proportional to displacement, whereas Hertzian theory requires that pressure be proportional to the square-root of displacement. The latter assumption gives a more correct estimate of penetration depth\(^{15}\). However, in its present form, eqn. (39) assumes the validity of a simple Voigt model to simulate the viscoelastic behaviour of the elastomeric plane surface, and the theory depicts concisely the nature of the asymmetry effect in pure rolling.

May, Morris and Atack\(^{16}\) have chosen first a Maxwell model and later a modified Maxwell model (as shown in Fig. 4 (b) and (d)) to simulate the viscoelastic behaviour of an elastomeric plane surface when a hard cylinder rolls over it. Although specific relationships between \( f_{\text{hyst}} \) and \( \tan \delta \) were not obtained, it is shown clearly that rolling friction reaches a maximum value at a particular value of the dimensionless relaxation time \((\tau / T)\), as indicated in Fig. 15.

Fig. 15. Rolling friction as function of dimensionless relaxation time.

Here, \( \tau \) is the ratio of dashpot viscosity \( \eta \) to spring modulus \( E \) in the Maxwell model, and \( T \) the time taken for the cylinder to move forward a distance equal to the semi-contact length \( a \) (see Fig. 14). The peak of the relaxation time distribution shown in Fig. 15 corresponds to a specific forward rolling speed, since the ratio \((\tau / T)\) is numerically equal to \((\eta V / Ea)\). We also note that the load required to maintain constant deflection of the elastomeric plane increases with \((\tau / T)\) and therefore with speed—this is a clear indication of the "stiffening" effect which occurs.
at higher speeds of deformation in elastomers. By using a Gaussian distribution of Maxwell models\(^1\), a more valid representation of viscoelastic behaviour is obtained, and the results can be interpreted as a refinement of the simpler models. In this case, curves similar to those shown in Fig. 15 may be obtained.

**Spherical asperities**

The rolling of a rigid sphere on a viscoelastic medium\(^2\) has been treated in a manner very similar to Norman's theory for a cylinder\(^3\), assuming a simple Voigt model simulation of viscoelastic behaviour. Figure 16 shows the coordinate system for a rolling sphere, with the origin at the centre of the contact circle.

![Image of a rigid sphere rolling on a viscoelastic plane](image)

The centre of the sphere has coordinates \((0, 0, -z_0)\) as shown, and the equation of the sphere is:

\[
x^2 + y^2 + (z + z_0)^2 = R^2
\]  
(46)

The circle of contact is described by the equation:

\[
x^2 + y^2 = R^2 - z_0^2 = a^2
\]  
(47)

The depth of indentation is denoted by \(z\), and by substituting for \(z_0\) from eqn. (47) into eqn. (46), the following expression for \(z\) emerges:

\[
(z_0 - z) = (R^2 - x^2 - y^2)^{\frac{1}{2}} - (R^2 - a^2)^{\frac{1}{2}}
\]  
(48)

which is analogous to eqn. (40) for a cylinder. Differentiating eqn. (48) with respect to time gives:

\[
\dot{z} = \frac{xV}{(R^2 - x^2 - y^2)^{\frac{1}{2}}}
\]  
(49)

where \(V\) is the forward velocity of rolling in the negative \(x\)-direction as before. By substituting from eqns. (48) and (49) into eqn. (39), we obtain the following expression for \(p\):

\[
p = \frac{E'}{L} \left\{ (R^2 - x^2 - y^2)^{\frac{1}{2}} - (R^2 - a^2)^{\frac{1}{2}} + \frac{\tau Vx}{(R^2 - x^2 - y^2)^{\frac{1}{2}}} \right\}
\]  
(50)

where the relaxation time \(\tau = \eta/E'\).
For small deflections, \( x^2 + y^2 < R^2 \), so that eqn. (50) can be expanded to give the simplified relationship:

\[
p = \frac{E' R}{2L} \left\{ \phi^2 - \frac{x^2}{R^2} - \frac{y^2}{R^2} + 2\beta \frac{x}{R} \left[ 1 + \frac{x^2}{2R} + \frac{y^2}{2R} \right] \right\}
\]

(51)

where \( \beta \) and \( \phi \) are defined as previously. To find the value of \( x(=x_0) \) at which the sphere loses contact with the viscoelastic material, we set \( p=0 \) in this equation, so that:

\[
\lambda = \beta - (\beta^2 + \phi^2 - y^2/R^2)^{\frac{1}{3}}
\]

(52)

where \( \lambda \) is again \((-x_0/R)\). We note that \( x \) is defined at any \( y \)-value and assumes the maximum value \( x_{0\text{max}} \) at \( y=0 \). If we substitute \( y=0 \) in eqn. (52) and then divide by \( \phi \), it becomes identical with the expression for a cylinder. When the \( y \)-dependence is taken into account, the distribution of \( x \) across the contact width is shown graphically in Fig. 17. The centre-line position for which \( y=0 \) is represented by the \( x \)-axis.

![Fig. 17. Limits of contact area for sphere rolling on viscoelastic plane.](image-url)

The left-hand side of eqn. (52) now becomes \((x_0/a)\), but we note that this ratio has no meaning in the case of a sphere, as seen from Fig. 17. The dimensionless contact asymmetry at any location \( y \) is given by the ratio \((x_0/a(1-y^2/a^2)^{\frac{1}{3}})\), and from eqn. (52):

\[
\frac{x_0}{a(1-y^2/a^2)^{\frac{1}{3}}} = \left(1 + \frac{\gamma^2}{(1-y^2/a^2)^{\frac{1}{3}}}\right)^{-\frac{1}{3}} - \frac{\gamma}{(1-y^2/a^2)^{\frac{1}{3}}}
\]

(53a)

We can also write in place of eqn. (38) which applies for a cylinder:

\[
\tan \delta = \frac{V\eta}{E'a(1-y^2/a^2)^{\frac{1}{3}}} = \frac{\gamma}{(1-y^2/a^2)^{\frac{1}{3}}}
\]

(54)

Substituting eqn. (54) in eqn. (53a) above gives:

\[
\frac{x_0}{a(1-y^2/a^2)^{\frac{1}{3}}} = (1 + \tan^2 \delta)^{\frac{1}{3}} - \tan \delta = \frac{x_{0\text{max}}}{a}
\]

(53b)
Thus, the dimensionless contact asymmetry ratio is constant across the contact width, and has the same value for a sphere or cylinder, being dependent only on the tangent modulus of the elastomer.

Experimental evidence in support of Fig. 17 is shown by the interference fringes surrounding the actual contact area for a glass sphere rolling on a viscoelastic resin base at different forward speeds (see Fig. 18). At low speeds, the contact area is circular and has its maximum value. An increase in $V$ produces size and shape effects within the contact area, both of which can be traced to the viscoelastic properties of the resin. The shape effect is given by the contact asymmetry ratio $[x_0/a(1-y^2/a^2)^{1/2}]$ previously discussed, which depends only on the tangent modulus. It is observed that this ratio constitutes only a relative measure of contact asymmetry, since the contact radius $a$ varies with speed $V$. The size effect occurs because of an effective "stiffening" of the elastomer at higher rolling speeds, which reduces the value of $a$.

Fig. 18. Experimental observation of variation in contact area: glass ball sliding at different speeds on epoxy resin.

Fig. 19. Dependence of width of contact area of glass ball on speed.
We note that the asymmetry effect takes place at modestly low forward travel speeds, and yields to the dominating influence of elastomer stiffening at higher speeds, which restores circularity to the contact area. Figure 19 shows that both the length and width of the contact area decrease with an increase in rolling speed. The circularity of the contact area is apparent from the convergence of these two curves at zero and high forward speeds.

To calculate the coefficient of hysteresis friction, it is necessary to return to eqn. (50), which describes the pressure distribution $p$ in the contact patch. The normal load $W$ and friction force $F_{\text{hyst}}$ are then readily computed as follows:

$$W = \int_{y=-a}^{y=a} \int_{x=-x_0}^{x=(a^2-y^2)^{1/2}} p \, dx \, dy$$

and:

$$F_{\text{hyst}} = \frac{M}{R} = \int_{y=-a}^{y=a} \int_{x=-x_0}^{x=(a^2-y^2)^{1/2}} px \, dx \, dy$$

where $M$ is the moment which opposes the rolling motion of the sphere. The coefficient of hysteresis friction, $f_h$, is then computed as the ratio $F_{\text{hyst}}/W$, and the final result$^{15}$ appears as:

$$f_h = C_2 \left[ \frac{W}{\eta VR} \right] \quad \text{(intermediate rolling speeds)}$$

$$f_h = C_3 \left[ \frac{W}{\eta VR} \right] \quad \text{(higher rolling speeds)}$$

As in the case of the cylinder, a more generalised treatment would necessitate the use of a combination of many elements, instead of a simple Voigt model as used here. This serves to broaden the peak of the friction versus retardation time curve shown in Fig. 15, and leads to a more realistic distribution of retardation times.

It is of interest to compare the above theory with the "elastic" theory of Greenwood et al.$^{11}$ at very high forward speeds. It has been shown$^{17}$ in the development of the Flom and Bueche theory that:

$$f_{\text{hyst}} = \frac{3\pi}{16} \phi$$

when the speed of travel is very great. Indeed, at such high speeds, the viscoelastic material does not have time to recover even partially behind the sphere, and the length of contact is reduced by one-half. It follows therefore that the work $\phi_2$ in the Greenwood theory which is expended in deforming the elastic base is never recovered, and $\alpha = 1$ in eqn. (16). Since $\phi$ in eqn. (57) is identical with $(a/R)$ in eqn. (16), it follows that the former estimate of the hysteresis or rolling coefficient exceeds the latter by a factor of $\pi$. Tabor explained this discrepancy in the case of a cylinder by pointing out that in his earlier work he underestimated $f_{\text{rolling}}$ (as it appears in eqn. (8)) by a factor of 3.5, and suggested that this might be less for a sphere. This appears to reconcile the two different treatments of rolling at higher sliding speeds, but it must again be pointed out that Greenwood's theory is inherently self-contradictory in attempting to apply elasticity theory to viscoelastic materials.

Figure 19 shows a sustained decrease in contact area with increasing speed for a sphere rolling on a viscoelastic resin base. Since the applied load is constant.
HYSTERESIS THEORIES

this implies an increase in the mean pressure in the contact area as rolling speed increases. Equation (39) indicates that at higher speeds a decrease in local deformation \( z \) is more than compensated for by increases in \( E', \eta \) and \( \dot{z} \), so that the Voigt model concept satisfactorily explains the required increase in pressure. Indeed, the restored circularity of the contact patch and its substantially decreased diameter indicate jointly the reduction in hysteresis at high sliding speeds, and the eight contact images in Fig. 19 show conclusively the rise, peak and fall in coefficient according as the speed of rolling is increased. Such resonance-type curves are typical of viscoelastic behaviour, as we have seen.

OTHER THEORIES

Unified theory

The unified theory of Kummer\(^{14}\) proposes semi-empirical and generalised equations by analogy with a corresponding theory for adhesion, and these equations are then combined to define a coefficient of hysteresis friction. We first express the hysteresis component of frictional force in the form:

\[
F_{\text{hyst}} = MJ
\]

where \( M \) is the total number of macroscopic sites or asperities, and \( J \) is the frictional resistance at each site. Equation (58) is analogous to eqn. (3) in a recent review paper on adhesion presented by the authors\(^3\). The number of asperities \( M \) encountered by the sliding elastomer is given by:

\[
M = \gamma M_0 = 4\gamma A/\lambda^2
\]

where \( M_0 \) is the maximum number of asperities in the given surface, \( \gamma \leq 1 \) is an asperity packing factor, \( A \) the actual area of contact and \( \lambda \) the mean wavelength of surface asperities.

Now, the frictional resistance \( J \) can be expressed as:

\[
J = E_d/\lambda = QE''/\lambda
\]

where \( E_d \) is the energy dissipated per site, \( Q \) the volume of elastomeric material deformed over each asperity during sliding, and \( E'' \) the loss modulus of the material. By writing \( W = \bar{p}A \), where \( \bar{p} \) is the mean areal pressure on an asperity, the coefficient of hysteresis friction from eqns. (58), (59) and (60) becomes:

\[
f_h = \frac{F_{\text{hyst}}}{W} = \left( \frac{M}{A} \right) \left( \frac{J}{\bar{p}} \right)
\]

\[
= \frac{4\gamma}{\lambda^3} \left( \frac{E_d}{\bar{p}} \right) = \frac{4\gamma Q}{\lambda^3} \left( \frac{E''}{\bar{p}} \right)
\]

Figure 20 indicates by shading the projection of \( Q \) in the direction of sliding for cone, sphere and cylinder. For each of these basic asperity shapes, the "draping" height \( \delta \) can be represented by an equation of the form:

\[
\delta = C_A(\bar{p}/E)^n
\]

If the deformed volume \( Q \) is now expressed in terms of \( \delta \), and then this substitution
TABLE II
VALUES OF $C_4$, $m$, $C_5$ AND $n$ FOR VARIOUS ASPERITY SHAPES

<table>
<thead>
<tr>
<th>Asperity shape</th>
<th>$C_4$</th>
<th>$m$</th>
<th>$C_5$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>$9\pi^2 R (1-v^2)^2/16$</td>
<td>2</td>
<td>$81\pi^5 (1-v^2)^4/2048$</td>
<td>3</td>
</tr>
<tr>
<td>Cylinder</td>
<td>$64R (1-v^2)^2/\pi^2$</td>
<td>2</td>
<td>$256\cdot2^4 (L/R) (1-v^2)^3$</td>
<td>2</td>
</tr>
<tr>
<td>Cone</td>
<td>$2\cdot2^4 \frac{(K/E) \cdot (W/E) \cdot (d/\bar{c})}{3 \cdot \pi} (1-v^2)^2 \frac{(L/R) (1-v^2)^3}{ER} \frac{(d/\bar{c})}{3 \cdot \pi}$</td>
<td>1</td>
<td>$2 \cdot 2^4 \frac{(L/R) (1-v^2)^3}{ER} \frac{(d/\bar{c})}{3 \cdot \pi} \frac{(K/E) \cdot (W/E) \cdot (d/\bar{c})}{3 \cdot \pi}$</td>
<td>2</td>
</tr>
</tbody>
</table>

for $\delta$ made in eqn. (62), the final form of eqn. (61) is as follows:

$$f_h = 4 C_5 \gamma (\bar{p}/E)^* \tan \delta$$

(63)

where the constants $C_4$ and $C_5$ and the indices $m$ and $n$ have the values listed in Table II.

The index $n$ in eqn. (63) according to this theory would then appear to lie between the values 2 and 3 for a random asperity, assuming that the latter exhibits a contour that lies between the extremes of infinite and zero sharpness. This theory can be improved by replacing the constant Young's modulus $E$ in eqn. (63) by the complex modulus $E^*$, which is both speed- and temperature-dependent, but there are severe limits to the speed range within which the theory might be considered valid.

The unified theory of hysteresis friction must be considered unsatisfactory for several reasons:

(a) The concept of a “deformed volume” is indefinite and therefore inconclusive. Indeed, the boundaries of deformed material extend well beyond the interface indicated by the shading in Fig. 20, and it would therefore be more appropriate to speak of grooved volume rather than deformed volume.

(b) Even with this correction, grooved volume is extremely insensitive to speed effects (particularly at lower speeds of sliding), and is therefore an inappropriate measure of hysteresis friction.

(c) In the case of conical asperities, the “constant” $C_5$ is a function of viscoelastic properties (see Table II above), so that the exact viscoelastic dependence of $f_h$ in eqn. (63) is not clearly evident. It is difficult, of course, to disprove this theory because of the generality of the terms used, and it must be regarded as approximate.

Relaxation theory

The relaxation theory of hysteresis proposed by Hegmon is based upon an energy method of analysis, and assumes a simple Maxwell model of viscoelastic behaviour, although this is not specifically mentioned in the theory. Because of inaccuracies in the original theory, the following modified form is presented. Consider an elongated and rigid spherical slider moving on the flat surface of an
elastomer, as shown in Fig. 21. The surface of the slider is taken to be perfectly smooth, and temperature effects are taken as constant within the contact area during sliding. It is also assumed that the adhesion component of friction is negligibly small.

The slider is divided into three sections for purposes of analysis. Corresponding to the first section, work is done by the slider on the elastomer which is deformed as shown. Within the second section, this deformation is preserved and no work is done either on the elastomer or the slider. In the third section, the elastomer recovers from the deformation and returns energy to the slider. An energy balance on the entire deformation process is as follows:

Section 1: Work in = Energy stored + Energy dissipated
Section 2: Work is zero, except for dissipation of part of the stored energy due to stress relaxation
Section 3: Work out = Energy stored − Energy dissipated during recovery of the deformation

Summing over the three sections, we find that:

Work out = Work in − Σ Energy dissipated

Thus, in principle there are two ways of finding the coefficient of hysteresis friction
from the energy balance. One may either try to sum the dissipated energies for each of the three sections, or one may calculate the difference between the work input to the elastomer during deformation and the work output during the recovery period. The second approach is summarised in the following paragraphs.

Applying eqn. (14) for the work $\phi_1$, done by the hemispherical slider in section 1 for unit forward distance, we can write:

$$\phi_1 = \frac{3W_1 a_1}{16R}$$

(14a)

using the notation in Fig. 21. In moving forward a distance $a_1$, the work done is given by:

$$(W.D.)_1 = \frac{3W_1 a_1^2}{16R}$$

(64)

A similar procedure applies to section 3, where work of amount $\phi_3$ is done by the elastomer on the slider in moving forward unit distance. Thus,

$$\phi_3 = \frac{3W_3 a_3}{16R}$$

(14b)

and for a travel distance $a_1$:

$$(W.D.)_3 = \frac{3W_3 a_1 a_3}{16R}$$

(65)

The energy loss or frictional energy can be written as the difference between the work input and the work returned to the slider. Thus, from eqns. (64) and (65):

$$W_f = (W.D.)_1 - (W.D.)_3$$

$$= (3/16R) [W_1 a_1^2 - W_3 a_1 a_3]$$

(66)

If we assume that the relaxation process occurring between sections 1 and 3 typifies a Maxwell model, we can then relate the mean normal pressures $\bar{p}_3$ and $\bar{p}_1$ in sections 3 and 1 respectively, thus:

$$\bar{p}_3 / \bar{p}_1 = \exp (-b \cdot \tau)$$

(67)

where $b$ is the time during which the pressure relaxes from $\bar{p}_3$ to $\bar{p}_1$, and $\tau = \eta / E$ is the relaxation time for the model. Putting $b = a_2 + 4(a_1 + a_2)^2/3\pi$, $\tan \delta = F_0 / \eta \omega$, and $t_b = b / V$, we obtain from eqn. (67):

$$\frac{\bar{p}_3}{\bar{p}_1} = \exp \left[ -\frac{b \omega \tan \delta}{V} \right]$$

(67a)

Within section 3, the elastomer recovers from the indentation $z_1$ through a vertical distance $z_3$, as shown in Fig. 21. In the absence of viscoelastic effects it is apparent that $z_1 = z_3$ and complete recovery occurs. However, for the general case of retarded elasticity, it can be shown that

$$\left( \frac{z_3}{z_1} \right) = \exp \left[ -\frac{t_a}{\tau} \right] = \exp \left[ -\frac{a_3 \omega \tan \delta}{V} \right]$$

(68)
Since $a_1^2 = 2Rz_1$ and $a_3^2 = 2Rz_3$, we can write also

$$\left(\frac{a_3}{a_1}\right)^2 = \frac{z_3}{z_1} \quad (69)$$

Now, the load ratio $W_3/W_1$ is given by

This equation is not yet complete, because we must relate the partial load $W_1$ acting on section 1 of the contact area to the total applied load $W$. We achieve this by assuming that $W_2 \sim a_2$ and $(W_1 + W_2) \sim R$. Thus, it has been shown that:

$$\frac{W_3}{W_1} = \left(\frac{\bar{p}_3}{\bar{p}_1}\right) \left(\frac{a_3}{a_1}\right)^2 \quad (70)$$

so that by substituting from eqns. (67a) and (69) in eqn. (70), we find that:

$$\frac{W_3}{W_1} = \exp \left[\frac{-(a_3 + b) \omega \tan \delta}{V}\right] \quad (71)$$

If we then arrange the energy loss eqn. (66), making appropriate substitutions from eqns. (67a) and (71), we obtain:

$$W_F = \frac{27\pi^2}{64} \left(1 - v^2\right)^2 \frac{W_1 R \bar{p}_1^2}{E^2} \left\{1 - \exp \left[\frac{-(a_3 + 2b) \omega \tan \delta}{V}\right]\right\} \quad (72)$$

$$W_1 = \frac{W}{(1 + a_2/R)\{1 + \exp[-(a_3 + b) \omega \tan \delta/V]\}} \quad (73)$$

This equation shows that for $a_2 = 0$ and $\tan \delta = 0$, $W_1 = W/2$ as should be the case. Substitution of eqn. (73) in eqn. (72) gives:

$$W_F = \frac{27\pi^2(1-v^2)^2 W R \bar{p}_1^2}{64E^2(1 + a_2/R)\{1 + \exp[-(a_3 + b) \omega \tan \delta/V]\}} \quad (74)$$

Now, the total sliding distance $L$ is given by:

$$L = a_1 + a_2 + a_3$$

$$= a_1 \left[1 + \frac{a_2}{a_1} + \frac{a_3}{a_1}\right] = \alpha a_1 \quad (75)$$

where $\alpha > 0$, and it can be shown from eqns. (10) and (11) that:

$$\alpha a_1 = \frac{3\pi}{4} \alpha (1 - v^2) \frac{\bar{p}_1 R}{E} \quad (76)$$

By dividing the work of friction $W_F$ in eqn. (74) by the sliding distance $\alpha a_1$, and by the total load $W$, we obtain finally the coefficient of hysteresis friction. Thus:

$$f_h = \frac{W_F}{\alpha a_1 W}$$

$$= \frac{9\pi(1-v^2)\bar{p}_1 \{1 - \exp[-(a_3 + 2b) \omega \tan \delta/V]\}}{16\alpha E(1 + a_2/R)\{1 + \exp[-(a_3 + b) \omega \tan \delta/V]\}} \quad (76)$$

The pressure $\bar{p}_1$ under the front of the slider cannot be measured easily, but it will
always be greater than the mean pressure $\bar{p}$ in the whole contact area. Let us define a factor $\beta$, such that:

$$\bar{p}_1 = \beta \bar{p}$$

where $\beta > 1$. With this substitution, eqn. (76) for an elongated spherical slider becomes:

$$f_h - \frac{9\pi(1-v^2)}{16(1+a_2/R)} \left( \frac{\beta}{\alpha} \right) \left( \frac{\bar{p}}{E} \right) \left( 1 - \exp \left[ -\frac{(a_3+2b)\omega \tan \delta}{V} \right] \right)$$

$$\left( 1 + \exp \left[ -\frac{(a_3+b)\omega \tan \delta}{V} \right] \right)$$

(77)

It must be stated here that the authors have found Hegmon's original theory unconvincing, partly because of the use of the "deformed volume" concept of Kummer and partly because of certain assumptions made in the derivation. We believe that the treatment presented here is more systematic and thorough. However, the mixture of elastic theory in Sections 1 and 3 (see Fig. 21) with elementary viscoelastic theory in Section 2 is somewhat disturbing, since we assume energy dissipation to occur only within the central elongated section. It is certain that all three sections contribute to an energy dissipative mechanism. Hegmon obtained an expression for $f_h$ which differs from eqn. (77) in that the numerical constants have different values and the Poisson's ratio $\nu$ does not appear. He has also claimed to have obtained similar expressions for conical and cylindrical asperities, but the details of the derivation have yet to appear in the published literature.

If we assume that blunt and elongated asperities would yield an equation similar to eqn. (77) above, then we might express $f_h$ in the following general form:

$$f_h = A(\beta/\alpha)(\bar{p}/E) \left( 1 - \exp \left[ -B\omega \tan \delta/V \right] \right)$$

(78)

where $A$ is a numerical factor dependent on the shape of the asperity, and the length dimension $B$ is proportional to contact length. If the first two terms of the expansion of the exponential term in eqn. (78) are taken, we can also write this equation as follows:

$$f_h = \zeta A(\beta/\alpha)(\bar{p}/E) \tan \delta$$

(79)

where $\zeta = B/\alpha L$ is a length ratio, and $L = V/\omega$ is approximately equal to the contact length, $x_{a1}$.

It is seen that the form of eqn. (79) is remarkably similar to that of eqn. (63), despite the different approaches taken in each theory and the lack of rigour in both. We note the proportionality between $f_h$ and $\tan \delta$ in each theory, which is a further confirmation of the viscoelastic nature of hysteresis friction. It also appears to be significant that whereas in all theories for adhesion and hysteresis the coefficient of friction is proportional to $\tan \delta$, the adhesion component is generally proportional to the ratio $(E/\bar{p})$ and the hysteresis component to its reciprocal. Perhaps also it is more correct to substitute the complex modulus $E^*$ for $E$ in the previous equations to allow for velocity and temperature variations, although $E$ can always be regarded as a mean value of $E^*$ during the sliding operation.

**RECENT THEORIES**

*Mechano-lattice or network theory*

The most significant theory of hysteretic friction appearing in the recent
literature is due to Yandell\textsuperscript{20}, who offered a relatively complex network or mechano-lattice analogy to explain the frictional resistance of rubber slipping on an irregular lubricated surface. The analysis permits large deformations and any value of Poisson's ratio, rigidity or damping factor. Figure 22 (a) shows a simplified model of a hysteresis loop using spring and Coulomb friction elements (see insert), and Fig. 22 (b) a unit of the mechano-lattice arrangement. We observe that the basic viscoelastic model used is a variation on the modified Maxwell unit, with the viscous element replaced by Coulomb damping (see Fig. 4 (d) earlier). Each lattice unit is made up of eight such basic model elements as shown, having two horizontal, two vertical, two shear and two volume elements. The volume and shear behaviour are represented by two crosses free to rotate. It is noted from this arrangement that shear strains in a unit are functions of differences in diagonal lengths and volume strains are functions of changes in the sum of diagonal lengths.

Figure 23 shows an assembly of 264 of the mechano-lattice units depicted in Fig. 22 (b). They are connected at their joints to simulate a long section of rubber sliding under conditions of plane stress. All the boundary joints except those at the lower edge are free to move, and those contacting the sawtooth asperity are deflected to the shape of the latter. It can be seen in Fig. 23 that generally the units to the left of the asperity have been loaded whereas those to the right are not.
As they move across the asperity during sliding, corresponding elements experience periods of loading and deflection and generate hysteresis loops which are qualitatively similar to that shown in Fig. 22 (a). The area of each hysteresis loop is governed by the damping factor of the rubber and the maximum elastic stress range experienced by the element. A computer programme was devised\textsuperscript{20} to calculate and store the lengths and rotations of all the elements and hence to compute elastic and frictional forces for each element during each stage of loading by means of subroutines. The coefficient of hysteretic friction $f_h$ is determined by dividing the vectorial sum of the horizontal forces acting on the joints which contact the asperity by the vertical forces on these joints. Each programme required extensive computer time because of its complexity, and various triangular prismatic and cylindrical asperities were used in combination with rubbers having different damping factors. The results generally confirm the predictions of earlier and simpler viscoelastic theories of hysteretic friction.

The rigour of Yandell's method is to be admired, and the network theory identifies precisely the characteristic length dimension $L$ in relationships of the type given previously by eqn. (30) or eqn. (39). This is because each lattice unit identifies particular directions of strain with its horizontal, vertical and shear elements. One wonders why the author chose Coulomb rather than viscous damping in the basic model element of Fig. 22 (a), since there is evidence that the bulk of internal hysteretic losses can be attributed to viscous damping effects and only a small fraction at most is caused by Coulomb or dry friction. Furthermore, the use of a simple Voigt element in place of that shown in the insert of Fig. 22 (a) permits a much closer simulation of the actual hysteresis loop shown in the same figure, and finally the mathematical complexity of the network theory is reduced.

A more serious criticism of the Yandell theory derives from a distinct downgrading of the complementary role of adhesion in establishing frictional forces. Whereas the author has been careful to specify lubricated conditions at frictional contacts, the corresponding reduction in adhesion does not imply that it can be neglected entirely. Consider the sliding of a rubber block past a textured asperity\textsuperscript{20} as shown in Fig. 24. It seems reasonable to claim that 80\% of hysteretic energy
HYSTERESIS THEORIES

Rubber

Total texture

Coarse texture

Fine texture

Rubber

Fig. 24. The separation of total texture into coarse and fine components during sliding friction.

is dissipated within the hatched areas for such a sliding system, and we may identify the separate contributions of coarse (or macro-) and fine (or micro-) texture to hysteresis friction by the superposition technique depicted in the figure. Yandell has used the term "texture saturation" to describe the interaction of the rubber and fine texture of the asperity as illustrated by the right-hand picture in Fig. 24, and has claimed to explain the area and load dependence of adhesion with this model. This conclusion is physically apparent and constitutes no advance nor change in fundamental thinking on this subject. Here we wish simply to emphasise that adhesion largely accounts for the friction corresponding to the fine texture schematic in Fig. 24, and the importance of the area effect has been stressed in a previous review.

In the case of a road surface, Yandell commendably maintains that there is a multiplicity of superimposed scales of texture, so that the resultant hysteretic friction force is made up from a large number of friction-speed curves with each having a peak at a different speed. For a given sliding speed of operation of a pneumatic tyre, the total friction force due to hysteresis is a superposition of the contributions from the different scales of texture on road asperities. Yandell insists, however, that the lesser or finer scales of texture contribute substantially more to the total hysteretic friction than the coarser textures, since for a given sliding speed \( V \) the frequency of deformation \( \omega \) is correspondingly greater. We emphatically disagree with this conclusion, since it leads ultimately to the absurd situation where a virtually smooth surface has a maximum hysteresis contribution! The very definition of hysteresis as the friction arising from energy dissipation in bulk implies necessarily a diminution for progressively finer scales of texture.

COMPLEX VOIGT MODEL THEORY

This theory is a simplified alternative to using an array of Voigt models in series or Maxwell-models in parallel to simulate the dynamic behaviour of real viscoelastic materials. In essence, the mechanical model used is a Voigt model as depicted in Fig. 4 but with variable rather than constant spring and dashpot parameters. The variability of the latter is designed to accommodate frequency and temperature effects in the following manner. Assuming initially a modified Maxwell model of viscoelastic behaviour (see Fig. 4), we can write:

\[
\dot{p} + \left( \frac{E_2}{\eta} \right) p = \left( \frac{E_1 + E_2}{L} \right) \dot{z} + \left( \frac{E_1 E_2}{L \eta} \right) z
\]
with the usual notation. This equation cannot be reduced to an explicit dependence of pressure \( p \) on extension \( \tau \), but we can surmount this difficulty by assuming a sinusoidal variation in pressure and deflection, thus:

\[
p = p^* e^{i\nu} \quad \text{and} \quad z = z^* e^{i\nu}
\]

where \( p^* \) and \( z^* \) are complex quantities. Substituting these relationships in eqn. (80) and defining the complex modulus \( E^* \), we obtain:

\[
E^* = \frac{pL}{z} = \frac{[E_1 E_2 + j\omega (E_1 + E_2)] L}{(E_2 / \eta + j\omega)} = \frac{[E_1 E_2^2 + \omega^2 \eta^2 (E_1 + E_2)] + j \left[ \frac{\omega \eta E_2^2}{E_2^2 + \omega^2 \eta^2} \right]}{E'} \quad \text{and} \quad \frac{\omega \eta E_2^2}{E_2^2 + \omega^2 \eta^2} = E''
\]

(81)

and the tangent modulus becomes:

\[
\tan \delta = \frac{E''}{E'} = \frac{E_2 \omega}{(E_1 E_2 / \eta) + \omega^2 \eta (E_1 + E_2 / E_2)}
\]

(82)

We observe at once that the real and imaginary parts of the complex modulus \( E' \) and \( E'' \) are now no longer constants but functions of frequency, and the same applies to the tangent modulus. Putting \( \eta' = E'' / \omega \) and re-writing eqn. (39) for a Voigt model (with \( E \) and \( \eta \) replaced by \( E' \) and \( \eta' \) respectively), we obtain:

\[
p = (E'/L)z + (E''/\omega L)z
\]

(83)

where the primed quantities are now functions of frequency in accordance with eqn. (81). This complex Voigt representation of viscoelastic performance closely simulates actual material behaviour with considerably less mathematical complexity than would be the case if arrays of simple Voigt models were used. This technique has been used successfully to predict the variation of contact area with increasing speed for cylinders and spheres rolling on a viscoelastic plane, and to demonstrate the characteristic features of hysteresis friction as speed is varied.

CONCLUSIONS

We conclude this extensive review article on hysteresis friction by summarising the most important theoretical advances made within the past two decades. The early “elastic” theory of Greenwood\(^\text{11}\) inherently contradicts itself, since if elasticity is assumed there will be complete recovery of elastomeric materials behind rolling spheres and cylinders, and the resulting symmetrical pressure distribution in the contact area precludes any coefficient of rolling friction. To explain this contradiction, the term “elastic hysteresis” has been used which is a further contradiction. The theory at best is applicable only at very low speeds of sliding
and is thus self-defeating. The various viscoelastic theories presented offer a more logical and systematic explanation of hysteretic frictional behaviour at speeds of practical interest. While these theories are similar in the sense that a simplified model of viscoelastic behaviour is initially assumed (usually a Voigt or Maxwell model), their very simplicity illustrates clearly the necessary viscoelastic implications of hysteresis. Hegmon’s relaxation theory\textsuperscript{19} contains several inaccuracies in the original presentation, and errors in energy loss calculations. The use of the “deformed volume” concept as used by both Kummer\textsuperscript{14} and Hegmon\textsuperscript{19} is not recommended, not only because the boundaries of such are difficult if not impossible to define, but it also must necessarily be insensitive to low speed variations. Yandell\textsuperscript{20,21} extended the simpler concepts to a relatively complex network of spring and dashpot elements, but while the approach and method used are commendable, the theory offers little additional insight beyond the early theories pioneered by Norman. A simplified Voigt modelling of viscoelastic behaviour developed by one of the authors has the advantage of simulating actual material performance with a minimum of mathematical complexity.

The role of Coulomb damping in internal hysteresis has not been discussed in this article, partly because of its relatively minor role in the total energy dissipative process and partly because of space limitations. There is no doubt that Coulomb damping is present in systems commonly identified by viscous damping alone, but the overall contribution to hysteretic friction is small and in most cases negligible.

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REFERENCES

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