Optimal consumption—portfolio problem with CVaR constraints

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The optimal portfolio selection is a fundamental issue in finance, and its two most important ingredients are risk and return. Merton’s pioneering work in dynamic portfolio selection emphasized only the expected utility of the consumption and the terminal wealth. To make the optimal portfolio strategy be achievable, risk control over bankruptcy during the investment horizon is an indispensable ingredient. So, in this paper, we consider the consumption-portfolio problem coupled with a dynamic risk control. More specifically, different from the existing literature, we impose a dynamic relative CVaR constraint on it. By the stochastic dynamic programming techniques, we derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation. Moreover, by the Lagrange multiplier method, the closed form solution is provided when the utility function is a logarithmic one. At last, an illustrative empirical study is given. The results show the distinct difference of the portfolio strategies with and without the CVaR constraints: the proportion invested in the risky assets is reduced over time with CVaR constraint instead of being constant without CVaR constraints. This can provide a good decision-making reference for the investors.

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1. Introduction

Portfolio selection is an interesting and important issue in finance. It studies how to allocate an investor’s wealth among a basket of securities to maximize the return and minimize the risk. In 1952, Markowitz firstly used variance to measure the risk and proposed the so-called mean-variance (MV) model for the static portfolio selection problem [1], which laid the foundation of modern portfolio theory and inspired a great number of extensions and applications. However, the MV model was criticized for its inapplicability. On the one hand, besides the difficulty of the covariance matrix’s computation, variance, which takes the deviation both up and down from the mean without discrimination as the risk, is incompatible with the actual portfolio situations. On the other hand, the static buy-and-hold portfolio strategy which makes a one-off decision at the beginning of the period and holds on until the end of the period is usually inappropriate for a long investment horizon. To overcome the classical MV model’s shortcomings, measures measuring the downside risk were proposed, such as semi-variance, value-at-risk (VaR) [2], conditional value-at-risk (CVaR) [3], etc. Meanwhile, the extension to the dynamic case is an issue which has been extensively studied as well.

Among all downside risk measures, VaR is popular in practice of risk management by virtue of its simplicity. It is defined as the quantile of the loss at a certain confidence level. But it has been criticized for its lose of subadditivity. And as its modification, CVaR, which is defined as the average value of the loss greater than VaR, has attracted increasing attention in recent years on the fact that it is a coherent risk measure [4] and mean-risk model based on it can be solved easily by linear programming method.

As for extensions from single-period portfolio to dynamic cases, multi-period setting and continuous-time circumstances are included. Merton [5] and Samuelson [6] extended the static model to a continuous-time setting by utility functions and stochastic control theory respectively. Since then, the literature on dynamic portfolio selection has been dominated by expected utility maximization model. Moreover, multi-period and continuous-time MV model have been tackled by embedding technique [7] and linear quadratic approach [8] respectively in 2000. Afterwards, dynamic MV model became another hot research topic. From existing literature, it is easy to find that the optimal consumption-portfolio problem in continuous time is an interesting and appealing issue [9–13]. And in this paper, we intend to study this problem further.

As we know, in continuous-time setting, the movement of risky assets is always assumed to follow some stochastic process, and based on this assumption, stochastic control theory as well as the martingale method is the main solving method. Among all stochastic processes, geometric Brownian motion is used widely. But as shown by empirical analysis, the actual return distribution of the risky assets has the properties of aiguiles and fat tails. Scholars engaged in econophysics have carried out many studies on this issue and proposed several more appropriate models, e.g. [14–19].
The martingale method decomposes the dynamic optimization problem into two sub-problems [20,21], while stochastic control method [22–24] obtains the optimal control by introducing an optimal value function and is more intuitively. It should be noticed that a bankruptcy occurs when the wealth falls below a predefined disaster level at any moment during the investment horizon. And when an investor is in bankruptcy, he is unable to pursue further investment due to his high liability and low credit. To control the risk of bankruptcy and execute the investment strategy, imposing a dynamic risk constraint on the instantaneous wealth throughout the investment is obviously needed. Fortunately, the optimal portfolio model coupled with a dynamic VaR constraint, which was proposed by Yiu recently [25], provides a new perspective on the risk management and gives us some inspiration. But, to our knowledge, literature on the dynamic risk constraint is still scarce [24,26,27]. So, in this paper, we study the optimal consumption and portfolio problem further. More specifically, we study the problem of maximization the expected discounted utility of both consumption and the terminal wealth by imposing a dynamic relative VaR constraint on it, which is more appropriate in practice.

The rest of this paper is organized as follows. In Section 2, we present the market setting and problem formulation, where include the analytical expression of VaR. In Section 3, we derive the Hamilton–Jacobi–Bellman (HJB) equation for general utility functions using the dynamic programming technique. In Section 4, we employ the Lagrange multiplier method to tackle the VaR constraint and present the closed-form solutions for logarithmic utility function. In Section 5, we give an illustrative empirical study. At last, we summarize the paper.

2. Market setting and problem formulation

Consider a financial market with $n + 1$ assets: one risk-free asset and $n$ risky assets. The price process $S_0(t)$ of the risk-free asset follows the following ordinary differential equation

$$
\begin{align*}
\frac{dS_0(t)}{S_0(t)} &= r, \quad t \in [0, T] \\
S_0(0) &= 1
\end{align*}
$$

where $r > 0$ is the risk-free rate, $T > 0$ is the terminal time of the investment. The price processes of the $n$ risky assets satisfy the following stochastic differential equations

$$
\begin{align*}
\frac{dS_i(t)}{S_i(t)} &= \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t) \\
S_i(0) &= S_i, \quad i = 1, \ldots, n
\end{align*}
$$

where $\mu = (\mu_1, \ldots, \mu_n)'$ is the appreciation rate of returns, $\sigma = (\sigma_{ij}, i, j = 1, \ldots, n)$ is the volatility rate of returns satisfies the non-degenerate condition, $B(t) = (B_1(t), \ldots, B_n(t))'$ is an $n$ dimensional standard Brownian motion with $B_1(t)$ and $B_2(t)$ mutually independent for $i \neq j$. Let $(\Omega, \mathcal{F}, P, \{F_t\}_{t \geq 0})$ be a filtered complete probability space, where $F = \{F_t; t \geq 0\}$ is the natural filtration generated by the $n$ dimensional Brownian motion $B(t)$, $F_t = \sigma(B(s); 0 \leq s \leq t)$ is a $\mathcal{F}$-field representing the information available up to time $t$. Let $L^2(0, T; R^n)$ be the set of all $R^n$ valued, $\{F_t; t \geq 0\}$-adapted and square integrable stochastic processes.

In what follows, we consider an investor entering the market with initial wealth $w_0$. The investor allocates his wealth in these $n + 1$ assets continuously and withdraws some funds out of the portfolio for consumption within the time horizon $[0, T]$, where the investor's objective is to maximize his expected discounted utility of consumption and terminal wealth. Denote the consumption process as $c(t)$ and the control process as $\alpha(t)$, $c(t) = \{c_1(t), \ldots, c_n(t), c(t)\}$, where the components of $c(t)$ are proportions of the investor's wealth invested in the risky assets, $c_i(t)$ is the consumption rate. A control strategy $\{\pi(t), c(t)\}$ is admissible if $[\pi(t), c(t)] \in L^2(0, T; R^{n+1})$ is $F_t$ progressively measurable. Let $W^{\pi,c}(t)$ be the investor’s wealth at time $t$. Then the wealth process satisfies the following stochastic differential equations

$$
\begin{align*}
\frac{dW^{\pi,c}(t)}{W^{\pi,c}(t)} &= \left\{ \sum_{i=1}^{n} \pi_i(t) \left[ \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t) \right] \\
&+ \left[ 1 - \sum_{i=1}^{n} \pi_i(t) \right] rd - c(t) dt \right\} dt + \pi'(t) dB(t) \\
W^{\pi,c}(t) &= W^{\pi,c}(0) + \int_{0}^{t} \pi'(s) dB(s)
\end{align*}
$$

where $\mu = (\mu_1 - r, \ldots, \mu_n - r)'$ is the excess rate of return. By Itô lemma, we can obtain the unique solution of (2.1) as

$$
\begin{align*}
W^{\pi,c}(t) &= w_0 \exp \left\{ \int_{0}^{t} Q(s, \pi'(s), c(s)) ds + \int_{0}^{t} \pi'(s) dB(s) \right\}
\end{align*}
$$

where $t \in [0, T]$ and $Q(t, \pi(t), c(t)) = \pi'(t) \mu + r - c(t) - (1/2)\pi'(t)\sigma^2$. For sufficiently small $t > 0$, we may deem that the control process $[\pi'(s), c(s)]_{[0,t]}$ is unchanged and stays at the present value all the time interval $[t, t + \tau]$, which is reasonable in practice. Then,

$$
\begin{align*}
W^{\pi,c}(t + \tau) &= W^{\pi,c}(t) \exp \left\{ Q(t, \pi'(t), c(t)) \tau + \frac{1}{2} \left\| \pi'(t) \sigma \right\|^2 \right\}
\end{align*}
$$

where $L(t) = W^{\pi,c}(t) - W^{\pi,c}(t + \tau) = W^{\pi,c}(t) \left[ 1 - \exp \left\{ Q(t, \pi'(t), c(t)) \tau + \frac{1}{2} \left\| \pi'(t) \sigma \right\|^2 \right\} \right]$. For the given time $t$, the random variable $\pi'(t)\sigma(B(t + \tau) - B(t))$ follows a normal distribution with mean zero and standard deviation $\left\| \pi'(t) \sigma \right\| \sqrt{\tau}$. Hence, for the given confidence level $\beta \in (0.5, 1]$, the VaR of the loss can be written as

$$
\text{VaR}_\beta(L(t)) = W^{\pi,c}(t) \left\{ 1 - c_{2}(t) \exp \left\{ Q(t, \pi'(t), c(t)) \tau + \frac{1}{2} \left\| \pi'(t) \sigma \right\|^2 \tau \right\} \right\}
$$

where $c_2(t) = \Phi^{-1}(1 - \beta) - \Phi^{-1}(1 - \beta)$. $\Phi$ and $\Phi^{-1}$ are the cumulative distribution function and its inverse function of the standard normal distribution, respectively. The derivation of (2.3) is given in the appendix. By introducing another parameter $\beta \in (0.1)$ to be a benchmark of VaR constraint, we have the relative VaR constraint as

$$
\begin{align*}
1 - c_{2}(t) \exp \left\{ Q(t, \pi'(t), c(t)) \tau + \frac{1}{2} \left\| \pi'(t) \sigma \right\|^2 \tau \right\} \leq \beta.
\end{align*}
$$

We call it relative VaR since here $\text{VaR}_\beta(L(t))/W^{\pi,c}(t)$ is used instead of VaR.

Let $U_1$ and $U_2$ be utility functions of the consumption and terminal wealth respectively. Both $U_1$ and $U_2$ are twice differentiable, increasing, concave functions and satisfying $U_1(0^+) = U_1(0^+) = +\infty$ and $U_1(+\infty) = U_1(+\infty) = 0$, where $U'(0^+) = \lim_{x \to 0^+} U'(x)$, $U'(+\infty) = \lim_{x \to +\infty} U'(x)$. Let $\rho > 0$ be the discount factor. Then the expected discounted utility is

$$
E \left[ \alpha \int_{0}^{T} e^{-\rho t} U_1(c(t)) dt + (1 - \alpha) e^{-\rho T} U_2(W^{\pi,c}(T)) \right],
$$

where $\alpha \in [0, 1]$ is a trade-off factor indicating the investor’s emphasis on consumption and terminal wealth. If $T = +\infty$, then
(2.5) is an infinite time horizon problem (i.e. lifetime consumption problem) and in this case it is sufficient to think about only consumption utility, see [9]. But it is more realistic to analyze a finite time horizon. So in this paper, we let T be a finite number. Sum up, we can obtain our model as follows:

$$
\max_{\{\pi(t), c(t)\in [0,1]\}} E \left[ \alpha \int_0^T e^{-\rho t} U_1(c(t)) dt + (1-\alpha)e^{-\rho T} U_2(W^{\pi_\rho c}(T)) \right] \\
\text{s.t.} (2.1), (2.4) 
$$

(2.6)

3. Hamilton–Jacobi–Bellman (HJB) equation

In order to employ the optimal control techniques of dynamic programming, we define the optimal function as

$$
V(t, w) = \max_{\{\pi(t), c(t)\in [0,1]\}} E_t \left[ \alpha \int_0^T e^{-\rho t} U_1(c(s)) ds + (1-\alpha)e^{-\rho T} U_2(W^{\pi_\rho c}(T)) \right] \\
\text{s.t. (3.1)}
$$

(3.1)

where $t \in [0, T], u(t)\in [0,1]$ represents all the admissible strategies in the time interval $[t, T], E_t$ stands for the conditional expectation and $W^{\pi_\rho c}(T) = w$ is known. Obviously, we obtain the whole portfolio strategy on condition that $t = 0$. By Ito lemma, we can obtain the HJB equation of (3.1) as

$$
V_t + \max_{\{\pi(t), c(t)\in [0,1]\}} \left\{ \alpha \frac{\partial}{\partial \pi} V(t, \pi(t), c(t)) + \frac{1}{2} \frac{\partial^2}{\partial \pi^2} V(t, \pi(t), c(t)) \right\} = 0
$$

(3.2)

with the boundary condition $V(T, w) = (1-\alpha)e^{-\rho T} U_2(w)$. Theoretically, once the utility function is given, the optimal control process can be obtained by solving the HJB equation.

There are many risk-averse utility functions, such as exponential function, logarithmic function, power function and HARA utility function, etc. In [5], Merton chose a power function and derived the analytical solution for the value function $V(t, w)$ by a trial function in the form of separable variables. And most existing literature employs power utility functions, see [26,27]. In the next section, we choose the logarithmic utility function, which is frequently used in economics, as our utility function and derive the closed form solution.

4. Optimal control process under logarithmic utility function

For simplicity, here we let $\alpha = 0.5$. That is, the consumption utility and the terminal wealth utility have equal importance. Then we may delete $\alpha$ and $(1-\alpha)$ in Problem (2.6). Let $U_1(\cdot) = U_2(\cdot) = \log(\cdot)$. Problem (2.6) can be reduced to

$$
\max_{\{\pi(t), c(t)\in [0,1]\}} \left\{ \frac{1 - e^{-\rho T}}{\rho} + e^{-\rho T} \right\} \log w_0 + E \left[ \int_0^T e^{-\rho t} \left[ \log c(t) + \frac{1}{\rho} \left(1 - (1 - \rho)e^{-\rho(T-t)} \right) \right] dt \right) \\
\times Q(t, \pi(t), c(t)) \right] \\
\text{s.t.} 1 - c_2(t) \exp \left( Q(t, \pi(t), c(t)) \right) \leq \beta.
$$

The derivation is given in the appendix. It is not difficult to find that the above problem is equivalent to the following one

$$
\max_{\{\pi(t), c(t)\}} \left\{ \log c(t) + \frac{1}{\rho} \left(1 - (1 - \rho)e^{-\rho(T-t)} \right) Q(t, \pi(t), c(t)) \right] \\
\text{s.t.} 1 - c_2(t) \exp \left( Q(t, \pi(t), c(t)) \right) \leq \beta.
$$

(4.1)

Theorem 1. The solution of the optimization problem (4.1) is

$$
\pi^*(t) = \begin{cases} 
\left( \sigma \sigma' \right)^{-1} \hat{\mu}, & \text{if } 1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) < \hat{\beta} \\
\left( k_2 \sigma \right)^{-1} \hat{\mu}, & \text{if } 1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) \geq \hat{\beta}
\end{cases}
$$

$$
c'(t) = \begin{cases} 
\frac{1}{1 - (1 - \rho)^2} & \text{if } 1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) < \hat{\beta} \\
\frac{k_2 c}{1 - (1 - \rho)^2} & \text{if } 1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) \geq \hat{\beta}
\end{cases}
$$

where $k_1^2$ is the root of Eq. (4.4) in the variable $x, k_2^2$ is defined by (4.3).

Proof. Obviously, the objective function is concave with respect to $\pi(t)$ and $c(t)$. The feasible region is a convex set. So, Problem (4.1) is a convex programming and its KKT point is the same as the optimal point. Define its Lagrange function as

$$
L(t, \pi(t), c(t), \lambda) = \log c(t) + \frac{1}{\rho} \left(1 - (1 - \rho)^2 e^{-\rho(T-t)} \right) Q(t, \pi'(t), c(t)) \\
\lambda \left[ 1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) - \hat{\beta} \right] \leq 0, \lambda \geq 0
$$

(4.2)

In what follows, we discuss Problem (4.1) in two different cases. Case 1. If $1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) < \hat{\beta}$, then the relative CVaR constraint is inactive at the optimal solution. In this case, Problem (4.1) is reduced to the following unconstrained problem

$$
\max \log c(t) + \frac{1}{\rho} \left(1 - (1 - \rho)^2 e^{-\rho(T-t)} \right) Q(t, \pi'(t), c(t)).
$$

According to first-order necessary conditions, the optimal solution $(\pi_M(t), c_M(t))$ must satisfy

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial \pi} L(t, \pi, c, \lambda) = 0 \\
\frac{\partial}{\partial c} L(t, \pi, c, \lambda) = 0
\end{array} \right.
\text{s.t.} \pi_M(t) = \left( \sigma \sigma' \right)^{-1} \hat{\mu}, \ c_M(t) = \frac{\beta}{1 - (1 - \rho)^2 e^{-\rho(T-t)}}.
$$

Hence,

$$
\pi_M(t) = \left( \sigma \sigma' \right)^{-1} \hat{\mu}, \ c_M(t) = \frac{\beta}{1 - (1 - \rho)^2 e^{-\rho(T-t)}}.
$$

(4.2)

Case 2. If $1 - c_2(t) \exp \left( Q(t + \frac{1}{2} \parallel \pi'(t) \sigma \parallel^2 \right) = \hat{\beta}$, then the relative CVaR constraint is active. Let $\frac{\partial}{\partial \pi} L(t, \pi, c, \lambda) = 0$, that is

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial \pi} L(t, \pi, c, \lambda) = 0 \\
\frac{\partial}{\partial c} L(t, \pi, c, \lambda) = 0
\end{array} \right.
\text{s.t.} \pi_M(t) = \left( \sigma \sigma' \right)^{-1} \hat{\mu}, \ c_M(t) = \frac{\beta}{1 - (1 - \rho)^2 e^{-\rho(T-t)}}.
$$

(4.3)

$$
\lambda \left[ \frac{\partial}{\partial \pi} L(t, \pi, c, \lambda) + c_M(t) \hat{\mu} \right] \exp \left( \parallel \pi'(t) \hat{\mu} + r - c(t) \parallel \right) = 0.
$$

(4.4)
By calculating the above equation, we find that the optimal portfolio $\pi^*(t)$ is parallel to $\pi_M(t)$. So, we let $\pi^*(t) = k_1^* \pi_M(t)$, then, there must exist the unique $(k_1^*, k_2^*)$ be the optimal solution of the following problem

$$\max \log k_2 c_{M}(t) + \frac{1}{\rho} (1 - (1 - \rho) e^{-(T-t)}) Q(t, k_1^* \pi_M(t), k_2 c_{M}(t))$$

s.t. $1 - c_2(t) \exp \left( Q_{\pi} + \frac{1}{2} \| k_1^* \pi_M(t) \| \| \tau \| \right) = \tilde{\beta}.$

By first-order necessary conditions, there must exist a unique Lagrange multiplier $\lambda^*$ such that

$$\nabla \left\{ \log k_2 c_{M}(t) + \frac{1}{\rho} (1 - (1 - \rho) e^{-(T-t)}) Q(t, k_1^* \pi_M(t), k_2 c_{M}(t)) \right\} = \lambda^* \left\{ 1 - c_2(t) \exp \left( Q_{\pi} + \frac{1}{2} \| k_1^* \pi_M(t) \| \| \tau \| \right) - \tilde{\beta} \right\}.$$ 

By eliminating $\lambda^*$, we obtain

$$k_2^* = f(k_1^*) = 1 + \frac{k_1^* \| \pi_M \| \| \tau \|}{\Phi(\tau - k_1^*)},$$

where $k_1^*$ is the root of the equation

$$(1 - \tilde{\beta})(1 - \beta) = \Phi(c_1 - x) \| \pi_M \| \| \tau \| \sqrt{\tau}$$

$$\times \exp \left( \tau \pi_M(t) + r - f(x) c_{M}(t) \right) \right)$$

in the variable of $x$.

5. Empirical analysis

In this section, we choose four stocks from Chinese financial market with codes 600611, 600796, 600054 and 600118, and use their daily closing prices from 01/04/2010 to 01/04/2016. By computation, we have $\mu = (0.0316, 0.1013, 0.0485, 0.1291)'$ and

$$\sigma = \begin{bmatrix} 0.5812 & 0 & 0 & 0 \\ 0.3184 & 0.4915 & 0 & 0 \\ 0.2445 & 0.1258 & 0.3631 & 0 \\ 0.3392 & 0.1769 & 0.967 & 0.6337 \end{bmatrix}.$$ 

Let $r = 0.017, \beta = 0.95, \tilde{\beta} = 0.06, T = 5 \text{ (year), } \tau = 0.02 \approx \text{ a week.}$

Fig. 1 presents the optimal portfolio of the risk-free asset at different time. When there is no relative CVaR constraint, the proportion of the wealth invested in the risk-free asset is constant. However, as the value of the Lagrange multiplier turns from zero to positive, the constraint becomes binding. And the investor will increase the proportion of the wealth invested in the risk-free asset, i.e., the investor will reduce the proportion invested in the risky assets. Fig. 2 presents the optimal consumption rate at different time. Since $k_2^*$ is very close to 1, the optimal consumption rate with relative CVaR constraint and without it is almost the same.

6. Conclusion

The optimal consumption-investment problem with a dynamic CVaR constraint has been studied. For general utility function, we derived the HJB equation by the dynamic programming technique. As for the logarithmic utility function, the method of Lagrange multiplier has been applied to tackle the constraint and the closed form solution is presented.

From the empirical study’s results, we find that the constrained problem invested less in the risky assets. This is due to the fact the CVaR constraint is imposed all the time. How to extend the idea of dynamic risk control to other portfolio problems, such as the benchmark process, dynamic mean-variance model, etc, is our future interest.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the manuscript, read and approved the final manuscript.

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Appendix

1. The derivation of formula (2.3)

We divide the derivation into three steps. Denote the loss $\Delta$, where $\Delta = L(t)$. For simplicity, we denote $W^{\pi^*(t)}$ by $W, \pi(t)$ by $\pi, Q_t(t, \pi^*(t), c(t))$ by $Q, B(t + \tau) - B(t)$ by $\Delta B$ respectively.

(1) Derivation the probability density function (PDF) of the loss.
Since the cumulative distribution function of the loss is
\[ F_\Delta(x) = P(\Delta \leq x) = P(W - W e^{\rho T} e^{\pi^2 \sigma^2 T} \leq x) = P \left( \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - x}{W e^{\rho T}} \leq \frac{\Delta B}{\sqrt{T}} \right) \]
\[ = 1 - \Phi \left( \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - x}{W e^{\rho T}} \right). \]
So, the PDF of the loss is
\[ f_\Delta(x) = f'(\Delta)(x) = -\Phi \left( \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - x}{W e^{\rho T}} \right) \cdot \frac{1}{\pi \sigma \sqrt{T}} \cdot \frac{W e^{\rho T}}{(W - x) \pi \sigma \sqrt{T}}. \]

(2) Derivation the VaR.

By definition,
\[ \beta = P(\Delta \leq \text{VaR}) = P(W - W e^{\rho T} e^{\pi^2 \sigma^2 T} \leq \text{VaR}) \]
\[ = P \left( \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - \text{VaR}}{W e^{\rho T}} \leq \frac{\pi^2 \sigma^2 \Delta B}{\pi \sigma \sqrt{T}} \right) \]
\[ = 1 - \Phi \left( \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - \text{VaR}}{W e^{\rho T}} \right), \quad \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - \text{VaR}}{W e^{\rho T}} = -\Phi^{-1}(\beta). \]

Therefore, \( \text{VaR} = W - W e^{\rho T} e^{\pi^2 \sigma^2 T} \) if \( \pi^2 \sigma^2 \parallel x \parallel C_1 \), where \( c_1 = \Phi^{-1}(1 - \beta) \).

(3) Derivation the CVaR.

By definition,
\[ \text{CVaR} = E(\Delta | \Delta \geq \text{VaR}) = \frac{1}{1 - \beta} \int_{\text{VaR}}^{+\infty} x f_\Delta(x) dx \]
\[ = 1 - \beta \int_{\text{VaR}}^{+\infty} \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - x}{W e^{\rho T}} \cdot e^{-\frac{1}{2\pi}(\frac{\log \frac{W - x}{W e^{\rho T}}}{\pi \sigma \sqrt{T}})^2} dx \]
\[ = \left[ 1 - \beta \right] \log \frac{W - \text{VaR}}{W e^{\rho T}} - \int_{\text{VaR}}^{+\infty} \frac{1}{\pi \sigma \sqrt{T}} \log \frac{W - x}{W e^{\rho T}} \cdot e^{-\frac{1}{2\pi}(\frac{\log \frac{W - x}{W e^{\rho T}}}{\pi \sigma \sqrt{T}})^2} dx \]
\[ \left( + \text{c}_2 \right). \]

2. The simplification of (2.6)

Taking the log on both sides of (2.2), we have
\[ \log(W(\pi, t)) = \log w_0 + \int_0^t Q(s, \pi, s, c(s)) ds + \int_0^t \pi(s) \sigma dW(s). \]

Then the utility of the consumption is
\[ \int_0^T e^{-\rho t} \log(c(t)W(\pi, c(t))) dt \]
\[ = \int_0^T e^{-\rho t} \log(c(t)) dt + \int_0^T e^{-\rho t} \log w_0 + \int_0^T e^{-\rho t} \int_0^t Q(s, \pi(s), c(s)) ds \]
\[ + \int_0^T e^{-\rho t} \int_0^t \pi(s) \sigma dW(s). \]

Exchanging the order of integral by Fubini’s theorem, we have
\[ \int_0^T e^{-\rho t} \int_0^t Q(s, \pi(s), c(s)) ds \]
\[ = \int_0^T \left[ Q(s, \pi(s), c(s)) ds \right] \int_0^t e^{-\rho t} dt \]
\[ = \int_0^T \frac{e^{-\rho t} - e^{-\rho T}}{\rho} Q(s, \pi(s), c(s)) ds. \]

Hence,
\[ \int_0^T e^{-\rho t} \log(c(t)W(\pi, c(t))) dt \]
\[ = \frac{1 - e^{-\rho T}}{\rho} \log w_0 \]
\[ + \int_0^T e^{-\rho t} \frac{1}{\rho} \left[ (1 - e^{-\rho(T+t)}) Q(t, \pi(t), c(t)) \right] dt \]
\[ + \int_0^T e^{-\rho t} \int_0^t \pi(s) \sigma dW(s). \]

By taking mathematical expectation on both sides, we have
\[ E \int_0^T e^{-\rho t} \log(c(t)W(\pi, c(t))) dt \]
\[ = \frac{1 - e^{-\rho T}}{\rho} \log w_0 \]
\[ + E \left[ \int_0^T e^{-\rho t} \frac{1}{\rho} \left[ (1 - e^{-\rho(T+t)}) Q(t, \pi(t), c(t)) \right] dt \right]. \]

Similarly, we can get
\[ E\left[ e^{-\rho T}U_2(W(\pi, \pi(T))) \right] \]
\[ = e^{-\rho T} \log w_0 + e^{-\rho T} E \int_0^T Q(t, \pi(t), c(t)) dt. \]

Problem (2.6) can be reduced to
\[ \max_{\pi(\cdot), c(\cdot)} \left[ \frac{1 - e^{-\rho T}}{\rho} + e^{-\rho T} \log w_0 \right. \]
\[ \left. + E \left[ \int_0^T e^{-\rho t} \right. \times \left[ \log(c(t)) + \frac{1}{\rho} (1 - (1 - \rho) e^{-\rho(T+t)}) Q(t, \pi(t), c(t)) \right] dt \right] \]
\[ \text{s.t.} \ 1 - c_2(t) \exp \left( Q(T) + \frac{1}{2} \parallel \pi(t) \parallel^2 \right) \leq \beta. \quad \Box \]

References
