Nonlinear model predictive control of a magnetic levitation system

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Abstract

The paper presents a fast nonlinear model predictive control (MPC) scheme for a magnetic levitation system. A nonlinear dynamical model of the levitation system is derived that additionally captures the inductor current dynamics of the electromagnet in order to achieve a high MPC performance both for stabilization and fast setpoint changes of the levitating mass. The optimization algorithm underlying the MPC scheme accounts for control constraints and allows for a time and memory efficient computation of the single iteration. The overall control performance of the levitation system as well as the low computational costs of the MPC scheme is shown both in simulations and experiments with a sampling frequency of 700 Hz on a standard dSPACE hardware.

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1. Introduction

Magnetic levitation systems are of growing importance in the industry. Typical applications are, for instance, frictionless bearings for high-speed machining (Knope, 2007), magnetically levitating trains (Hasirci, Balikci, Zabar, & Birenbaum, 2011) like the Transrapid, micro-manipulation of levitating objects (Kummer et al., 2010), or nanoscale positioning systems (Khamseen & Shameli, 2005; Kim, Verma, & Shakir, 2007).

Magnetic levitation systems with a single axis are widely used as benchmark problems for advanced control strategies due to their inherently nonlinear and unstable open loop nature. In recent years, a variety of control methods have been proposed for these systems. For instance, feedforward and feedback linearization for trajectory tracking of the levitating mass was considered by Morales and Sira-Ramirez (2010) and El Hajjaji and Ouladsine (2001). Additional robustness regarding parameter uncertainties for this approach can be provided by means of backstepping (Yang & Minashima, 2001). Further nonlinear control methods applied to magnetic levitation systems are, for instance, adaptive control (Yang, Kunitoshi, Kanae, & Wada, 2008; Yang & Tateishi, 2001), sliding mode controllers (Elahi & Nekoubin, 2011; Shieh, Siao, & Liu, 2010), and neuronal networks (Chen, Lin, & Shyu, 2009; Lin, Chen, & Shyu, 2009).

Another modern control method that became increasingly popular over the last years is model predictive control (MPC). MPC relies on the solution of an optimal control problem on a receding horizon (Camacho & Bordons, 2007; Grünne & Pannek, 2011; Mayne, Rawlings, Rao, & Scokaert, 2000) and is well suited for nonlinear multiple-input systems and to account for state or control constraints. The drawback of MPC is the high computational effort that is usually required to solve the underlying optimal control problem (OCP) in each sampling step. MPC schemes for fast systems therefore often rely on approximations or tailored algorithms to reduce the computational load, see e.g. Ohtsuka (2004), Ferreau, Bock, and Diehl (2008), DeHaan and Guay (2007), and Graichen and Kugi (2010).

For magnetic levitation systems, which typically necessitate sampling times in the (sub)millisecond range, the MPC design was investigated by several authors. An explicit MPC scheme is presented by Ulbig, Olaru, and Dumur (2008, 2010) based on a piecewise affine, linear system approximation. Further MPC approaches for magnetic levitation systems concern linear (discrete-time) MPC in combination with linear matrix inequalities (Matos, Galvão, & Yoneyama, 2010) and feedback linearization (Maia & Galvão, 2007) as well as bit-stream based MPC (Camasca, Swain, & Patel, 2011) and networked MPC (Wang, Liu, & Rees, 2009). All these MPC approaches rely on a linear or linearized model of the levitation system and various specializations with the goal to minimize the computational effort and to achieve real-time feasibility. However, an accurate model capturing the nonlinearities of the unstable system in combination with a real-time nonlinear MPC scheme is essential if a high control performance as well as fast setpoint transitions over a wide operational region of a magnetic levitation system are desired.

This contribution describes a nonlinear MPC scheme for an experimental magnetic levitation system with a maximum levitation height of 70 mm. Special attention is paid to the modeling and identification of a mathematical model with state-dependent parameters, using a similar approach as employed by Truong, Wang, and Huang (2007). The experimental magnetic levitation system provides several challenges for the modeling process as well as for the MPC scheme. On the
one hand, the electromagnet’s core is highly conductive, which has to be accounted for in the model. On the other hand and in order to achieve the fastest possible control performance, no cascaded current controllers are used in the MPC design. Instead, the constrained duty cycles of the pulse width modulated (PWM) terminal voltages serve as control inputs in the model. This, however, necessitates an accurate model of the nonlinear inductor current dynamics. A further challenge for the MPC scheme arises from the comparatively high sampling frequency of 700 Hz corresponding to the PWM frequency.

The nonlinear MPC scheme employed in this paper to control the magnetic levitation system is based on a recently presented real-time gradient algorithm that is tailored to nonlinear MPC with control constraints (Graichen & Käpernick, 2012; Graichen & Kugi, 2010) and allows for a memory and time efficient execution of the single iterations. A first version of this real-time MPC scheme was successfully used to control a laboratory crane in the millisecond range on a standard PC (Graichen, Egretzberger, & Kugi, 2010). For the magnetic levitation system considered in this contribution, however, the nonlinear MPC design is significantly more challenging due to the strong nonlinearity, fast current dynamics, and considerable stiffness of the overall system. The runtime efficiency of the nonlinear MPC scheme is demonstrated by a constant computation time of approximately 900 μs on a dSPACE real-time hardware which is well below the sampling time of 1.43 ms (700 Hz). In addition to the mathematical modeling of the magnetic levitation system and the actual MPC design, a robustness analysis is carried out and a comparison with linear MPC shows the superior control performance which underlines the importance of a nonlinear MPC setup.

The paper is organized as follows. The experimental setup and the nonlinear model of the magnetic levitation system are described in Section 2. Section 3 introduces the MPC scheme as well as the real-time optimization algorithm. Section 4 presents the numerical and experimental results for the magnetic levitation system. Finally, conclusions are drawn in Section 5.

2. Experimental setup and mathematical model

This section describes the experimental setup of the levitation system and derives a mathematical model that is used for the MPC design. In order to achieve a high MPC performance, the model of the system dynamics needs to be sufficiently accurate while requiring only moderate computational effort to allow for a high sampling frequency of the overall control system. In a first step, a detailed model is derived based on physical considerations that gives insight into the levitation system. In a second step, a reduced order model is identified that is suitable for real-time purposes and which will be used in the remainder of the paper.

2.1. Magnetic levitation system

A picture and a schematic drawing of the experimental levitation system are shown in Fig. 1. The levitating mass is a hollow object made of constructional steel with the mass \( m = 134 \text{ g} \). The electromagnet consists of a steel cup core and two coils with 2000 windings each. The electromagnetic field applies the electromagnetic force \( F_{\text{mag}} \) to the levitating mass in opposite direction to the gravitational force \( mg \) with the acceleration due to gravity \( g = 9.81 \text{ m/s}^2 \). The intention behind the two coils setup is to generate the major part of the electromagnetic field by means of the outer coil with the larger inductance, while the inner coil with the smaller inductance is more suitable to rapidly adapt the electromagnetic field for stabilization purposes. The experimental setup is designed for large levitation heights of up to \( z = -70 \text{ mm} \), corresponding to a maximum electrical power consumption of 800 W. Overheating of the coils is avoided by pressurized air that flows through cooling channels inside the electromagnet.

The control inputs of the electromagnet are the duty cycles \( r_1, r_2 \in [0, 1] \) of two PWM controlled buck converters that adjust the voltage applied to the coil terminals. Both buck converters are supplied by a shared direct voltage source \( U_s \) and allow the terminal voltages to be controlled individually within the interval \([0, U_s]\), while both coil currents \( i_1, i_2 \) are restricted to positive values. The levitation height \( z \) and the currents \( i_1, i_2 \) are measured with a laser sensor (see Fig. 1) and resistive current sensors. The levitation system is controlled by a dSPACE MicroAutoBox I real-time hardware equipped with an 800 MHz PowerPC processor.

The dynamics of the levitation system comprise the electrical and the mechanical subsystems. In general, the distance of the levitating mass affects the inductance of the coils. This property can, for instance, be taken advantage of for sensor-free position estimation (Glück, Kemmetmüller, Tump, & Kugi, 2011). For the experimental setup in this contribution, however, FE simulations\(^1\) have shown that the air gap between the levitating mass and the electromagnet has only limited effect on its inductances for the considered operational region of \( z \in [-70 \text{ mm}, -40 \text{ mm}] \). To illustrate this point, Fig. 2 shows the impedances \( Z_1 \) and \( Z_2 \) of both coils as computed in FE simulations for the currents \((i_1, i_2) = (1 \text{ A}, 0 \text{ A})\) and \((i_1, i_2) = (0 \text{ A}, 1 \text{ A})\) at a frequency of 5 Hz.

\(^1\) The software FEMM (Meeker, 2010) was used to develop a 2D axisymmetric finite element (FE) simulation of the experimental setup.
2.2. Inductor current dynamics—physical considerations

Both coils of the experimental setup in Fig. 1 are wound on the same core causing a mutual dynamical influence on each other. This setup is similar to a transformer and accordingly can be modeled using the equivalent circuit in Fig. 3. The resistor \( R_f \) models the iron losses due to the conductive core, while \( R_1 \) and \( R_2 \) represent the copper resistance of each coil. The mutual flux linking both coils and the leakage flux of each single coil are included in the circuit by the three inductors \( L_{1}, L_{2}, \) and \( L_{2a} \). The terminal voltages applied to both gates are individually adjusted within the interval \([0, U_s] \) by the real-time hardware controlling the buck converters.

Choosing the magnetizing current \( i_1 \) and the coil currents \( i_1 \) and \( i_2 \) as states, the state space model of the transformer circuit in Fig. 3 is given by

\[
\begin{bmatrix}
\dot{i}_1 \\
\dot{i}_2 \\
\dot{i}_3
\end{bmatrix}
= \begin{bmatrix}
\frac{R_2 + R_1}{L_1} & -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\
-\frac{R_1}{L_1} & \frac{R_1 + R_2}{L_2} & -\frac{R_2}{L_2} \\
0 & -\frac{R_2}{L_{2a}} & \frac{R_2}{L_{2a}}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix}
+ \begin{bmatrix}
\frac{U_s}{L_1} & 0 & 0 \\
0 & \frac{U_s}{L_2} & 0 \\
0 & 0 & \frac{U_s}{L_{2a}}
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix}
\tag{1}
\]

The linear system (1) accurately models the dynamics of the currents with respect to the duty cycles \( r_1, r_2 \in [0, 1] \). However, the assumption of constant values for the inductances \( L_{1}, L_{2}, \) \( L_{2a} \) and for the voltage \( U_s \) is too idealistic for the experimental setup. Moreover, the direct voltage source \( U_s \) is not stabilized and therefore depends on \( i_1 \) and \( i_2 \) as well. These nonlinearities are accounted for by including the dependency of \( L_{1}, L_{2}, \) \( L_{2a} \) and \( U_s \) on the total current \( i_{1,2} = i_1 + i_2 \), \( i_1, i_2 \geq 0 \)

\[
\begin{bmatrix}
\dot{i}_1 \\
\dot{i}_2 \\
\dot{i}_3 
\end{bmatrix}
= \mathbf{A}(i_{1,2})
\begin{bmatrix}
i_1 \\
\dot{i}_2 \\
\dot{i}_3 
\end{bmatrix}
+ \mathbf{B}(i_{1,2})
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix}
\tag{2}
\]

The values of the state-independent resistors \( R_1 = 8.2 \Omega \) and \( R_2 = 8.8 \Omega \) in (1) were measured separately. In addition, FE simulations reveal that \( R_f = 20 \Omega \) is a reasonable value for the resistor \( R_f \) that models the iron losses in the conductive core.

The remaining dependencies in \( \mathbf{A}(i_{1,2}) \) and \( \mathbf{B}(i_{1,2}) \) are determined by identifying the values of \( L_{1}, L_{2}, \) \( L_{2a} \) and \( U_s \) in (1) at various operating points of the total current \( i_{1,2} \) over the range \([2, 11, 11] A\). The identification is based on random step series in \( r_1 \) and \( r_2 \) corresponding to an interval for each operating point of \( i_{1,2} \) and using a nonlinear least-squares solver to identify \( L_{1}, L_{2}, \) \( L_{2a} \) and \( U_s \) in the linear dynamics (1) for each data set, see Fig. 4. For small values of \( i_{1,2} (< 4 A) \), the identified parameters are almost independent of \( i_{1,2} \), whereas increasing values of \( i_{1,2} \) lead to a linear decline of \( U_s \) due to the internal resistance of the voltage source. The inductors \( L_{1}, L_{2} \) show a corresponding behavior over \( i_{1,2} \) due to the nonlinear core material. The leakage flux inductance \( L_{2a} \) of the inner coil is comparatively small due to the strong magnetic coupling to the outer coil and exhibits only a minor dependency on \( i_{1,2} \). The nonlinear model (3) for the inductor current dynamics is finally obtained using appropriate interpolations of the parameters between the identified values.

While the aforementioned modeling approach is well founded on physical considerations and yields good modeling results, the resulting differential equations (3) are considerably stiff. This is illustrated in Fig. 5, where the eigenvalues of the system matrix \( \mathbf{A} \) are plotted for their associated operating points (cf. Fig. 4). The decline in \( \lambda_1 \) and \( \lambda_2 \) is caused by the degrading inductances for larger values of \( i_{1,2} \). On the other hand, \( \lambda_3 \) is dominated by the inverse of the leakage inductance \( L_{2a} \), which vanishes for small values of \( i_{1,2} \). This leads to an increasingly faster dynamics of \( i_2 \) (see the equivalent circuit in Fig. 3) corresponding to large absolute values of the eigenvalue \( \lambda_3 \), which increases the stiffness of the system (3).

2.3. Inductor current dynamics—simplified model for real-time MPC

An alternative to the physically motivated (gray-box) model of Section 2.2 is to directly identify a second-order model of the form

\[
\begin{bmatrix}
\dot{i}_1 \\
\dot{i}_2
\end{bmatrix}
= \mathbf{A}(i_{1,2})
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
+ \mathbf{B}(i_{1,2})
\begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}
\tag{4}
\]

using a black-box approach. Based on the measurement data sets used for the physically motivated gray-box approach in Section 2.2, a linear instrumental variable least-squares solver is employed to estimate the elements of the \((2 \times 2)\)-matrices \( \mathbf{A} = [\mathbf{a}_{ij}] \) and \( \mathbf{B} = [\mathbf{b}_{ij}] \) for the aforementioned operation points of the total inductance \( L \) [H] and voltage \( U_s \) [V].
current \(i_{1,2}\) over the range \([2 A, 11 A]\). The system model (4) is finally obtained by using polynomials of fourth order to interpolate between the identified values of \([\hat{\alpha}_g]\) and \([\hat{\beta}_g]\) over \(i_{1,2}\), which are shown in Fig. 6.

The validation of this model is shown in Fig. 7 for a random step series in \(r_1\) and \(r_2\). In particular, it demonstrates the mutual dynamical influence between the coil currents \(i_1\) and \(i_2\) caused by the flux linkage. A transient current in one coil causes an opposing transient in the other coil as predicted by the equivalent circuit in Fig. 3. Obviously, the model shows good accuracy both concerning steady state and dynamical behavior over the considered total current range \([2 A, 11 A]\).

2.4. Electromagnetic force and levitating mass

The mechanical submodel of the levitation system is given by the equation of motion of the levitating mass in the \(z\)-direction (see Fig. 1)

\[
\frac{d^2 z}{dt^2} = \frac{1}{m} F_{\text{mag}}(z, i_1, i_2) - g,
\]

where the electromagnetic force \(F_{\text{mag}}\) depends on the inductor currents \(i_1\) and \(i_2\) and the levitation height \(z\) of the mass.

The validation of this model is shown in Fig. 7 for a random step series in \(r_1\) and \(r_2\). In particular, it demonstrates the mutual dynamical influence between the coil currents \(i_1\) and \(i_2\) caused by the flux linkage. A transient current in one coil causes an opposing transient in the other coil as predicted by the equivalent circuit in Fig. 3. Obviously, the model shows good accuracy both concerning steady state and dynamical behavior over the considered total current range \([2 A, 11 A]\).

\[
F_{\text{mag}} = \frac{\partial}{\partial z} \left( \frac{1}{2} L_{1h} i_1^2 + \frac{1}{2} L_{2h} i_2^2 + \frac{1}{2} L_{1f} i_2^2 \right).
\]

Moreover, a closer look at the equivalent circuit in Fig. 3 reveals that in steady state \(i_2\) corresponds to the total current \(i_{1,2}\). Under these considerations, the following semiparametric approach:

\[
F_{\text{mag}}(z, i_1, i_2) = e^{\rho(z)} \left( P_k^1(i_1) + P_k^2(i_2) + P_k^3(i_{1,2}) \right)
\]

based on the structure of (6) is used to obtain a model of the electromagnetic force \(F_{\text{mag}}\), where \(P_k\) represents polynomials of degree \(k\).

The electromagnetic force data that is used to identify the parameters of (7) are computed from FE simulations and is validated by measurements at the experimental setup.\(^2\) The parameters of the polynomials \(P_k\) in (7) are fitted to the numerical data using a nonlinear least-squares solver. Fig. 8 shows surface plots of the measured electromagnetic force compared to the fitted function (7) for different levitation heights, which reveal the good correspondence between the model and the experimental setup.

In summary, the overall nonlinear model of the levitation system consists of the inductor current dynamics (4) and the equations of motion (5) with the electromagnetic force (7), i.e.

\[
\dot{x} = f(x, u) = \begin{bmatrix}
\frac{\partial}{\partial z} F_{\text{mag}}(x_1, x_2, x_4) - g \\
\tilde{A}(x_3 + x_4) x_3 + \tilde{B}(x_3 + x_4) u
\end{bmatrix}
\]

with the state \(x = [z, \dot{z}, i_1, i_2]^T\) and the constrained duty cycles \(r_1, r_2 \in [0, 1]\) as the control \(u = [r_1, r_2]^T\) of the system.

\(^2\) The electromagnetic force \(F_{\text{mag}}\) was measured with an additional apparatus consisting of a copy of the levitating object mounted to a (non-ferromagnetic) frame that is connected to a load cell.
3. Model predictive control

This section shortly describes the receding horizon principle of MPC as well as a real-time optimization algorithm in order to apply the MPC scheme to the magnetic levitation system based on the nonlinear model (8).

3.1. Receding horizon principle

The MPC scheme for the levitation system relies on the solution of an optimal control problem of the form

\[
\min_j \ J_k (x_k, u) = V(x(t)) + \int_{t_k}^{t_{k+1}} l(x(t), u(t)) \, dt
\]

\[
\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t)), \quad x(t_k) = x_k
\]

with the optimal cost function \( J_k(x_k) = j(x_k, u_k) \) is available, the first part of the control trajectory \( u_k(t) \) is used as control input

\[
u(t) = u_k(t), \quad t \in [t_k, t_{k+1} + \Delta t]
\]

for the system (8). In the next sampling instant \( t_{k+1} = t_k + \Delta t \) with the sampling time \( \Delta t > 0 \), the OCP (9)-(11) is solved again with the state \( x_{k+1} \) as the new initial condition in (10).

A well-known drawback of MPC is the high numerical effort that is usually required to compute the optimal solution (12). This issue is particularly challenging for complex systems or fast dynamics with sampling times in the (sub)millisecond range—as it is the case for the levitation system in this paper. In order to keep the MPC complexity within reasonable bounds, the OCP (9)-(11) does not include state constraints as well as terminal constraints which are often considered to guarantee stability. In the absence of terminal constraints, stability can still be guaranteed if the terminal cost \( V(x) \) is a control Lyapunov function [CLF] (Chen & Allgöwer, 1998; Graichen & Kugi, 2010; Limon, Alonso, Salas, & Camacho, 2006), if the prediction horizon \( T \) is sufficiently long (Jadbabaie, Yu, & Hauser, 2001), or if a certain controllability condition is satisfied (Reble & Allgöwer, 2012).

Moreover, there exist further MPC formulations based on discrete-time dynamics and/or sample-and-hold control updates (Mayne et al., 2000). However, for the levitation system and the real-time optimization algorithm presented in the following lines, the continuous-time MPC formulation is the most appropriate.

3.2. Real-time optimization algorithm

The efficient numerical solution of the OCP (9)-(11) is of importance in view of the sampling time \( \Delta t \approx 1.43 \text{ ms} \) (700 Hz) of the levitation system. A suitable algorithm is the gradient method that is easy to implement as well as efficient and memory efficient executable. The gradient method takes advantage of the special structure of the optimality conditions that results from the OCP formulation without terminal constraints.

With the Hamiltonian

\[
H(x, \lambda, u) = l(x, u) + \lambda^T f(x, u), \quad \lambda \in \mathbb{R}^d
\]

and initial trajectories \( \mathbf{X}_k(0) \in [0, 1], \mathbf{U}_k(0), t \in [t_k, t_k + T] \), the gradient method consists of the following steps:

- **Initialization**
  1. Choose an initial control trajectory \( \mathbf{U}_k(0) \in [0, 1], t \in [t_k, t_k + T] \).
  2. Integrate in forward time

\[
\mathbf{X}_k(0) = f(X_k(0), U_k(0)), \quad U_k(0) = x_k
\]

- **Gradient iteration: While \( j < N \) Do**
  1. Integrate in backward time

\[
\dot{X}_k(t) = -H_k(X_k(t), U_k(t), \lambda), \quad X_k(t_k + T) = V_k(X_k(t_k + T)), \quad H_k = \partial H / \partial x, \quad V_k = \partial V / \partial x.
\]

where \( H_k = \partial H / \partial x \) and \( V_k = \partial V / \partial x \).

  2. Compute the search direction

\[
\dot{X}_k(t) = -H_k(X_k(t), U_k(t), \lambda), \quad X_k(t_k + T) = V_k(X_k(t_k + T)), \quad \text{t} \in [t_k, t_k + T]
\]

with \( H_k = \partial H / \partial u \).

  3. Compute the step size by solving

\[
\alpha_k = \arg \min_{\alpha > 0} J(X_k, U_k + \alpha S_k),
\]

where \( S_k = \psi (X_k, U_k + \alpha S_k) \) is a (point-wise in time) projection function defined by

\[
\psi_i(u_i) = \begin{cases} u_i & \text{if } u_i \in [0, 1) \\ 0 & \text{if } u_i \leq 0 \\ 1 & \text{if } u_i \geq 1 \end{cases}
\]

  4. Compute the next control trajectory

\[
\mathbf{U}_k(i+1) = \psi (\mathbf{U}_k(i) + \alpha \mathbf{S}_k), \quad \text{t} \in [t_k, t_k + T].
\]

  5. Integrate in forward time

\[
\dot{X}_k(t) = f(X_k(t), U_k(i+1), \lambda), \quad X_k(t_k + T) = x_k
\]

- 6. Stop if \( J(X_k, U_k(i+1)) - J(X_k, U_k(i)) \leq \epsilon_j \) for some \( \epsilon_j > 0 \). Otherwise, set \( j = j + 1 \).

One gradient step basically consists of two integrations of the canonical equations (16) and (21) and the line search (18) that represents a scalar optimization problem. The computational effort for solving (18) can be significantly reduced by computing an approximate solution of the step size \( \alpha_k \) via a polynomial fitting with three sample points \( \alpha_1 < \alpha_2 < \alpha_3 \) that are adapted over the single gradient iterations (Graichen & Käpnerick, 2012).

A sufficiently accurate solution close to the optimal one (12) can be obtained by iterating the gradient loop until the convergence criterion for the cost functional \( J \) in Step 6 or a similar criterion is fulfilled. This, however, typically requires an unpredictable number of iterations, which is unsuitable for a real-time MPC implementation. To this end, the gradient algorithm is stopped after a fixed number of iterations \( N \) and the first part of the control trajectory \( \mathbf{U}_k(0) \) is used as control input to the system (8), i.e.

\[
u(t) = U_k(0), \quad t \in [t_k, t_k + \Delta t]
\]
In the next MPC step $k+1$, $\mathbf{u}_k^{(N)}(t)$ is used to reinitialize the gradient algorithm. Due to the premature stop of the gradient algorithm after $N$ iterations, the trajectories $\mathbf{x}_k^{(N)}(t)$, $\mathbf{u}_k^{(N)}(t)$ and $\mathbf{Q}_k^{(N)}(t)$ are suboptimal in the sense that the associated cost value will in general be larger than the optimal one, i.e. $J(x_k, \mathbf{u}_k^{(N)}(t)) \geq J(x_k)$. 

The suboptimal character of this real-time MPC scheme is investigated in more detail in Graichen and Kugi (2010) and Graichen and Käpernick (2012), where it is shown (under certain assumptions) that asymptotic stability as well as incremental decay of the suboptimality can be obtained if the number of iterations is sufficiently large, i.e. if $N$ satisfies a lower bound of the form $N \approx \tilde{N}$ that depends on the system dynamics and cost functional. A conservative estimate of the minimum number of iterations $\tilde{N}$ can be obtained by means of Lipschitz estimates as shown in Graichen and Kugi (2010) and Graichen (2012a). For practical purposes, the actual value of $\tilde{N}$ has been determined in simulations (see Sections 4.1 and 4.2).

4. Numerical and experimental results

The real-time MPC scheme is applied to the levitation system described in Section 2. The computational speed and performance in closed loop are demonstrated in nominal simulations and robustness scenarios. In addition, experimental results on a standard real-time hardware with the sampling time $T=1.43$ ms (700 Hz) corresponding to the PWM frequency reveal the applicability of the MPC scheme as well as the good correspondence between simulation and experiment due to the accuracy of the nonlinear model.

4.1. Simulation results (nominal case)

The simulation scenario for evaluating the MPC is to stabilize a desired levitation height $\hat{z}$ and to perform setpoint changes between $\hat{z} = -70$ mm and $\hat{z} = -40$ mm. To this end, the cost functions in (9) are chosen as

$$V(x) = 0, \quad l(x, u) = (x - \hat{x})^T Q (x - \hat{x}) + (u - \hat{u})^T R (u - \hat{u})$$

(24)

where $\hat{u} = [\hat{z}, \hat{i}]^T$ denotes the stationary duty cycle corresponding to a desired setpoint $\hat{x} = [\hat{z}, 0, \hat{i}, \hat{i}]^T$. The terminal cost $V(x)$ is set to zero, which simplifies the adjoint terminal conditions of the gradient algorithm in (16) to $\mathbf{x}_0(t_k + T) = 0$. In this case, the prediction time $T$ has to be chosen sufficiently long in order to guarantee asymptotic stability (Jadbabaie et al., 2001). Since the rigorous computation of this threshold is hardly feasible in practice, the prediction horizon $T$ as well as the weighting matrices $Q$ and $R$ in (24) are determined in simulations and validated in experiments.

A good compromise between robustness and performance is obtained for $T = 70$ ms and the weights $Q = \text{diag}(10^8 \, 1^2/\text{m}^2 \, s, \ 40 \, \text{s}^2/\text{m}^2, 0, 0)$, $R = \text{diag}(0.1/\text{s}, 0.1 \, 1/\text{s})$. In order to allow for an aggressive control performance, $Q$ only penalizes the levitation height $z$ and velocity $\dot{z}$ ($Q$ is positive semi-definite) with a dominant value for $z$ compared to the weights $R$. The weight $40 \, \text{s}^2/\text{m}^2$ for $z$ was determined as a trade-off between an aggressive speed for setpoint changes and robustness (also see Section 4.2).

The gradient-based MPC scheme was implemented as a C-mex-function in MATLAB. The numerical integrations in the gradient algorithm (see Section 3.2) are performed using an Euler forward scheme over the discretized MPC interval $[t_k, t_k + T]$ with 35 equidistant points. The polynomial functions in (4) and (7) that are part of the overall system dynamics (8) are implemented using the Horner scheme, see for instance Kastner, Hosangadi, & Fallah (2010), which leads to a significant reduction of the computational load in the MPC scheme.

Fig. 9 shows the MPC simulation results for successive setpoint changes between the levitation height setpoints $\hat{z} = [-40 \, \text{mm}, -55 \, \text{mm}, -70 \, \text{mm}]$. The simulations are obtained by using $N = 4$ gradient iterations per MPC step and assuming full state measurement in order to provide the MPC scheme with the entire state vector $x(t_k)$ at each sampling instant $t_k$. See (10). The trajectories in Fig. 9 show an excellent and aggressive MPC performance with a bang-bang-like control action that fully exploits the constraints $[0, 1]$ of the duty cycles, in particular for the large setpoint changes between $\hat{z} = -70$ mm and $\hat{z} = -40$ mm. Moreover, the trajectories of the currents $i_1$ and $i_2$ show that the total current $i_1 + i_2$ varies over the whole range for which the inductor current model (4) is identified. This is also apparent from the variations of the matrix entries in Fig. 6, which clearly reveal the nonlinear behavior of the system for the considered transition scenario.

4.2. Performance and robustness

The performance and computational efficiency can be evaluated in terms of the computation time and the overall cost value

$$J_{\text{sim}} = \int_0^{t_{\text{sim}}} (x(t) - \hat{x})^T Q (x(t) - \hat{x}) + (u(t) - \hat{u})^T R (u(t) - \hat{u}) \, dt$$

(25)

that is computed using the simulated closed-loop trajectories. Table 1 summarizes the respective data in dependence of the number of gradient iterations $N$ per MPC step. The results are obtained on an Intel Core i7 CPU (M620, 2.67 GHz) running with Windows 7 and MATLAB 2012a (both 64 bit). The computational speed of the algorithm is demonstrated by the CPU times ranging from 32 $\mu$s for one gradient iteration up to 187 $\mu$s for $N = 10$. The overall cost value $J_{\text{sim}}$ as a measure for the control performance ("nominal" case in Table 1) decreases only marginally for larger values of $N$, which shows that an excellent performance is already obtained for small values of $N$. 

Fig. 9. MPC simulation results for successive transitions between the setpoints $\hat{z} = [-40 \, \text{mm}, -55 \, \text{mm}, -70 \, \text{mm}]$ of the levitation height.
To further illustrate the robustness of the MPC scheme and as a step towards the experimental validation in Section 4.4, the simulations are additionally carried out under the influence of noise and model mismatch. The noise of the measurements \( \mathbf{y} = [z, \dot{z}, \dot{i}_1, \dot{i}_2]^T \) that are used in the experiments is modeled as zero-mean Gaussian noise with the standard deviation \( \sigma(z) = 10^{-5} \text{ m} \) and \( \sigma(i_k) = 0.15 \text{ A} \). These values are approximately two times larger than the actual noise levels in the experimental setup and therefore can be seen as a conservative estimate for the robustness analysis. The MPC is supplemented by an unscented Kalman filter (UKF) to estimate the state vector \( \mathbf{x} = [z, \dot{z}, i_1, i_2]^T \) based on the noisy sensor measurements. In addition, an identification error of 15% in the electromagnetic force \( F_{\text{mag}} \) is considered as mismatch in the MPC model (8).

Table 1 lists the overall cost value (25) of the MPC simulation subject to the measurement noise, the model error and the combination of both effects. It turns out that \( N=4 \) gradient iterations per MPC step are necessary to cope with all robustness scenarios. In addition, Fig. 10 shows the simulation results for the levitation height \( z \) for \( N=4 \). The model error induces a steady state offset in the levitation height, whereas the noise generally leads to a more nervous behavior. Overall, the choice of \( N=4 \) gradient iterations per sampling step represents a good compromise between computation time, performance, and robustness.

### 4.3. Comparison with linear MPC

It has already been pointed out that the magnetic levitation system behaves strongly nonlinear. To account for the nonlinearity of the system in the MPC design is the more important as the system is inherently unstable. To illustrate this point and to underline the necessity of using a nonlinear MPC to control the magnetic levitation system, the control performance of the presented control scheme is compared to a linear MPC that is based on the linearization of the nonlinear model (8) about the stationary setpoints for \( z \in [-40 \text{ mm}, -55 \text{ mm}, -70 \text{ mm}] \). The linear MPC was implemented and simulated using the ACADO TOOLKIT (Houska, Ferreau, & Diehl, 2011) with the same sampling time \( \Delta t \), prediction horizon \( T \), and cost functional setup (24) as in the nonlinear MPC case.

Fig. 11 shows the resulting trajectories for the three setpoints \( z \in [-40 \text{ mm}, -55 \text{ mm}, -70 \text{ mm}] \) with different starting points. Especially for the setpoints \( z = -55 \text{ mm} \) and \( z = -70 \text{ mm} \), the linear MPC degrades in control performance and eventually turns unstable if the initial levitation position is 10 mm or farther away. The decrease in control performance in the linear MPC case is also reflected in the overall cost value (25) that is computed with respect to the simulated closed-loop trajectories. Fig. 11 shows the values of \( J_{\text{sim}} \) for both the linear and nonlinear MPC scheme, whereby the values for \( J_{\text{sim}} \) are omitted in the case of instability.

### 4.4. Experimental results

In addition to the numerical investigations above, the nonlinear MPC scheme was implemented on the dSPACE hardware of the experimental setup described in Section 2. The computation time on the dSPACE MicroAutoBox 1 with an 800 MHz PowerPC processor amounts to 900 \( \mu \text{s} \) for one MPC step \( (N=4) \), which is well below the sampling time \( \Delta t = 1.43 \text{ ms} \) (700 Hz) of the levitation system. The UKF mentioned in Section 4.2 is again used to estimate the current state vector \( \mathbf{x}_k = \mathbf{x}(t_k) \) at each sampling instant \( t_k \) based on the measurements of the levitation height \( z \) and the currents \( i_1, i_2 \). Since the computation time for the UKF is marginal compared to the MPC, the real-time capability of the overall control scheme is not diminished by including the UKF.

Fig. 12 shows the measurement results for the setpoint changes considered in Fig. 9. The noise in the profiles of Fig. 12 is mainly caused by the fast control action of the MPC in connection with the measurement noise, although the noise level is already considerably reduced by the UKF. The noise also leads to persistent perturbations in the initial conditions (10) of the MPC scheme, which demonstrates the numerical robustness of the gradient algorithm (also see Section 4.2). However, the noise as well as modeling inaccuracies naturally lead to steady state offsets in
setpoints (−70 mm, −55 mm, −40 mm) in Fig. 12 amounts to (1.5 mm, 1.0 mm, −3.0 mm), which closely resembles the offsets observed in Fig. 10.

In addition, Fig. 13 shows the disturbance rejection of the MPC in the experiments. In this scenario, the levitating mass was deflected by hand from the setpoint $z = −55$ mm. The disturbances are quickly rejected in both directions after the mass is set free again.

A video of the levitation system showing the performance of the MPC scheme is available at (http://youtu.be/01q4TUVrToc).

5. Conclusions

The numerical and experimental results for the inherently nonlinear and unstable levitation system reveal the performance as well as the feasibility of the presented nonlinear MPC scheme for fast mechatronics systems. An important aspect concerns the detailed modeling of the system, especially of the inductor current dynamics in order to achieve accurate predictions by the MPC. The accuracy of the nonlinear model is confirmed by the consistency of numerical and experimental results.

The MPC scheme that is used to control the levitation system utilizes a tailored real-time version of the well-known gradient method in optimal control that is easy to implement and computationally efficient. The numerical and experimental results as well as the computation times of approximately 80 $\mu$s and 900 $\mu$s on a standard computer and real-time hardware, respectively, show the robustness as well as the excellent control performance and efficiency of the MPC scheme.

Moreover, a variant of the gradient algorithm based on fixed-point iterations was recently presented for a class of nonlinear systems (Graichen, 2012a). Its potential to further reduce the computation time in nonlinear MPC was demonstrated in Graichen (2012b). Further research focuses on the combination of the real-time algorithm with a transformation technique for state constraints (Graichen & Petit, 2009) in order to extend the framework to state constrained MPC formulations.

References

Graichen, K. (2012b). Nichtlineare modellprädiktive Regelung basierend auf Fixpunktkriterien. at – Automatisierungstechnik, 60(8), 442–451 (in German).
Khamesee, M., & Shameli, E. (2005). Regulation technique for a large gap magnetic
Graichen, K., & Petit, N. (2009). Incorporating a class of constraints into the dynamics