Analysis of the Hollow Inclusion Technique for Measuring In Situ Rock Stress

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The complete mathematical analysis is given for a method of measuring the in situ state of stress in a rock mass. The technique consists of cementing a tubular probe, cast from epoxy resin and in which strain gauges have been embedded, into a hole predrilled into the rock mass. The probe is then overcored releasing the strains in the rock (apart from small residual strains) and inducing strains into the epoxy probe. As the probe has different elastic properties to the rock an analysis is required which accounts for the differing elastic properties of the two media and also the position of the strain gauges relative to the surface of the hole in the piece of rock.

The details of this technique are compared with those of two related techniques, the first in which the strain gauges are cemented directly to the surface of the rock, the second in which a solid epoxy probe, with embedded strain gauges, is cemented into the hole. In fact the equations derived in the paper reduce to those of the other two techniques as special cases.

INTRODUCTION

Knowledge of the in situ state of stress is important in the design of underground openings. This paper is concerned with three similar techniques developed to measure these stresses. All involve the steps illustrated in Fig. 1 (described in detail in Section 3).

The purpose of this paper is threefold:

(i) to present the exact mathematical analysis which describes all three techniques,
(ii) to compare the three techniques,
(iii) to set out their relative advantages and disadvantages.

The first technique of this type was developed by Leeman [1]. Three rosette strain gauges were cemented directly to the surface of the rock in the pilot hole. This represented a significant improvement over previous methods in that one set of measurements from a single borehole provided sufficient data to enable the complete state of stress to be determined at the measurement point. Earlier methods required at least two pilot holes in different directions.

The Leeman technique is satisfactory in dry conditions, but there is a practical difficulty in cementing the strain gauges directly to the rock in wet conditions. Leeman [1] presented the exact mathematical solution to this situation, and set out clearly a procedure for obtaining in situ stresses from measured strains.

Rocha & Silverio [2] developed a second technique in which the strain gauges were embedded in a solid epoxy probe. This is bonded into the pilot hole and overcored as above. This technique, which clearly overcomes any strain gauge water-proofing problem, was also utilized by Blackwood [3-5]. A feature of the mathematical solution for the solid probe is that the stresses and strains are constant throughout the probe. Blackwood does not give an explicit formula relating the in situ rock stresses to the strains measured in the probe. His analysis seems cumbersome. Rocha and Silverio arrived at explicit relationships between the strains in the probe and the in situ rock stresses. Their expressions for the strains, \( e_x, e_y, e_z, \gamma_{xy} \) agree exactly with those derived here for the special case where the inner radius of the hollow probe is set to zero (i.e. a solid probe).
The analysis of the solid probe is complicated by the fact that the size of the overcoring diameter must be included unless it is substantially larger than the pilot hole diameter. For values less than 0.05 of the parameter \( \varepsilon \) (ratio of the shear modulus of the epoxy to that of the rock), and large enough overcoring diameter (greater than three times the pilot hole diameter) the effect is negligible, Duncan Fama [6]. Thus the results reported by Rocha & Silverio [2] are close to the true result. Blackwood et al. [4], whose measurements were made in coal, may have incurred some errors by ignoring the effect of overcoring diameter.

The major disadvantage of the solid probe is that, on overcoring, tensile stresses are developed at the epoxy/rock interface which may be sufficiently large to break the bond between the probe and the rock. Blackwood [5] used an epoxy with a low Young's modulus and succeeded in reducing these tensile stresses. This also reduces the effect of ignoring overcoring diameter discussed above.

To overcome the problem of the failure of the epoxy/rock bond, a third technique was developed independently both by Rocha et al. [7] and Worotnicki & Walton [8]. This technique was also used by Pender [9]. Instead of a solid epoxy cylinder, they embedded the strain gauges in a thin-walled epoxy cylinder. This has been referred to as a "hollow inclusion triaxial stress gauge". This method solves many of the practical difficulties of the first two methods. One version of this instrument is illustrated in Fig. 2. As far as the analysis is concerned, although both papers overlooked the fact that a complete analytical solution is possible, Worotnicki and Walton found the main part of the solution presented by Savin [10], and used an excellent approximation for the shear strains \( \gamma_{xz}, \gamma_{yz} \).

In this paper the exact solution is given for the strains induced by the full set of \textit{in situ} stresses. The stresses obtained from using the exact solution are compared with those obtained by using the Leeman analysis. The latter is shown to be adequate in most cases. However, once the \( K \) factors introduced by Worotnicki and Walton have been calculated from the exact solution, the Leeman analysis has no advantage over the exact analysis.

**NOTATION**

- \( c_1, c_2, \ldots, c_6 \) \( d_1, d_2, \ldots, d_6 \)
- \( D \)
- \( e_1, e_2, e_3, e_4, e_5, e_6 \)
- \( E_z \)
- \( E_1, E_2 \)
- \( G_1, G_2 \)
- \( K_1, K_2, K_3, K_4 \)
- \( m = R_3/R_2 \)
- \( p \)
- \( r, n \)
- \( R_1, R_2 \)
- \( u, v, w \)
- \( x, y, z \)
- \( \theta \)
- \( \beta \)
- \( \phi \)
- \( \psi \)
- \( \gamma_{xz}, \gamma_{yz} \)
- \( \varepsilon \)
- \( \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \)
- \( \sigma_{5}, \sigma_{6} \)
- \( \sigma_{1}^{0}, \sigma_{2}^{0}, \sigma_{3}^{0}, \sigma_{4}^{0} \)
- \( \sigma_{5}^{0}, \sigma_{6}^{0} \)
- \( \tau_{x}^{0}, \tau_{y}^{0}, \tau_{z}^{0} \)
- \( \tau_{xy}, \tau_{xz}, \tau_{yz} \)
- \( \rho_1, \rho_2, \rho_3, \rho_4 \)
- \( \psi_1, \psi_2, \psi_3, \psi_4 \)
- \( \Theta_1, \Theta_2, \Theta_3, \Theta_4 \)
- \( \theta_1, \theta_2, \theta_3, \theta_4 \)

**ENGINEERING SHEAR STRAINS**

- \( \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \)
- \( \sigma_{5}, \sigma_{6} \)
- \( \sigma_{1}^{0}, \sigma_{2}^{0}, \sigma_{3}^{0}, \sigma_{4}^{0} \)
- \( \sigma_{5}^{0}, \sigma_{6}^{0} \)
- \( \tau_{x}^{0}, \tau_{y}^{0}, \tau_{z}^{0} \)
- \( \tau_{xy}, \tau_{xz}, \tau_{yz} \)

**DIRECT STRESSES IN THE EPOXY**

- \( \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \)
- \( \sigma_{5}, \sigma_{6} \)
- \( \sigma_{1}^{0}, \sigma_{2}^{0}, \sigma_{3}^{0}, \sigma_{4}^{0} \)
- \( \sigma_{5}^{0}, \sigma_{6}^{0} \)
- \( \tau_{x}^{0}, \tau_{y}^{0}, \tau_{z}^{0} \)
- \( \tau_{xy}, \tau_{xz}, \tau_{yz} \)

**IN SITU STRESSES TO BE MEASURED**

- \( \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \)
- \( \sigma_{5}, \sigma_{6} \)
- \( \sigma_{1}^{0}, \sigma_{2}^{0}, \sigma_{3}^{0}, \sigma_{4}^{0} \)
- \( \sigma_{5}^{0}, \sigma_{6}^{0} \)
- \( \tau_{x}^{0}, \tau_{y}^{0}, \tau_{z}^{0} \)
- \( \tau_{xy}, \tau_{xz}, \tau_{yz} \)

**IN SITU SHEARS TO BE MEASURED**

- \( \gamma_{xz}, \gamma_{yz} \)
- \( \varepsilon \)
- \( \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \)
- \( \sigma_{5}, \sigma_{6} \)
- \( \sigma_{1}^{0}, \sigma_{2}^{0}, \sigma_{3}^{0}, \sigma_{4}^{0} \)
- \( \sigma_{5}^{0}, \sigma_{6}^{0} \)
- \( \tau_{x}^{0}, \tau_{y}^{0}, \tau_{z}^{0} \)
- \( \tau_{xy}, \tau_{xz}, \tau_{yz} \)
At this stage we assume that the rock far away from chamber itself on the $p_0$ concentration of stress about the hole. The resulting this hole is in a state of uniform compression. Thus the from the investigation chamber that the effect of the axes. The hole is drilled to a distance sufficiently far direction of the $z$ axis, see Fig. 3 for the coordinate of generality, by considering the measurement of an elastic stress state near the middle section of the pilot hole is then:

$$\sigma_z = p, \quad \sigma_r = p\left(1 - \frac{R_z^2}{r^2}\right), \quad \sigma_\theta = p\left(1 + \frac{R_z^2}{r^2}\right)$$

where $R_z$ is the radius of the pilot hole and compression is taken as positive. The pilot hole must be long enough not only to neglect the effects of its own ends, but also to neglect the disturbance in stress caused by the larger overcored hole, Fig. 1(a).

Next, Fig. 1(c), one of the various measuring devices is cemented into the pilot hole. In the case of the Leeman technique three rosette strain gauges are merely fixed to the surface of the rock. In the case of the triaxial hollow inclusion stress gauge the thin walled epoxy probe is cemented into the hole. Similarly with the solid inclusion stress gauge. At this stage in all three methods there is circumferential strain in the rock but no strain in the measuring instrument.

Finally, Fig. 1(d), the instrument is overcored. The diameter of the overcored piece of rock typically ranges between 75 and 150 mm, i.e. from about 2 to 4 times the diameter of the pilot hole. It is clear that, when there is no probe in the hole, i.e. for the Leeman technique, the overcoring releases the strains not only due to the in $in situ$ stress state but also those strains due to the introduction of the pilot hole. This is true regardless of the overcorering diameter.

Thus strains equal but opposite in sign to those on the surface of the pilot hole at the stage shown in Fig. 1(c), are induced in the strain gauges, and from these measured strains the $in situ$ stress state can be inferred provided the linear elastic constants of the rock can be measured [1].

When a hollow probe is cemented to the rock, the strains induced in the probe by overcoreing, can be related to the $in situ$ stress state by a linear elastic analysis. Because of the “soft” nature of the hollow probe the stresses and strains in the rock are disturbed only in a region very close to the probe (about a probe diameter), see Fig. 8. Hence provided we overcore away from this region we can use the same argument as in the Leeman technique that the strains observed in the probe are the same as if the overcoreing diameter were very large.

The above considerations are not true for the solid probe, Duncan Fama [6].

**OPERATION OF THE MEASURING INSTRUMENT**

The three techniques can be illustrated, with no loss of generality, by considering the measurement of an $in situ$ stress state that is hydrostatic.

Figure 1 illustrates the steps involved.

In Fig. 1(a) a large diameter hole is drilled in the direction of the $z$ axis, see Fig. 3 for the coordinate axes. The hole is drilled to a distance sufficiently far from the investigation chamber that the effect of the chamber itself on the $in situ$ stresses can be neglected. At this stage we assume that the rock far away from this hole is in a state of uniform compression. Thus the stress state is specified completely by the parameter, $p$.

When the pilot hole is drilled, Fig. 1(b), there is a concentration of stress about the hole. The resulting elastic stress state near the middle section of the pilot hole is then:

$$\sigma_z = p, \quad \sigma_r = p\left(1 - \frac{R_z^2}{r^2}\right), \quad \sigma_\theta = p\left(1 + \frac{R_z^2}{r^2}\right)$$

where $R_z$ is the radius of the pilot hole and compression is taken as positive. The pilot hole must be long enough not only to neglect the effects of its own ends, but also to neglect the disturbance in stress caused by the larger overcored hole, Fig. 1(a).

The three techniques can be illustrated, with no loss of generality, by considering the measurement of an $in situ$ stress state that is hydrostatic.

**STRAINS IN THE INSTRUMENT**

Strains, occasioned by the release of the $in situ$ stresses $\sigma_\theta^0, \sigma_r^0, \tau_{xy}^0, \tau_{yz}^0, \tau_{zx}^0$, are induced in the instrument during overcoreing.

The full solution for the stresses and strains in both the probe and the host rock is presented in Appendix A.

The following are the strains in the probe at position $(\rho, \theta)$, see Fig. 3 (the minus sign is inserted because the strains actually measured are negative; in the remainder of the paper the minus sign is omitted).

$$-E_2 e_\rho = [\sigma_\rho^0 + \sigma_\theta^0]K_1(\rho) - v_2\sigma_\rho^0 K_4(\rho) - 2[1 - v_2^2]$$

$$\times [\{\sigma_\rho^0 - \sigma_\theta^0\} \cos 2\theta + 2\tau_{xy}^0 \sin 2\theta]K_2(\rho)$$

$$-E_2 e_\theta = \sigma_\rho^0 - v_2[\sigma_\rho^0 + \sigma_\theta^0]$$

$$-E_2 e_\phi = 4(1 + v_2)[-r_\rho^0 \sin \theta + r_\phi^0 \cos \theta]K_3(\rho)$$

where

$$K_1(\rho) = d_1(1 - v_1 v_2)\left[1 - 2v_1 + \frac{R_z^2}{\rho^2}\right] + v_1 v_2$$

$$K_2(\rho) = (1 - v_1)d_2\rho^2 + d_3 + v_1 d_4 + \frac{d_5}{\rho^4}$$

$$K_3(\rho) = d_6\left[1 + \frac{R_z^2}{\rho^2}\right]$$

$$K_4(\rho) = -\frac{(v_1 - v_2)d_1}{v_2}\left[1 - 2v_1 + \frac{R_z^2}{\rho^2}\right] + v_1 v_2$$

where: $\rho$ is the radial coordinate at the position of the strain gauges. These $K$ factors were introduced by Worotnicki & Walton [8].

Note that the expressions above for the strains reduce to the strains on the inside surface of the rock for the Leeman solution when $K_1 = K_2 = K_3 = K_4 = 1$. Thus the $K$ factors are a measure of the difference between the strains in the probe at a small distance from the epoxy probe interface and the strains measured in the Leeman technique on the inside of the pilot hole.

The constants $d_1, \ldots, d_6$ are given in Appendix A. The $K$ factors for two different instruments are tabulated in Appendix B. The instrument developed by Worotnicki and Walton has an inner radius of 16 mm, while the instrument developed by Pender has an inner radius of 14 mm. In both cases the strain gauges are at
The effect of increasing the ratio \( E_1/E_2 \) is shown in Fig. 4. The table given by Worotnicki and Walton [8] is somewhat misleading as it appears that the \( K \) factors tend to unity as the ratio \( E_1/E_2 \) goes to 1. Figure 4 shows that this is not the case. When measurements are made in hard rock, \( E_1/E_2 \) is typically less than 0.10. When measurements are made in soft rocks \( E_1/E_2 \) may be close to unity.

An interesting feature of Fig. 4 is the fact that the curves for \( K_1, K_2, K_4 \) all take the value unity for \( \epsilon \) close to 0.2. From Appendix A:

\[
d_1 = \frac{1}{[1 - 2v_1 + m^2 + \epsilon(1 - m^2)]}
\]

and

\[
d_6 = \frac{1}{1 + m^2 + \epsilon(1 - m^2)}
\]

so then \( K_3(\rho) \) (equation 6) is also unity. Thus in Fig. 4 the three curves \( K_1, K_2 \) and \( K_4 \) all take the value unity when \( \epsilon = 0.212 \) (for this case with \( R_1 = 14, \rho = 17.5, R_2 = 19 \) giving \( m = 0.737 \)). For \( R_1 = 16 \) the corresponding value of \( \epsilon \) is 0.440. Figure 5 shows the variation of the \( K \) factors with the Poisson's ratio of the epoxy. Figure 6 gives the effect on the \( K \) factors of the variation of the Poisson's ratio of the rock. Figure 7
shows the dependence of the $K$ factors on the position of the strain gauges for the Pender probe. The dashed line shows the corresponding curve for $K_z$ when $R_1 = 16\, \text{mm}$ (Worotnicki). The other three curves for this probe lie close to the corresponding ones shown.

**COMPUTATION OF ROCK STRESSES FROM MEASURED STRAINS**

For the Pender probe the strain gauges are positioned around the circumference of the probe at positions $\theta = 0$, $\theta = \pi/2$ and $\theta = 5\pi/4$. These positions were also used by Rocha et al. [7]. As will be seen below these particular $\theta$-values give rise to a compact analysis. The analysis herein can be developed for any $\theta$-values, but the resulting equations are not as convenient as equation 12-15 and 19 and 20 below. At $\theta = 0$ and $\theta = 5\pi/4$ the three gauges are in directions $\phi = 0$, $\pi/2$, $\pi/4$, at $\theta = \pi/2$ the directions are $\phi = 0$, $\pi/2$, $-\pi/4$.

The gauge $\phi = 0$ measures $e_z$, $\phi = \pi/2$ measures $e_\theta$ and $\phi = \pi/4$ measures $e_{45} = \frac{1}{2}(e_\theta + e_z + \gamma_{\theta z})$.

The three measurements for $e_\theta$ are averaged, to give a most probable value for $e_\theta$. Then this value together with the three $e_\theta$ values determine $\sigma_\theta^0\sigma_\theta^0\sigma_\theta^0$ and $\tau_{yz}^0$ from the following equations:

\begin{equation}
E_2e_\theta = \sigma_\theta^0 - v_2[\sigma_x^0 + \sigma_y^0] (8)
\end{equation}

\begin{equation}
E_2e_\theta(0) = [\sigma_x^0 + \sigma_y^0]K_1 - [\sigma_x^0 - \sigma_y^0]K_2 - \sigma_\theta^0K_4 (9)
\end{equation}

\begin{equation}
E_2e_\theta \left[ \frac{\pi}{2} \right] = [\sigma_x^0 + \sigma_y^0]K_1 + [\sigma_x^0 - \sigma_y^0]K_2 - \sigma_\theta^0K_4 (10)
\end{equation}

\begin{equation}
E_2e_\theta \left[ \frac{5\pi}{4} \right] = [\sigma_x^0 + \sigma_y^0]K_1 - 2e_{45}K_2 - \sigma_\theta^0K_4 (11)
\end{equation}

where

\begin{equation}
K_1 = K_1(\rho)
\end{equation}

\begin{equation}
K_2 = 2[1 - v_2^2]K_4(\rho)
\end{equation}

and

\begin{equation}
K_4 = v_2K_4(\rho)
\end{equation}

From these we obtain in closed form:

\begin{equation}
\sigma_\theta^0 = \frac{E_2}{K_1 - v_2K_4} \left( K_1e_\theta + \frac{v_2}{2} \left[ e_\theta(0) + e_\theta \left( \frac{\pi}{2} \right) \right] \right) (12)
\end{equation}

\begin{equation}
\sigma_x^0 = \frac{K_4\sigma_x^0}{2K_1} + \frac{E_2}{4K_1K_2} \left( K_1e_\theta + \frac{v_2}{2} \left[ e_\theta(0) + e_\theta \left( \frac{\pi}{2} \right) \right] \right) (13)
\end{equation}

\begin{equation}
\sigma_y^0 = \frac{K_4\sigma_y^0}{2K_1} + \frac{E_2}{4K_1K_2} \left( K_1e_\theta + \frac{v_2}{2} \left[ e_\theta(0) + e_\theta \left( \frac{\pi}{2} \right) \right] \right) (14)
\end{equation}

The three measurements for the two shears $\tau_{yz}^0$ and $\tau_{xz}^0$ are:

\begin{equation}
\tau_{yz}^0 = -\frac{1}{2K_2} \left( E_2e_\theta \left[ \frac{5\pi}{4} \right] + a_2^0\bar{K}_A - [\sigma_x^0 + \sigma_y^0]\bar{K}_1 \right) (15)
\end{equation}

The three measurements for $\tau_{yz}^0$ and $\tau_{xz}^0$ are:

\begin{equation}
\tau_{yz}^0 = \frac{E_2\gamma_{\theta z}(0)}{4[1 + v_2]K_3} (16)
\end{equation}

\begin{equation}
\tau_{xz}^0 = -\frac{E_2\gamma_{\theta z}(\frac{\pi}{2})}{4[1 + v_2]K_3} (17)
\end{equation}

\begin{equation}
\tau_{yz}^0 - \tau_{xz}^0 = \sqrt{2E_2\gamma_{\theta z}(\frac{5\pi}{4})} \frac{5\pi}{4} \frac{1}{4[1 + v_2]K_3} (18)
\end{equation}

where $K_3 = K_3(\rho)$.

The "best fit" (least squares) solution is:

\begin{equation}
tan_x^0 = \frac{E_2}{16[1 + v_2]K_3} \left\{ -3\gamma_{\theta z}(0) + \gamma_{\theta z}(\frac{\pi}{2}) + \sqrt{2\gamma_{\theta z}(\frac{5\pi}{4})} \right\} (19)
\end{equation}

\begin{equation}
tan_y^0 = \frac{E_2}{16[1 + v_2]K_3} \left\{ -3\gamma_{\theta z}(0) - \gamma_{\theta z}(\frac{\pi}{2}) - \sqrt{2\gamma_{\theta z}(\frac{5\pi}{4})} \right\} (20)
\end{equation}

These equations were used to analyse five successful sets of strain measurements taken at the site of the Rangipo powerhouse, near Turangi, New Zealand, Pender [9]. One set of measurements is:

\begin{equation}
\begin{array}{cccccc}
e_\theta & e_x(0) & e_\theta \left( \frac{\pi}{2} \right) & e_{45}(0) & e_{45} \left( \frac{\pi}{2} \right) & e_{45} \left( \frac{5\pi}{4} \right) & \text{unit micro-strain} \\
50 & 10 & 250 & -10 & -50 & 200 & 74
\end{array}
\end{equation}

The stresses calculated from equations (12–15, 19 and 20) are:

\begin{equation}
\begin{array}{cccccc}
\sigma_x^0 & \sigma_y^0 & \sigma_z^0 & \tau_{xy}^0 & \tau_{xz}^0 & \tau_{yz}^0 \\
7.0 & 2.5 & 5.8 & 2.6 & 1.0 & -1.8 \\
7.4 & 3.0 & 6.1 & 2.6 & 1.0 & -1.8 \\
\end{array}
\end{equation}

The results in the row marked P are for the full analysis with the appropriate $K$ factors. The results in the row marked L are obtained using the Leeman analysis, i.e. assuming that the strain gauges are cemented directly to the rock.

The principal stresses for each set, with the Leeman results in brackets, are as follows:

\begin{equation}
\begin{array}{cccccc}
\sigma_1 & \sigma_2 & \sigma_3 \\
8.2(8.6) & 6.6(6.9) & 0.5(1.0) \\
23.8 & 85.1 & 113.1 \\
66.7 & 112.3 & 33.2 \\
85.6 & 22.9 & 67.6 \\
\end{array}
\end{equation}

The directions obtained from the Leeman analysis

\begin{equation}
\begin{array}{cccccc}
\text{Direction} & \text{angles} \\
23.8 & 66.7 & 85.6 \\
85.1 & 112.3 & 22.9 \\
113.1 & 33.2 & 67.6 \\
\end{array}
\end{equation}
were virtually identical to those with the full analysis. The orientation of the principal stress directions for the results are shown on an equal angle upper hemisphere stereographic projection in Fig. 11.

The calculations were based on a Poisson's ratio of 0.25 and a Young's modulus of 69.0 GPa for the rock, and, for the epoxy, a Poisson's ratio of 0.40 and a Young's modulus of 3.45 GPa.

**RADIAL STRESSES DEVELOPED AT THE EPOXY-ROCK INTERFACE**

Of interest for the two techniques in which an epoxy inclusion is cemented into the pilot hole is the radial stress at the epoxy/rock bond. Because compressive stresses are released during the overcoring the residual radial stress at the interface is tensile. These stresses are shown in Fig. 8. The stresses in the hollow inclusion gauge are, as expected, relatively small because the radial stress at the inner surface of the inclusion must be zero. For the solid inclusion the stress is constant across the diameter, Muskhetishvili [11], and may be as high as the levels of in situ stress normal to the probe, Duncan Fama [6].

Figure 9 shows the dependence of the bond stress on the parameter \( \varepsilon \) for \( v_1 = 0.45 \). In Fig. 10 the bond stress is plotted against \( v_1 \) for \( \varepsilon = 0.1 \). The parameter \( n \) for the solid probe is the ratio of the overcoring diameter to the diameter of the probe [6]. In Fig. 10 the dashed curve shows the bond stress for a solid probe with \( \varepsilon = 0.017 \)—it coincides with the Pender (hollow) curve up to about \( v_1 = 0.35 \). This shows the substantial reduction in bond stress achieved for the solid probe when \( \varepsilon \) is very small, provided \( v_1 \) is below about 0.45. For these figures \( v_2 = 0.2 \) and the in situ stress state is \( \sigma_2^o = 4, \sigma_3^o = \sigma_y^o = 1 \). Figure 9 shows that in “soft” rocks (\( \varepsilon > 0.4 \)) the bond stress will be large, of the order of the in situ stresses, even for the hollow probe. However, for the most commonly encountered range of \( \varepsilon \) (\( \varepsilon < 0.2 \)), there should be no problems unless the in situ stresses normal to the probe are high.

Clearly the bond stresses for a given in situ stress state will be minimized by a favourable orientation of the probe (along the axis of maximum principal stress). It is of interest to note that, according to Blackwood et
al. [4] calculated in situ stresses, was the instrument oriented in the most favourable direction.

CONCLUSIONS

The mathematical analysis of a technique for the measurement of in situ rock stress has been set out. Previous workers have introduced various simplifying assumptions into the analysis. These are shown to be unnecessary so that the analysis presented is exact.

The procedure for obtaining the stresses from the strains measured during overcoring is based on the analysis developed by Leeman [1]. In the Leeman technique three rosette strain gauges are cemented directly to the surface of the rock. The inclusion technique discussed in this paper has the strain gauges positioned a little distance from the rock embedded in a different material. The analysis arrives at equations of the same form as those used by Leeman. A series of factors, originally devised by Worotnicki & Walton [8], are a measure of the difference between the strains observed in this technique and those in the Leeman technique. Detailed equations for these K factors are introduced in the paper. By means of diagrams and tables the sensitivity of the K factors to variations in the properties of the rock and the probe are given.

The ratio of the shear modulus of the rock to that of the probe material has a most significant effect on the analysis. It was found that when measurements are made in hard rock (i.e. high modulus) the effect of the epoxy probe is very small. To a very good approximation the presence of the epoxy can be ignored and the technique can be used in soft rock (i.e. rock modulus \( \approx \) epoxy modulus), in this case the full analysis is needed.

The radial tensile stress at the epoxy/rock bond is considered in detail. It is found, not surprisingly, that this stress is a function not only of the properties of the rock and epoxy but also of the orientation of the instrument with respect to the stress system.

It is shown that a solid probe should not be used unless the ratio of the shear modulus of the probe to that of the host rock is extremely small [5]—or the level of in situ stress normal to the probe is known to be low. When these conditions are not satisfied a hollow probe should be employed, and where the shear modulus ratio is not small and the in situ stress level high it may be necessary to use a hollow probe of minimal thickness [7,8].

APPENDIX A

The mathematical solution is required in order to relate the strains measured in the probe to the in situ stresses in the host rock. The region of interest, where the strain gauges are attached to the circular probe, is far enough away from the ends of the probe so that we can assume that there is no variation in strains or stresses along the length of the probe, which we take as the z direction for cylindrical polar coordinates, \( r, \theta, z \).

The equilibrium equations are then as follows:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (A1)
\]

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \quad (A2)
\]

\[
\frac{\partial \sigma_\theta}{\partial r} - r \tau_{r\theta} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} = 0 \quad (A3)
\]

Solutions to these equations are to be found in the two regions:

1. \( R_1 \leq r \leq R_2 \) occupied by the probe
2. \( R_2 \leq r \) occupied by the host rock

The boundary conditions are as follows:

1. At \( r = R_1 \), \( \sigma_r = 0 \), \( \tau_{r\theta} = 0 \), \( \tau_{\theta\theta} = 0 \)
2. At \( r = R_2 \), \( \sigma_r = \sigma_r^p \), \( \tau_{r\theta} = \tau_{r\theta}^p \), \( \tau_{\theta\theta} = \tau_{\theta\theta}^p \) and
   \[
   u_r = u_r^p \quad e_r = e_r^p \quad u_\theta = u_\theta^p
   \]
3. As \( r \to \infty \), the stresses in the rock approach the in situ stresses present before the introduction of the pilot hole.

In addition we have the following strain-displacement relations:

\[
\varepsilon_r = \frac{\partial u}{\partial r} \quad (A4)
\]
and stress-strain relations (linear)
\[ e_e = \frac{1}{E} [\sigma_e - \nu (\sigma_s + \sigma_\theta)] \]
\[ e_\theta = \frac{1}{E} [\sigma_\theta - \nu (\sigma_s + \sigma_e)] \]
\[ e_\theta = \frac{1}{E} [\sigma_\theta - \nu (\sigma_s + \sigma_\theta)] \]

\[ 2e_e = \gamma_e = \frac{1}{G} \tau_{ee} \]
\[ 2e_\theta = \gamma_\theta = \frac{1}{G} \tau_{e\theta} \]

Let us consider the implications of boundary condition (2). Referring to the strain-displacement conditions we see that \( w_0 = w_o \) at \( r = R_2 \) implies that \( e_0 = e_o \) at \( r = R_2 \). We assume plane strain conditions where the gauges are positioned.

This implies that \( e_s = e_s = E_3 \) is constant in both regions. Furthermore boundary condition (2) implies that

\( e_0 = e_o = e_0 = E_3 \) at \( r = R_2 \)

(since \( u' = u'' \) then \( \frac{\partial u'}{\partial \theta} = \frac{\partial u''}{\partial \theta} \) at \( r = R_2 \) etc.).

Thus the strains measured by the strain gauges (directly or indirectly) are continuous across the interface \( r = R_2 \). But the strains \( e_s, e_\theta, e_\theta \) are definitely not continuous at \( r = R_2 \)—in fact in the case of the latter two since \( \gamma = \gamma = G_1/2 \) and \( G_1/G_2 \) may be as little as 0.017—it is clear that these strains are highly discontinuous.

The full solution to the equations is now obtained as the superposition of three solutions:

(a) that which arises from the stresses \( \sigma_s, \sigma_\theta, \tau_{r\theta} \)

(b) that which arises from \( S_s = \sigma_s - \nu (\sigma_s + \sigma_\theta) \)

(c) that which arises from \( T_{e\theta} \)

This is derived here, and is as straightforward as the contribution from (b). Previous analyses have approximated this part of the solution—in particular Worotnicki's extrapolation of the shear strain \( \gamma_\theta \) from the Leeman solution turns out to be within 2% of the exact solution at the position of the strain gauges, for his device.

Figure A1 shows a plot of this shear strain showing the three solutions including Leeman's extrapolated into the epoxy. Also shown is Rocha's (and Blackwood's) approximation to the shear strain in the solid probe compared with exact solution.†

The full solution in the epoxy follows:

\[ \sigma_s = \frac{E_1}{E_3} [\sigma_\theta - \nu (\sigma_s + \sigma_\theta)] + \frac{2\nu_1\varepsilon_1}{1 + \nu_2} [(1 - \nu_1\nu_2)(\sigma_s + \sigma_\theta) + (\nu_1 - \nu_2)\sigma_\theta] + 2\nu_1\varepsilon_1(1 - \nu_2) \left[ -d_3r_\theta - \frac{d_3d_5}{r_\theta^2} \right] (\sigma_s - \sigma_\theta) \cos 2\theta + 2s_\theta^2 \sin 2\theta] 

\[ \tau_{e\theta} = 2\varepsilon_1 (1 - \nu_2) \left[ -d_3r_\theta - \frac{d_3d_5}{r_\theta^2} \right] (\sigma_s - \sigma_\theta) \cos 2\theta + 2s_\theta^2 \sin 2\theta] 

† Blackwood (personal communication) later also derived this part of the solution for the solid probe.
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\[ W_l = \frac{2d_4}{G_2} \left[ \frac{r + R_1}{r} \right] \left[ \sigma^0_{\theta} \cos \theta + \sigma^0_{\phi} \sin \theta \right] \]

\[ + \frac{z}{E_2} \left[ \sigma^0_{\theta} - \sigma^0_{\phi} \right] \]

where the constants \(d_1, \ldots, d_4\) are given by

\[ d_1 = \frac{1}{1 - 2v_1 + \epsilon(1 - m^2)} \]

\[ d_2 = \frac{12(1 - \epsilon)\pi t^2 (1 - m^2)}{R_1^2 D} \]

\[ d_3 = \frac{1}{D} \left[ m^4(4m^2 - 3)(1 - \epsilon) + \chi_1 + \epsilon \right] \]

\[ d_4 = -\frac{4R_1^2}{D} \left[ m^4(1 - \epsilon) + \chi_1 + \epsilon \right] \]

\[ d_5 = \frac{3R_1^2}{D} \left[ m^4(1 - \epsilon) + \chi_1 + \epsilon \right] \]

\[ d_6 = \frac{1}{1 + m^4 + \epsilon(1 - m^2)} \]

where

\[ \epsilon = \frac{G_1}{G_2} \]

Note that usually \(\epsilon \ll 1\) (typically \(\epsilon = 0.05\) for hard rocks)

\[ m = \frac{R_1}{R_2} \]

for the hollow cylinder \(m\) is close to 1

\[ \chi_i = 3 - 4v_i, \quad i = 1, 2 \]

so

\[ 1 + \chi_i - 4(1 - v_i) \]

\[ \chi_i - 1 = 2(1 - 2v_i) \quad i = 1, 2 \]

and

\[ D = (1 + 2\epsilon)[(1 + \epsilon) + (1 - \epsilon)(3m^2 - 6m^4 + 4m^6)] \]

\[ + (\chi_1 - \chi_2)2m^4(1 - \epsilon) + \chi_1 + \epsilon] \]

Note that \(e_1, e_2, \chi_0\) are written out in Section 2.

In region II (the rock) the solution is:

\[ \sigma^0_{\theta} = \frac{1 - C_4}{r^2} \sigma^0_{\theta} + \frac{C_4}{r^2} \]

\[ \sigma^0_{\phi} = \frac{1 + C_4}{r^2} \sigma^0_{\phi} + \frac{C_4}{r^2} \]

\[ + \frac{1}{2} \frac{[\sigma^0_{\theta} - \sigma^0_{\phi}] \cos \theta + 2\sigma^0_{\theta} \sin \theta]}{r^2} \]

\[ \sigma^0_{\theta} = \sigma^0_{\phi} + \frac{v_2 C_4}{2r^2} \left[ (\sigma^0_{\theta} - \sigma^0_{\phi}) \cos \theta + 2\sigma^0_{\phi} \sin \theta \right] \]

\[ t^0_{\theta} = \frac{1}{2} \left[ 1 + \frac{C_4}{2} \right] \left[ (\sigma^0_{\ theta} - \sigma^0_{\phi}) \cos 2\theta + 2\sigma^0_{\phi} \sin 2\theta \right] \]

\[ t^0_{\phi} = \frac{1}{2} \left[ 1 + \frac{C_4}{2} \right] \left[ (\sigma^0_{\theta} - \sigma^0_{\phi}) \cos 2\theta + 2\sigma^0_{\phi} \sin 2\theta \right] \]

\[ u^0 = \frac{\sigma^0_{\theta} + \sigma^0_{\phi}}{2E_2} \left[ (1 - v_2) \frac{r + (v_1 - v_2)L}{r} \right] \]

where

\[ C_1 = \frac{R_2^2}{1 - 2(1 - m^2)(1 - v_1 v_2)} \]

\[ C_3 = R_2^2 \left[ 1 - \frac{(1 - \epsilon)(3m^2 - 3m^4 + m^6) + \chi_1 + \epsilon]}{D} \right] \]

\[ C_5 = 3R_2^2 \left[ 1 - \frac{(1 - m^2)(1 + \chi_1)}{D} \right] \]

\[ C_6 = R_2^2 \left[ 1 - (1 - m^2)(1 + \chi_1) \right] \]

\[ C_7 = \frac{R_2^2}{1 - 2(1 - m^2)(1 - v_1 v_2)} \]

\[ \epsilon\left[ \epsilon(1 + \chi_1) \right] \]

\[ \epsilon\left[ \epsilon(1 + \chi_1) \right] \]

\[ \epsilon\left[ \epsilon(1 + \chi_1) \right] \]

\[ \epsilon\left[ \epsilon(1 + \chi_1) \right] \]

Special cases

(i) Leeman's solution is obtained by setting \(R_1 = R_2\), i.e. \(m = 1\).

The rock solution is given by the above with

\[ C_1 = -R_2^2 \]

\[ C_3 = -4R_2^2 \]

\[ C_5 = 3R_2^2 \]

\[ C_6 = -R_2^2 \]

Hence, since

\[ e_0 = \frac{u^0}{r} + \frac{1}{r} \frac{\partial u^0}{\partial \theta} \]

at \(r = R_2\):

\[ E_2\epsilon_0 - (\sigma^0_{\theta} + \sigma^0_{\phi}) - v_2 \sigma^0_{\phi} - 2(1 - v_2)1(\sigma^0_{\theta} - \sigma^0_{\phi}) \cos 2\theta + 2\sigma^0_{\phi} \sin 2\theta \]

etc.

(ii) The solution for the solid probe is obtained by setting \(R_1 = 0\), i.e. \(m = 0\) also. Hence

\[ d_1 = \frac{1}{1 - 2v_1 + \epsilon} \]

\[ d_2 = d_4 = d_5 = 0 \]

\[ d_3 = \frac{1}{1 + 2\epsilon} \]

\[ d_6 = \frac{1}{1 + \epsilon} \]

and

\[ c_1 = R_2^2 \left[ 1 - (1 - m^2)(1 - v_1 v_2) \right] \]

\[ c_4 = -4R_2^2 \left[ 1 - \frac{(1 - \epsilon)(3m^2 - 3m^4 + m^6) + \chi_1 + \epsilon]}{D} \right] \]

\[ c_5 = 3R_2^2 \left[ 1 - \frac{(1 - m^2)(1 + \chi_1)}{D} \right] \]

\[ c_6 = R_2^2 \left[ 1 - (1 - m^2)(1 + \chi_1) \right] \]

\[ c_7 = \frac{R_2^2}{1 - 2(1 - m^2)(1 - v_1 v_2)} \]
where there is no $r$ dependence in the solution and the $	heta$ dependence is such that the stresses $a_t$, $a_r$, $a_\theta$, $z_t$, $z_r$, and their corresponding strains are constant.

The strains in the probe are then:

$$E_2 e_{r} = a_0 (K_1 + K_2) + a \theta (K_1 - K_2) - a + \theta K_1$$

$$E_2 e_{\theta} = a_\theta (K_1 - K_2) + a_\theta (K_1 + K_2) - a_\theta K_2$$

$$E_2 e_\varphi = \lambda (1 - v_\varphi) + \gamma (1 - v_\varphi)$$

$$E_2 e_r \gamma_{r\varphi} = \gamma_{r\varphi} (\theta = 0) = \frac{1}{G_1} t_{r\varphi} (\theta = 0)$$

where

$$K_1 = \frac{(1 - v_1 v_2)(1 - 2v_1)}{1 - 2v_1 + \epsilon} + v_1 v_2$$

$$K_2 = \frac{2(1 - v_1)}{1 - v_1}$$

$$K_4 = \frac{(v_1 - v_2)(1 - 2v_1)}{1 - 2v_1 + \epsilon} + v_1$$

$$e = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$$

For the solid probe these equations can be inverted as before and the stresses are given by equations (12-15, 19 and 20) with $K_1 K_2 K_3$ above and $K_3 = \frac{1}{1 + \epsilon}$, provided the strain gauges are positioned as in the Pender probe.

APPENDIX B

| TABLE BI. (WOROTNICKI PROBE, $R_1 = 16$ mm, $\rho = 17.5$ mm) |
|---|---|---|---|---|
| $E_1/E_2$ | $v_1$ | $v_2$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ |
| 0.05 | 0.4 | 0.2 | 1.114 | 1.112 | 1.066 | 0.876 |
| 0.05 | 0.4 | 0.25 | 1.111 | 1.112 | 1.066 | 0.926 |
| 0.05 | 0.45 | 0.25 | 1.123 | 1.108 | 1.066 | 0.889 |
| 0.05 | 0.49 | 0.25 | 1.107 | 1.081 | 1.058 | 0.935 |
| 0.1 | 0.45 | 0.25 | 1.108 | 1.075 | 1.059 | 0.902 |
| 0.1 | 0.5 | 0.25 | 1.122 | 1.067 | 1.059 | 0.860 |
| 0.2 | 0.4 | 0.25 | 1.070 | 1.023 | 1.042 | 0.953 |
| 0.2 | 0.45 | 0.25 | 1.079 | 1.013 | 1.043 | 0.929 |
| 0.33 | 0.4 | 0.25 | 1.036 | 0.954 | 1.022 | 0.976 |
| 0.5 | 0.4 | 0.25 | 0.997 | 0.880 | 0.998 | 1.002 |
| 0.8 | 0.4 | 0.25 | 0.935 | 0.773 | 0.958 | 1.044 |
| 1.0 | 0.4 | 0.25 | 0.898 | 0.714 | 0.932 | 1.068 |

| TABLE BII. (PENDER PROBE, $R_1 = 14$ mm, $\rho = 17.5$ mm) |
|---|---|---|---|---|
| $E_1/E_2$ | $v_1$ | $v_2$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ |
| 0.05 | 0.4 | 0.2 | 1.093 | 0.972 | 1.050 | 0.898 |
| 0.05 | 0.4 | 0.25 | 1.090 | 0.973 | 1.049 | 0.940 |
| 0.05 | 0.45 | 0.25 | 1.104 | 0.959 | 1.050 | 0.907 |
| 0.05 | 0.49 | 0.25 | 1.117 | 0.945 | 1.050 | 0.872 |
| 0.1 | 0.5 | 0.25 | 1.121 | 0.943 | 1.050 | 0.861 |
| 0.1 | 0.45 | 0.25 | 1.075 | 0.912 | 1.036 | 0.932 |
| 0.1 | 0.5 | 0.25 | 1.089 | 0.894 | 1.037 | 0.898 |
| 0.2 | 0.4 | 0.25 | 1.017 | 0.848 | 1.010 | 0.899 |
| 0.2 | 0.45 | 0.25 | 1.022 | 0.830 | 1.011 | 0.980 |
| 0.33 | 0.4 | 0.25 | 0.980 | 0.758 | 0.977 | 1.027 |
| 0.5 | 0.4 | 0.25 | 0.997 | 0.880 | 0.998 | 1.002 |
| 0.8 | 0.4 | 0.25 | 0.897 | 0.545 | 0.877 | 1.129 |
| 1.0 | 0.4 | 0.25 | 0.757 | 0.482 | 0.841 | 1.162 |

Note added in proof

We are indebted to Blackwood (personal communication) for elucidation of the reasons for his analysis (1977). His aim is to arrive at a method of incorporating data from ten measurements involving the six strains $e_x$, $e_y$, $e_z$, $e_{rx}$, $e_{ry}$, $e_{rz}$. Rocha & Silverio [2] were also doing this—and Worotnicki and Walton [8] included it in their analysis. The statistics involved are clearly set out by Panek [12]. In order to clear up what has caused some controversy we analyse the measured strains given by Rocha & Silverio [2].

We replace their equations (1)-(6) p. 119 by the following ten equations derived directly from the measurements:

$$e_x = e_1$$

$$e_y = e_2$$

$$e_z = e_3$$

$$e_\varphi = e_4$$

$$e_\theta = e_5$$

$$e_r + e_\theta = e_6 + e_9$$

$$e_\varphi + e_r = e_7 + e_{10}$$

$$e_x + e_\varphi = e_4 + e_6$$

We then get the “best fit” solution for the three strains $e_x$, $e_y$, $e_z$ from the first seven of these equations ($e_{rx}$, $e_{ry}$, $e_{rz}$ are given directly by the last three equations). The results for their measured strains on p. 131 are as follows:

$$e_x = -288.6 \quad e_y = -283.1 \quad e_z = -13.1$$

$$\gamma_{rx} = 47 \quad \gamma_{ry} = 62 \quad \gamma_{rz} = 31$$

These are all very close to Rocha and Silverio’s results. The only slight change of any interest in the computed stresses is a drop in $a_0$ from 90 to 84 kg/cm$^2$.

In the light of these remarks we could improve the simple analysis presented in this paper for computation of rock stresses from measured strains to incorporate the “best fit” solution for the nine strain measurements. In our case we think it would make very little difference.

An interesting and important question then arises as to the “best” positioning of the strain gauges for the probes—and the possibility of attaching more gauges than nine or ten.